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Research Article

Oscillation Results for Second-Order Nonlinear Damped Dynamic Equations on Time Scales

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This paper is concerned with second-order nonlinear damped dynamic equations on time scales of the following more general form $(p(t)k_1(x(t),x^\Delta(t)))^\Delta + r(t)k_2(x(t),x^\Delta(t))x^\Delta(t) + f(t,x(\sigma(t))) = 0$. New oscillation results are given to handle some cases not covered by known criteria. An illustrative example is also presented.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{T}$ a time scale, that is, a nonempty closed subset of $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The theory of time scales was introduced by Hilger in his Ph.D. thesis [1] in 1988, and for a comprehensive treatment of the subject, see [2]. Much recent attention has been concerned with the oscillation of dynamic equations on time scales; see, for example, [1–15]. In [9], Došlý and Hilger studied the second-order dynamic equation

$$\left(p(t)x^\Delta(t)\right)^\Delta + q(t)x(\sigma(t)) = 0. \tag{1}$$

The authors gave a necessary and sufficient condition for the oscillation of all solutions of (1) on time scales. In [7, 8], Del Medico and Kong used the Riccati transformation as

$$u(t) = \frac{p(t)x^\Delta(t)}{x(t)} \tag{2}$$

and obtained some sufficient conditions for oscillation of (1). In [14], Wang considered the nonlinear second-order damped differential equation

$$\begin{aligned}&\left(a(t)\psi(x(t))k\left(x'(t)\right)\right)' + p(t)k\left(x'(t)\right)\\&\quad + q(t)f(x(t)) = 0, \quad t \geq t_0,\end{aligned} \tag{3}$$

and established new oscillation criteria. In [13], Tiryaki and Zafer considered the second-order nonlinear differential equation with nonlinear damping

$$\left(r(t)k_1\left(x,x'\right)\right)' + p(t)k_2\left(x,x'\right)x' + q(t)f(x) = 0 \tag{4}$$

and gave interval oscillation criteria of (4). In [10], Huang and Wang considered the second-order nonlinear dynamic equation

$$\left(p(t)x^\Delta(t)\right)^\Delta + f(t,x(\sigma(t))) = 0. \tag{5}$$

The authors gave some new oscillation criteria of (5) and extended the results in [7, 8]. In [11], Qiu and Wang studied the second-order nonlinear dynamic equation

$$\left(p(t)\psi(x(t))x^\Delta(t)\right)^\Delta + f(t,x(\sigma(t))) = 0. \tag{6}$$

By employing the Riccati transformation as

$$u(t) = A(t)\frac{p(t)\psi(x(t))x^\Delta(t)}{x(t)} + B(t), \tag{7}$$

where $A \in C^1_{rd}(\mathbb{T},(0,\infty))$, $B \in C^1_{rd}(\mathbb{T},\mathbb{R})$, the authors established interval oscillation criteria for (6). And in [12], Qiu and Wang obtained some new Kamenev-type oscillation

criteria for dynamic equations of the following more general form:

$$\left(p(t)\psi(x(t))k\circ x^{\Delta}(t)\right)^{\Delta}+f(t,x(\sigma(t)))=0, \tag{8}$$

by using the transformation

$$u(t)=A(t)\frac{p(t)\psi(x(t))k\circ x^{\Delta}(t)}{x(t)}+B(t). \tag{9}$$

In this paper, we consider second-order nonlinear damped dynamic equations of the form

$$\begin{aligned}&\left(p(t)k_1\left(x(t),x^{\Delta}(t)\right)\right)^{\Delta}+r(t)k_2\left(x(t),x^{\Delta}(t)\right)x^{\Delta}(t)\\&\quad+f(t,x(\sigma(t)))=0\end{aligned} \tag{10}$$

on a time scale $\mathbb{T}$. We will employ functions of the form $H(t,s)$ and a generalized Riccati transformation as (7) and (9) which was used in [14, 15] and derive oscillation criteria for (10) in Section 2. An example is presented to demonstrate the obtained results in the final section.

Definition 1. A solution x of (10) is said to have a generalized zero at $t^*\in\mathbb{T}$ if $x(t^*)x(\sigma(t^*))\le 0$, and it is said to be nonoscillatory on $\mathbb{T}$ if there exists $t_0\in\mathbb{T}$ such that $x(t)x(\sigma(t))>0$ for all $t>t_0$. Otherwise, it is oscillatory. Equation (10) is said to be oscillatory if all solutions of (10) are oscillatory.

2. Main Results

In this section, we establish some oscillation criteria for (10). Our work is based on the application of the Riccati transformation. Throughout this paper we will assume that $\sup\mathbb{T}=\infty$ and

(C1) $p\in C_{rd}(\mathbb{T},(0,\infty))$;

(C2) $r\in C_{rd}(\mathbb{T},\mathbb{R})$;

(C3) $k_1,k_2\in C(\mathbb{R}^2,\mathbb{R})$, and there exist $\alpha_1\ge\alpha_2>0$ and $\alpha_3>0$ such that $0<\alpha_2 vk_1(u,v)\le k_1^2(u,v)\le\alpha_1 vk(u,v)$ and $\alpha_3k_1^2(u,v)\le uvk_2(u,v)$ for all $(u,v)\in(\mathbb{R}\setminus\{0\})^2$;

(C4) for p,r,α_1,α_3 above, we always have $\alpha_1\alpha_3r(t)+p(t)>0$;

(C5) $f\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$.

Preliminaries about time scale calculus can be found in [3–6] and are omitted here. For simplicity, we denote $(a,b)\cap\mathbb{T}=(a,b)_{\mathbb{T}}$ throughout this paper, where $a,b\in\mathbb{R}$ and $[a,b]_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ are denoted similarly.

Now, we give the first theorem.

Theorem 2. *Assume that (C1)–(C5) hold and that there exists a function $q\in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u)\ge q(t)u^2$. Also, suppose that $x(t)$ is a solution of (10) satisfying $x(t)>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ with $t_0\in\mathbb{T}$. For $t\in[t_0,\infty)_{\mathbb{T}}$, define*

$$u(t)=A(t)\frac{p(t)k_1\left(x(t),x^{\Delta}(t)\right)}{x(t)}+B(t), \tag{11}$$

where $A\in C^1_{rd}(\mathbb{T},(0,\infty))$, $B\in C^1_{rd}(\mathbb{T},\mathbb{R})$, and $(\alpha_1A-(\alpha_1-\alpha_2)A^{\sigma})p+\alpha_1\alpha_2\alpha_3rA^{\sigma}>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. Then, $u(t)$ satisfies

$$\mu(t)u(t)-\mu(t)B(t)+\alpha_1A(t)p(t)>0, \tag{12}$$

$$\begin{aligned}&u^{\Delta}(t)+\Phi_0(t)\\&\quad+\frac{\Phi_1(t)u^2(t)-\Phi_2(t)u(t)+\Phi_3(t)}{\alpha_1A(t)p(t)\left(\mu(t)u(t)-\mu(t)B(t)+\alpha_1A(t)p(t)\right)}\le 0,\end{aligned} \tag{13}$$

where

$$\begin{aligned}\Phi_0(t)&=A^{\sigma}(t)\left(q(t)-\left(\frac{B(t)}{A(t)}\right)^{\Delta}\right),\\\Phi_1(t)&=\left(\alpha_1A(t)-(\alpha_1-\alpha_2)A^{\sigma}(t)\right)p(t)\\&\quad+\alpha_1\alpha_2\alpha_3r(t)A^{\sigma}(t),\\\Phi_2(t)&=\left((2\alpha_2-\alpha_1)A^{\sigma}(t)+\alpha_1A(t)\right)p(t)B(t)\\&\quad+\alpha_1^2p^2(t)A^{\Delta}(t)A(t)+2\alpha_1\alpha_2\alpha_3r(t)A^{\sigma}(t)B(t),\\\Phi_3(t)&=\alpha_2\left(\alpha_1\alpha_3r(t)+p(t)\right)A^{\sigma}(t)B^2(t),\\A^{\sigma}(t)&=A(\sigma(t)).\end{aligned} \tag{14}$$

Proof. By (C3) we see that x^{Δ} and $k_1(x,x^{\Delta})$ are both positive or both negative or both zero. When $x^{\Delta}>0$, which implies that $k_1(x,x^{\Delta})>0$, it follows that

$$\begin{aligned}\mu u-\mu B+\alpha_1Ap&\ge\mu A\frac{pk_1^2\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_2Ap\\&\ge\alpha_2\mu Ap\frac{x^{\Delta}k_1\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_2Ap\\&=\alpha_2Ap\frac{x^{\sigma}}{x}>0.\end{aligned} \tag{15}$$

When $x^{\Delta}<0$, which implies that $k_1(x,x^{\Delta})<0$, it follows that

$$\begin{aligned}\mu u-\mu B+\alpha_1Ap&=\mu Ap\frac{k_1^2\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_1Ap\\&\ge\mu Ap\frac{\alpha_1x^{\Delta}k_1\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_1Ap\\&=\alpha_1Ap\frac{x^{\sigma}}{x}\ge\alpha_2Ap\frac{x^{\sigma}}{x}>0.\end{aligned} \tag{16}$$

When $x^{\Delta}=0$, which implies that $k_1(x,x^{\Delta})=0$ and $x=x^{\sigma}$, it follows that

$$\mu u-\mu B+\alpha_1Ap=\alpha_1Ap\ge\alpha_2Ap\frac{x^{\sigma}}{x}>0. \tag{17}$$

Hence, we always have

$$\mu u - \mu B + \alpha_1 A p > 0,$$
$$\frac{x}{x^\sigma} \geq \frac{\alpha_2 A p}{\mu u - \mu B + \alpha_1 A p}, \tag{18}$$

so (12) holds. Then differentiating (11) and using (10), it follows that

$$\begin{aligned}
u^\Delta &= A^\Delta \left(\frac{p k_1 \left(x, x^\Delta\right)}{x} \right) + A^\sigma \left(\frac{p k_1 \left(x, x^\Delta\right)}{x} \right)^\Delta + B^\Delta \\
&= \frac{A^\Delta}{A} (u - B) \\
&\quad + A^\sigma \frac{\left(p k_1 \left(x, x^\Delta\right)\right)^\Delta x - p k_1 \left(x, x^\Delta\right) x^\Delta}{x x^\sigma} + B^\Delta \\
&= \frac{A^\Delta}{A} u + B^\Delta - \frac{A^\Delta}{A} B - A^\sigma \frac{f\left(t, x^\sigma\right)}{x^\sigma} \\
&\quad - A^\sigma \frac{r k_2 \left(x, x^\Delta\right) x^\Delta}{x^\sigma} - A^\sigma \frac{p k_1 \left(x, x^\Delta\right) x^\Delta}{x x^\sigma} \\
&\leq \frac{A^\Delta}{A} u + A^\sigma \left(\frac{B}{A} \right)^\Delta - A^\sigma q \\
&\quad - A^\sigma r \frac{k_2 \left(x, x^\Delta\right) x x^\Delta}{x x^\sigma} - A^\sigma p \frac{k_1^2 \left(x, x^\Delta\right)}{\alpha_1 x x^\sigma} \\
&\leq \frac{A^\Delta}{A} u - \Phi_0 - \alpha_3 A^\sigma r \frac{k_1^2 \left(x, x^\Delta\right)}{x^2} \frac{x}{x^\sigma} \\
&\quad - \frac{1}{\alpha_1} A^\sigma p \frac{k_1^2 \left(x, x^\Delta\right)}{x^2} \frac{x}{x^\sigma} \\
&\leq \frac{A^\Delta}{A} u - \Phi_0 - \left(\alpha_3 r + \frac{p}{\alpha_1} \right) A^\sigma \left(\frac{u - B}{A p} \right)^2 \\
&\quad \times \frac{\alpha_2 A p}{\mu u - \mu B + \alpha_1 A p} \\
&= \frac{A^\Delta}{A} u - \Phi_0 - \frac{\alpha_2 \left(\alpha_1 \alpha_3 r + p\right)}{\alpha_1 p} \frac{A^\sigma}{A} \frac{(u - B)^2}{\mu u - \mu B + \alpha_1 A p} \\
&= \frac{-\Phi_1 u^2 + \Phi_2 u - \Phi_3}{\alpha_1 A p \left(\mu u - \mu B + \alpha_1 A p\right)} - \Phi_0,
\end{aligned} \tag{19}$$

so (13) holds. Theorem 2 is proved. □

Remark 3. In Theorem 2, the condition $(\alpha_1 A - (\alpha_1 - \alpha_2) A^\sigma) p + \alpha_1 \alpha_2 \alpha_3 r A^\sigma > 0$ ensures that the coefficient of u^2 in (13) is always negative. The condition is obvious and easy to be fulfilled. For example, when $A^\Delta(t) \leq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, we have $A^\sigma = A + \mu A^\Delta \leq A$; by (C4) we see that

$$\begin{aligned}
&\left(\alpha_1 A - \left(\alpha_1 - \alpha_2\right) A^\sigma\right) p + \alpha_1 \alpha_2 \alpha_3 r A^\sigma \\
&\geq \left(\alpha_1 A^\sigma - \left(\alpha_1 - \alpha_2\right) A^\sigma\right) p + \alpha_1 \alpha_2 \alpha_3 r A^\sigma \\
&= \alpha_2 A^\sigma \left(\alpha_1 \alpha_3 r + p\right) > 0.
\end{aligned} \tag{20}$$

Let $D_0 = \{s \in \mathbb{T} : s \geq 0\}$ and $D = \{(t, s) \in \mathbb{T}^2 : t \geq s \geq 0\}$. For any function $f(t, s)$: $\mathbb{T}^2 \to \mathbb{R}$, denote by f_2^Δ the partial derivatives of f with respect to s. For $E \subset \mathbb{R}$, denote by $L(E)$ the space of functions which are integrable on any compact subset of E. Define

$$\begin{aligned}
(\mathscr{A}, \mathscr{B}) = \Big\{ &(A, B) : A(s) \in C_{rd}^1 \left(D_0, (0, \infty)\right), \\
&B(s) \in C_{rd}^1 \left(D_0, \mathbb{R}\right), \\
&\left(\alpha_1 A(s) - \left(\alpha_1 - \alpha_2\right) A^\sigma(s)\right) p(s) \\
&+ \alpha_1 \alpha_2 \alpha_3 r(s) A^\sigma(s) > 0, \\
&\alpha_1 A(s) p(s) \pm \mu(s) B(s) > 0, s \in D_0 \Big\}; \\
\mathscr{H} = \Big\{ &H(t, s) \in C^1(D, [0, \infty)) : \\
&H(t, t) = 0, \\
&H(t, s) > 0, H_2^\Delta(t, s) \leq 0, t > s \geq 0 \Big\}.
\end{aligned} \tag{21}$$

These function classes will be used throughout this paper. Now, we are in a position to give the second theorem.

Theorem 4. *Assume that (C1)–(C5) hold and that there exists a function $q \in C_{rd}(\mathbb{T}, \mathbb{R})$ such that $uf(t, u) \geq q(t) u^2$. Also, suppose that there exist $(A, B) \in (\mathscr{A}, \mathscr{B})$ and $H \in \mathscr{H}$ such that $M(t, \cdot) \in L([0, \rho(t)]_{\mathbb{T}})$ and for any $t_0 \in \mathbb{T}$,*

$$\begin{aligned}
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \Bigg[&\int_{t_0}^{t} H(t, \sigma(s)) \Phi_0(s) \Delta s - \int_{t_0}^{\rho(t)} M(t, s) \Delta s \\
&+ H_2^\Delta(t, \rho(t)) \\
&\times \left(\alpha_1 A(\rho(t)) p(\rho(t)) \right. \\
&\left. - \mu(\rho(t)) B(\rho(t))\right) \Bigg] = \infty,
\end{aligned} \tag{22}$$

where Φ_0 is defined as before, and

$$\begin{aligned}
M(t, s) &= \frac{\Phi_4^2(t, s)}{4 \alpha_1 A(s) p(s) \min\left\{\Phi_5(t, s), \Phi_6(t, s)\right\}}, \\
\Phi_4(t, s) &= \alpha_1 p(s) H(t, s) A(s) B(s) \\
&\quad + \left(\left(2\alpha_2 - \alpha_1\right) p(s) + 2\alpha_1 \alpha_2 \alpha_3 r(s)\right) \\
&\quad \times H(t, \sigma(s)) A^\sigma(s) B(s) \\
&\quad + \alpha_1^2 p^2(s) A(s) \left(H(t, s) A(s)\right)^{\Delta_s}, \\
\Phi_5(t, s) &= \alpha_2 H(t, \sigma(s)) A^\sigma(s) \left(\alpha_1 \alpha_3 r(s) + p(s)\right) \\
&\quad \times \left(\alpha_1 A(s) p(s) + \mu(s) B(s)\right),
\end{aligned}$$

$$\begin{aligned}\Phi_6(t,s) = {} & (\alpha_1 p(s) H(t,s) A(s) \\ & - (\alpha_1 - \alpha_2) p(s) H(t,\sigma(s)) A^{\sigma}(s) \\ & + \alpha_1\alpha_2\alpha_3 r(s) H(t,\sigma(s)) A^{\sigma}(s)) \\ & \times (\alpha_1 A(s) p(s) - \mu(s) B(s)).\end{aligned} \tag{23}$$

Then, (10) *is oscillatory.*

Proof. Assume that (10) is not oscillatory. Without loss of generality we may assume that there exists $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Let $u(t)$ be defined by (11). Then by Theorem 2, (12) and (13) hold.

For simplicity in the following, we let $H_\sigma = H(t,\sigma(s))$, $H = H(t,s)$, and $H_2^\Delta = H_2^\Delta(t,s)$ and omit the arguments in the integrals. For $s \in \mathbb{T}$, $H_\sigma - H = \mu H_2^\Delta$.

Multiplying (13), where t is replaced by s, by H_σ and integrating it with respect to s from t_0 to t with $t \in \mathbb{T}$ and $t \geq \sigma(t_0)$, we obtain

$$\begin{aligned}&\int_{t_0}^{t} H_\sigma \Phi_0 \Delta s \\ &\quad \leq -\int_{t_0}^{t}\left(H_\sigma u^\Delta + H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u + \Phi_3}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)}\right)\Delta s,\end{aligned} \tag{24}$$

where Φ_1, Φ_2, Φ_3 are defined as before.

Noting that $H(t,t) = 0$, by the integration by parts formula we have

$$\begin{aligned}&\int_{t_0}^{t} H_\sigma \Phi_0 \Delta s \\ &\quad \leq H(t,t_0) u(t_0) \\ &\qquad + \int_{t_0}^{t}\left(H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u + \Phi_3}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)}\right)\Delta s \\ &\quad \leq H(t,t_0) u(t_0) \\ &\qquad + \int_{t_0}^{t}\left(H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)}\right)\Delta s \\ &\quad = H(t,t_0) u(t_0) + \int_{\rho(t)}^{t} H_2^\Delta u \Delta s \\ &\qquad + \int_{t_0}^{\rho(t)}\left(H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)}\right)\Delta s.\end{aligned} \tag{25}$$

Since $H_2^\Delta \leq 0$ on D, from (12) we see that, for $t \geq \sigma(t_0)$,

$$\begin{aligned}\int_{\rho(t)}^{t} H_2^\Delta u \Delta s &= H_2^\Delta(t,\rho(t)) u(\rho(t)) \mu(\rho(t)) \\ &\leq -H_2^\Delta(t,\rho(t))(\alpha_1 A(\rho(t)) p(\rho(t)) \\ &\qquad - \mu(\rho(t)) B(\rho(t))).\end{aligned} \tag{26}$$

Since $H_2^\Delta \leq 0$ on D, we see that $H_\sigma \leq H$. For $t \geq \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, from $(\alpha_1 A - (\alpha_1 - \alpha_2)A^\sigma)p + \alpha_1\alpha_2\alpha_3 r A^\sigma > 0$ and (C4), we have

$$\begin{aligned}&\alpha_1 pHA - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma \\ &\quad \geq \alpha_1 pH_\sigma A - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma \\ &\quad = ((\alpha_1 A - (\alpha_1 - \alpha_2) A^\sigma) p + \alpha_1\alpha_2\alpha_3 r A^\sigma) H_\sigma > 0.\end{aligned} \tag{27}$$

For $t \geq \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, and $u(s) \leq 0$, from (27) we have

$$\begin{aligned}&H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)} \\ &\quad = \frac{-[\alpha_1 pHA - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma] u^2 + \Phi_4 u}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)} \\ &\quad = -\frac{\alpha_1 pHA - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)} u^2 \\ &\qquad + \frac{\Phi_4}{\alpha_1 A p(\alpha_1 A p - \mu B)} u \\ &\qquad - \frac{\Phi_4}{\alpha_1 A p(\alpha_1 A p - \mu B)} \frac{\mu u^2}{\mu u - \mu B + \alpha_1 A p} \\ &\quad = -\frac{\Phi_5}{\alpha_1 A p(\alpha_1 A p - \mu B)(\mu u - \mu B + \alpha_1 A p)} u^2 \\ &\qquad + \frac{\Phi_4}{\alpha_1 A p(\alpha_1 A p - \mu B)} u \\ &\quad \leq -\frac{\Phi_5}{\alpha_1 A p(\alpha_1 A p - \mu B)^2} u^2 + \frac{\Phi_4}{\alpha_1 A p(\alpha_1 A p - \mu B)} u \\ &\quad = -\frac{\Phi_5}{\alpha_1 A p(\alpha_1 A p - \mu B)^2}\left(u - \frac{(\alpha_1 A p - \mu B)\Phi_4}{2\Phi_5}\right)^2 \\ &\qquad + \frac{\Phi_4^2}{4\alpha_1 A p \Phi_5} \\ &\quad \leq \frac{\Phi_4^2}{4\alpha_1 A p \min\{\Phi_5, \Phi_6\}} = M.\end{aligned} \tag{28}$$

For $t \geq \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, and $u(s) > 0$, from (27) we have

$$\begin{aligned}&H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)} \\ &\quad = \frac{-[\alpha_1 pHA - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma] u^2 + \Phi_4 u}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)} \\ &\quad = -\frac{\alpha_1 pHA - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma}{\alpha_1 A p(\mu u - \mu B + \alpha_1 A p)} \\ &\qquad \times\left(u - \frac{\Phi_4}{2(\alpha_1 pHA - (\alpha_1 - \alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma)}\right)^2\end{aligned}$$

$$+\frac{\Phi_4^2}{4\alpha_1 Ap\Phi_6}$$
$$\leq \frac{\Phi_4^2}{4\alpha_1 Ap\min\{\Phi_5,\Phi_6\}} = M. \tag{29}$$

Therefore, for all $t \geq \sigma(t_0)$, $s \in [t_0, \rho(t))_{\mathbb{T}}$, we have

$$H_2^{\Delta}u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \leq M. \tag{30}$$

Then, from (25), (26), and (30) we obtain that, for $t \in \mathbb{T}$ and $t > \sigma(t_0)$,

$$\begin{aligned}\int_{t_0}^{t} H_\sigma \Phi_0 \Delta s &\leq H(t,t_0)u(t_0) + \int_{t_0}^{\rho(t)} M\Delta s \\ &\quad - H_2^{\Delta}(t,\rho(t))(\alpha_1 A(\rho(t))p(\rho(t)) \\ &\quad -\mu(\rho(t))B(\rho(t))).\end{aligned} \tag{31}$$

Hence,

$$\begin{aligned}&\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\Phi_0(s)\Delta s - \int_{t_0}^{\rho(t)} M(t,s)\Delta s\right. \\ &\quad + H_2^{\Delta}(t,\rho(t)) \\ &\quad \left.\times(\alpha_1 A(\rho(t))p(\rho(t)) - \mu(\rho(t))B(\rho(t)))\right] \\ &\leq u(t_0) < \infty,\end{aligned} \tag{32}$$

which contradicts (22) and completes the proof. □

Remark 5. If we change the condition $(\alpha_1 A - (\alpha_1 - \alpha_2)A^\sigma)p + \alpha_1\alpha_2\alpha_3 rA^\sigma > 0$ in the definition of $(\mathscr{A}, \mathscr{B})$ to a stronger one $A^{\Delta}(t) \leq 0$, (27) in the proof of Theorem 4 will be changed to

$$\begin{aligned}&\alpha_1 pHA - (\alpha_1 - \alpha_2)pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 rH_\sigma A^\sigma \\ &\quad \geq \alpha_1 pH_\sigma A^\sigma - (\alpha_1 - \alpha_2)pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 rH_\sigma A^\sigma \\ &\quad = \alpha_2 H_\sigma A^\sigma(\alpha_1\alpha_3 r + p) > 0.\end{aligned} \tag{33}$$

Then the definition of M can be simplified as

$$\begin{aligned}M(t,s) &= \left(\Phi_4^2(t,s)\right) \\ &\quad \times(4\alpha_1\alpha_2 p(s)H(t,\sigma(s))A^\sigma(s)A(s) \\ &\quad \times(\alpha_1\alpha_3 r(s) + p(s))\min\{\Phi_7(s),\Phi_8(s)\})^{-1},\end{aligned} \tag{34}$$

where

$$\begin{aligned}\Phi_7(s) &= \alpha_1 A(s)p(s) - \mu(s)B(s), \\ \Phi_8(s) &= \alpha_1 A(s)p(s) + \mu(s)B(s).\end{aligned} \tag{35}$$

When $(A, B) = (1, 0)$, Theorem 4 can be simplified as Corollary 6.

Corollary 6. *Assume that (C1)–(C5) hold and that there exists a function $q \in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$. Also, suppose that there exists $H \in \mathscr{H}$ such that, for any $t_0 \in \mathbb{T}$,*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)} \\ &\quad \times\left[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s\right. \\ &\quad - \frac{\alpha_1^2}{4\alpha_2}\int_{t_0}^{\rho(t)}\frac{(p(s)H_2^{\Delta}(t,s))^2}{(\alpha_1\alpha_3 r(s) + p(s))H(t,\sigma(s))}\Delta s \\ &\quad \left.+\alpha_1 H_2^{\Delta}(t,\rho(t))p(\rho(t))\right] = \infty.\end{aligned} \tag{36}$$

Then, (10) *is oscillatory.*

When $r(t) \equiv 0$, (10) will be simplified as

$$\left(p(t)k_1\left(x(t),x^{\Delta}(t)\right)\right)^{\Delta} + f(t,x(\sigma(t))) = 0. \tag{37}$$

Then Theorem 4 can be simplified as Corollary 7.

Corollary 7. *Assume that (C1)–(C5) hold and that there exists a function $q \in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$. Also, suppose that there exist $(A, B) \in (\mathscr{A}, \mathscr{B})$ and $H \in \mathscr{H}$ such that, for any $t_0 \in \mathbb{T}$,*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\Phi_0(s)\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\Delta s\right. \\ &\quad + H_2^{\Delta}(t,\rho(t)) \\ &\quad \times(\alpha_1 A(\rho(t))p(\rho(t)) \\ &\quad \left.-\mu(\rho(t))B(\rho(t)))\right] = \infty,\end{aligned} \tag{38}$$

where

$$\begin{aligned}M_1(t,s) &= \frac{\Phi_9^2(t,s)}{4\alpha_1 A(s)\min\{\Phi_{10}(t,s),\Phi_{11}(t,s)\}}, \\ \Phi_9(t,s) &= \alpha_1 H(t,s)A(s)B(s) \\ &\quad + (2\alpha_2 - \alpha_1)H(t,\sigma(s))A^\sigma(s)B(s) \\ &\quad + \alpha_1^2 p(s)A(s)(H(t,s)A(s))^{\Delta_s}, \\ \Phi_{10}(t,s) &= \alpha_2 H(t,\sigma(s))A^\sigma(s)(\alpha_1 A(s)p(s) + \mu(s)B(s)), \\ \Phi_{11}(t,s) &= (\alpha_1 H(t,s)A(s) - (\alpha_1 - \alpha_2)H(t,\sigma(s))A^\sigma(s)) \\ &\quad \times(\alpha_1 A(s)p(s) - \mu(s)B(s)).\end{aligned} \tag{39}$$

Then, (37) *is oscillatory.*

Remark 8. When $r(t) \equiv 0$, $k_1(u,v) = \Psi(u)k(v)$, and (C3) is replaced by

(C6) $\psi \in C(\mathbb{R}, (0,\eta])$, where η is a fixed positive constant;

(C7) $k \in C(\mathbb{R},\mathbb{R})$, and there exists $\gamma_1 \geq \gamma_2 > 0$ such that $0 < \gamma_2 yk(y) \leq k^2(y) \leq \gamma_1 yk(y)$ for all $y \neq 0$.

Theorem 4 is reduced to [12, Theorem 4].

3. Example

In this section, we will give an example to demonstrate Corollary 7.

Example 1. Consider the equations

$$\left[\frac{1}{t^2}\frac{2+x^2(t)}{1+x^2(t)}x^{\Delta}(t)\right]^{\Delta} + t^2(2+\sin t)\,x(\sigma(t)) = 0, \tag{40}$$

$$\left[\frac{1}{t^2}\frac{1+2x^2(t)\left(x^{\Delta}(t)\right)^2}{1+x^2(t)\left(x^{\Delta}(t)\right)^2}x^{\Delta}(t)\right]^{\Delta} + t^2(2+\sin t)x(\sigma(t)) = 0, \tag{41}$$

where $p(t) = 1/t^2$, $r(t) \equiv 0$, $q(t) = t^2$, $k_1(u,v) = ((2+u^2)/(1+u^2))v$ in (40), and $k_1(u,v) = ((1+2u^2v^2)/(1+u^2v^2))v$ in (41), so we have both $\alpha_1 = 2$, $\alpha_2 = 1$. Letting $H(t,s) = (t-s)^2$, we have

(1) $\mathbb{T} = [1,\infty)$, $(A,B) = (s^2, 1/s^2)$,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\,\Phi_0(s)\,\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\,\Delta s + H_2^{\Delta}(t,\rho(t)) \times \left(\alpha_1 A(\rho(t))\,p(\rho(t)) - \mu(\rho(t))\,B(\rho(t))\right)\right] \\
&= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\left[\int_1^t (t-s)^2\left(s^4+\frac{4}{s^3}\right)ds - \int_1^t \frac{\left(t-s+4ts-8s^2\right)^2}{4s^4}ds\right] \\
&= \infty.
\end{aligned} \tag{42}$$

That is, (38) holds. By Corollary 7 we see that (40) and (41) are oscillatory. Consider

(2) $\mathbb{T} = \mathbb{N}$, $(A,B) = (1,0)$,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\,\Phi_0(s)\,\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\,\Delta s + H_2^{\Delta}(t,\rho(t))\left(\alpha_1 A(\rho(t))\,p(\rho(t)) - \mu(\rho(t))\,B(\rho(t))\right)\right] \\
&= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\left[\int_1^t (t-s-1)^2 s^2\,\Delta s - \int_1^{t-1}\frac{(2t-2s-1)^2}{s^2(t-s-1)^2}\Delta s - \frac{2}{(t-1)^2}\right] \\
&= \limsup_{n\to\infty}\frac{1}{(n-1)^2}\left[\sum_{k=1}^{n-1}(n-k-1)^2k^2 - \sum_{k=1}^{n-2}\frac{(2n-2k-1)^2}{k^2(n-k-1)^2} - \frac{2}{(n-1)^2}\right] \\
&= \infty.
\end{aligned} \tag{43}$$

That is, (38) holds. By Corollary 7 we see that (40) and (41) are oscillatory.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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