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Research Article

Global Existence of Solutions to the 2D Incompressible Generalized Liquid Crystal Flow

Mingxuan Zhu

School of Mathematical Sciences, Xiamen University, Xiamen, China

Correspondence should be addressed to Mingxuan Zhu; zhumingxuan587@gmail.com

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We consider the global existence of solutions to the 2D incompressible generalized liquid crystal flow. It is proved that the local solution exists globally with $\beta = 0, \alpha \geq 2$.

1. Introduction

In this paper, we consider the following 2D liquid crystal flow:

$$u_t + u \cdot \nabla u + \nabla p + \Lambda^{2\alpha} u = -\nabla d \cdot \Delta d, \quad (1)$$

$$d_t + u \cdot \nabla d + \Lambda^{2\beta} d = -f(d), \quad (2)$$

$$\operatorname{div} u = 0, \quad (3)$$

$$(u, d)|_{t=0} = (u_0, d_0), \quad (4)$$

where $\alpha \geq 0$, $\beta \geq 0$ are real parameters and u is the velocity, d is a vectorial function modeling the orientation of the crystal molecules, and p is the scalar pressure. Here $f(d) := (|d|^2 - 1)d$ and $\Lambda = (-\Delta)^{1/2}$ is defined in terms of Fourier transform by

$$\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi). \quad (5)$$

When $\alpha = \beta = 1$, it has been shown that (1)–(4) has unique global weak and smooth solutions [1–3]. In [4], global regularity for this system with mixed partial viscosity is proved. Some regularity criteria are established for the system with zero dissipation in [5].

The aim of this paper is to establish the following global regularity for the 2D liquid crystal model with fractional diffusion.

Theorem 1. *Assume $(u_0, d_0) \in H^3(\mathbb{R}^2) \times H^4(\mathbb{R}^2)$. Let (u, d) be the local strong solution to the problem (1)–(4). If α and β satisfy $\beta = 0$, $\alpha \geq 2$, then the 2D liquid crystal model has a unique global classical solution (u, d) satisfying*

$$u \in L^\infty\left(0, T; H^3\left(\mathbb{R}^2\right)\right), \qquad u \in L^2\left(0, T; H^{3+\alpha}\left(\mathbb{R}^2\right)\right),$$
$$d \in L^\infty\left(0, T; H^4\left(\mathbb{R}^2\right)\right). \quad (6)$$

Remark 2. This work is partially motivated by the recent progress on the 2D incompressible MHD system with fractional diffusion; we refer to [6–10] and references therein. In [7], Tran et al. obtained the global regularity of 2D GMHD equations for the following three cases: (1) $\alpha \geq 1$, $\beta \geq 1$; (2) $0 \leq \alpha < 1/2$, $2\alpha + \beta > 2$; (3) $\alpha \geq 2$, $\beta = 0$. Combining them with the result in [10], we know that if $\alpha + \beta \geq 2$, 2D incompressible MHD system with fractional diffusion possesses a global smooth solution. Fan et al. [8] proved the global existence of smooth solutions with $\alpha > 0$, $\beta = 1$. Global regularity for the case $\alpha = 0$, $\beta > 1$ was established by Jiu and Zhao [9] which improves the result in [6]. Very recently, the authors improved the case $\alpha = 0$, $\beta > 1$ for the 2D liquid crystal model in [11].

2. Proof of Theorem 1

It is sufficient to prove Theorem 1 with $\alpha = 2$, $\beta = 0$.

We will prove Theorem 1 if we can demonstrate the boundedness of $\|u\|_{H^3}^2 + \|d\|_{H^4}^2$. In order to reach our purpose, we will show this by contradiction: assume

$$\lim_{t \to T} \sup \|u\|_{H^3}^2 + \|d\|_{H^4}^2 = \infty \quad (7)$$

for some finite time $T > 0$. Our thought is that when T_0 is close enough to T, $\|u\|_{H^3}^2 + \|d\|_{H^4}^2$ remains uniformly bounded for $T_0 < t < T$ under such assumption, thus reaching a contradiction.

First, we do L^2 estimate for d. Multiplying (2) by d and using (3), after integration by parts, we see that

$$\frac{1}{2}\frac{d}{dt}\|d\|_{L^2}^2 + \|d\|_{L^4}^4 = \|d\|_{L^2}^2. \tag{8}$$

By using the Gronwall inequality, we have

$$\|d\|_{L^2} + \int_0^T \|d\|_{L^4}^4 d\tau \le C. \tag{9}$$

Then, we will show the L^2 estimate for u and ∇d. Multiplying (1) and (2) by u and $-\Delta d$, respectively, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \left\|\Lambda^2 u\right\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^2} \nabla f(d)\nabla d\, dx \\ &\le -3\int_{\mathbb{R}^2} |d|^2|\nabla d|^2 dx + \|\nabla d\|_{L^2}^2.\end{aligned} \tag{10}$$

Thanks to Gronwall's inequality and (9), we have

$$\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \int_0^T \left\|\Lambda^2 u\right\|_{L^2}^2 d\tau \le C, \tag{11}$$

which means $\nabla u \in L^2(0,T;\mathrm{BMO})$.

The H^1 estimate for u and H^2 estimate for d will be shown as follows. Multiplying (1) by Δu, applying Δ to (2), multiplying by Δd, and then summing them up, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) + \|\Delta\nabla u\|_{L^2}^2 \\ &\le \int_{\mathbb{R}^2} \nabla d\cdot\Delta d\cdot\Delta u - \Delta(u\cdot\nabla d)\cdot\Delta d \\ &\qquad - \Delta f(d)\cdot\Delta d\, dx \\ &\le C\|\Delta d\|_{L^2}^2\|\nabla u\|_{L^\infty} \\ &\quad + \int_{\mathbb{R}^2} -3|d|^2|\Delta d|^2 - d|\nabla d|^2\Delta d + |\Delta d|^2 dx \\ &\le C\|\Delta d\|_{L^2}^2\|\nabla u\|_{L^\infty} - 2\|d\Delta d\|_{L^2}^2 + C\|\nabla d\|_{L^4}^4 + \|\Delta d\|_{L^2}^2 \\ &\le C\|\Delta d\|_{L^2}^2\left(\|\nabla u\|_{L^\infty} + 1\right) - 2\|d\Delta d\|_{L^2}^2 \\ &\le C\|\Delta d\|_{L^2}^2\left(\|\nabla u\|_{L^\infty} + 1\right).\end{aligned} \tag{12}$$

Let us introduce the following commutator and bilinear estimates established in [12, 13]:

$$\begin{aligned}&\left\|\Lambda^s(fg) - f\Lambda^s g\right\|_{L^p} \\ &\quad\le C\left(\|\nabla f\|_{L^{p_1}}\left\|\Lambda^{s-1}g\right\|_{L^{q_1}} + \|g\|_{L^{p_1}}\left\|\Lambda^s f\right\|_{L^{q_1}}\right), \\ &\left\|\Lambda^s(fg)\right\|_{L^p} \\ &\quad\le C\left(\|f\|_{L^{p_1}}\left\|\Lambda^s g\right\|_{L^{q_1}} + \left\|\Lambda^s f\right\|_{L^{p_2}}\|g\|_{L^{q_2}}\right),\end{aligned} \tag{13}$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

Now, we do the H^2 estimate for u and H^3 estimate for d. Applying Λ^2 to (1), multiplying by $\Lambda^2 u$, and dealing with (2) in the same way by Λ^3 and $\Lambda^3 d$, after summing them up, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\left\|\Lambda^2 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2\right) + \left\|\Lambda^4 u\right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^2} -\Lambda^2(u\cdot\nabla u)\Lambda^2 u - \Lambda^2(\nabla d\cdot\Delta d)\Lambda^2 u \\ &\qquad - \Lambda^3(u\cdot\nabla d)\Lambda^3 d - \Lambda^3 f(d)\Lambda^3 d\, dx \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{14}$$

Using Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, and (13), we have the following estimates:

$$\begin{aligned}\left|I_1\right| &= \left|\int_{\mathbb{R}^2}\left(\Lambda^2(u\cdot\nabla u) - u\cdot\nabla\Lambda^2 u\right)\Lambda^2 u\, dx\right| \\ &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^2 u\right\|_{L^2}^2, \\ \left|I_2\right| &\le C\left\|\Lambda^4 u\right\|_{L^2}\|\nabla d\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4} \\ &\le C\left\|\Lambda^4 u\right\|_{L^2}\|\nabla d\|_{L^2}\left\|\Lambda^3 d\right\|_{L^2} \\ &\le \frac{1}{4}\left\|\Lambda^4 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2, \\ \left|I_3\right| &\le C\left|\int_{\mathbb{R}^2}\Lambda^3(u\cdot\nabla d)\Lambda^3 d - u\cdot\nabla\Lambda^3 d\Lambda^3 d\, dx\right| \\ &\le C\int_{\mathbb{R}^2}\left|\Lambda^3 u\right||\Lambda d|\left|\Lambda^3 d\right| + \left|\Lambda^2 u\right|\left|\Lambda^2 d\right|\left|\Lambda^3 d\right| \\ &\qquad + |\Lambda u|\left|\Lambda^3 d\right|^2 dx \\ &=: II_1 + II_2 + II_3.\end{aligned} \tag{15}$$

Now we estimate II_1, II_2, and II_3 one by one:

$$\begin{aligned}II_1 &\le C\left\|\Lambda^3 u\right\|_{L^4}\|\Lambda d\|_{L^4}\left\|\Lambda^3 d\right\|_{L^2} \\ &\le C\left\|\Lambda^2 u\right\|_{L^2}^{1/4}\left\|\Lambda^4 u\right\|_{L^2}^{3/4}\|\Lambda d\|_{L^2}^{3/4}\left\|\Lambda^3 d\right\|_{L^2}^{5/4} \\ &\le \frac{1}{8}\left\|\Lambda^4 u\right\|_{L^2}^2 + C\left\|\Lambda^2 u\right\|_{L^2}^{2/5}\left\|\Lambda^3 d\right\|_{L^2}^2, \\ II_2 &\le C\left\|\Lambda^2 u\right\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4}\left\|\Lambda^3 d\right\|_{L^2} \\ &\le C\left\|\Lambda^2 u\right\|_{L^2}^{3/4}\left\|\Lambda^4 u\right\|_{L^2}^{1/4}\|\Lambda d\|_{L^2}^{1/4}\left\|\Lambda^3 d\right\|_{L^2}^{7/4} \\ &\le \frac{1}{8}\left\|\Lambda^4 u\right\|_{L^2}^2 + \left\|\Lambda^2 u\right\|_{L^2}^{6/7}\left\|\Lambda^3 d\right\|_{L^2}^2, \\ II_3 &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^3 d\right\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
I_4 &= \left\|\Lambda^3 d\right\|_{L^2}^2 - \int_{\mathbb{R}^2} \Lambda^3\left(|d|^2 d\right)\Lambda^3 d \\
&\le \left\|\Lambda^3 d\right\|_{L^2}^2 - 3\left\|d\Lambda^3 d\right\|_{L^2}^2 \\
&\quad + C\int_{\mathbb{R}^2}\left|\Lambda^2 d\right||\Lambda d||d|\left|\Lambda^3 d\right| + C\int_{\mathbb{R}^2}|\Lambda d|^3\left|\Lambda^3 d\right| \\
&=: \left\|\Lambda^3 d\right\|_{L^2}^2 - 3\left\|d\Lambda^3 d\right\|_{L^2}^2 + K_1 + K_2.
\end{aligned}
\tag{16}
$$

K_1 and K_2 can be estimated as follows:

$$
\begin{aligned}
K_1 &\le C\|\Lambda d\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4}\left\|d\Lambda^3 d\right\|_{L^2} \\
&\le C\|\Lambda d\|_{L^2}^{1/4}\left\|\Lambda^3 d\right\|_{L^2}^{3/4}\|\Lambda d\|_{L^2}^{3/4}\left\|\Lambda^3 d\right\|_{L^2}^{1/4}\left\|d\Lambda^3 d\right\|_{L^2} \\
&\le C\left\|\Lambda^3 d\right\|_{L^2}^2 + 3\left\|d\Lambda^3 d\right\|_{L^2}^2,
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
K_2 &\le C\|\Lambda d\|_{L^6}^3\left\|\Lambda^3 d\right\|_{L^2} \\
&\le C\left(\|\Lambda d\|_{L^2}^{2/3}\left\|\Lambda^3 d\right\|_{L^2}^{1/3}\right)^3\left\|\Lambda^3 d\right\|_{L^2} \le C\left\|\Lambda^3 d\right\|_{L^2}^2.
\end{aligned}
$$

Combining K_1 and K_2, we have

$$
I_4 \le C\left\|\Lambda^3 d\right\|_{L^2}^2. \tag{18}
$$

Summing all the above estimates to (14), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\Lambda^2 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2\right) + \left\|\Lambda^4 u\right\|_{L^2}^2 \\
&\quad \le C\left(\|\nabla u\|_{L^\infty} + \left\|\Lambda^2 u\right\|_{L^2}\right)\left(\left\|\Lambda^2 u\right\|_{L^2}^2 + \left\|\Lambda^3 d\right\|_{L^2}^2\right).
\end{aligned}
\tag{19}
$$

Now, we will show the H^3 estimate for u and H^4 estimate for d. Applying Λ^3 to (1), multiplying by $\Lambda^3 u$, and dealing with (2) in the same way by Λ^4 and $\Lambda^4 d$, after summing them up, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^3 u\right\|_{L^2}^2 + \left\|\Lambda^4 d\right\|_{L^2}^2\right) + \left\|\Lambda^5 u\right\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}^2} -\Lambda^3(u\cdot\nabla u)\Lambda^3 u - \Lambda^3(\nabla d\cdot\Delta d)\Lambda^3 u \\
&\qquad - \Lambda^4(u\cdot\nabla d)\Lambda^4 d - \Lambda^4 f(d)\Lambda^4 d\,dx \\
&\quad =: J_1 + J_2 + J_3 + J_4.
\end{aligned}
\tag{20}
$$

Using Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, and (13), we have the following estimates:

$$
\begin{aligned}
|J_1| &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^3 u\right\|_{L^2}^2, \\
|J_2| &\le C\int_{\mathbb{R}^2}|\Lambda(\nabla d\Delta d)|\left|\Lambda^5 u\right|dx \le \|\Lambda(\nabla d\cdot\Delta d)\|_{L^2}\left\|\Lambda^5 u\right\|_{L^2} \\
&\le C\left\|\Lambda^5 u\right\|_{L^2}\left(\|\Delta d\|_{L^4}^2 + \|\Lambda d\|_{L^4}\left\|\Lambda^3 d\right\|_{L^4}\right) \\
&\le C\left\|\Lambda^5 u\right\|_{L^2}\left(\|\Lambda d\|_{L^2}\left\|\Lambda^4 d\right\|_{L^2} + \|\Lambda d\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{1/6}\right. \\
&\qquad \left.\times\|\Lambda d\|_{L^2}^{1/6}\left\|\Lambda^4 d\right\|_{L^2}^{5/6}\right) \\
&\le C\left\|\Lambda^4 d\right\|_{L^2}^2 + \frac{1}{4}\left\|\Lambda^5 u\right\|_{L^2}^2, \\
|J_3| &= C\left|\int_{\mathbb{R}^2}\left(\Lambda^4(u\cdot\nabla d) - u\cdot\nabla\Lambda^4 d\right)\Lambda^4 d\,dx\right| \\
&\le C\int_{\mathbb{R}^2}\left|\Lambda^4 u\right||\nabla d|\left|\Lambda^4 d\right| + \left|\Lambda^3 u\right|\left|\Lambda^2 d\right|\left|\Lambda^4 d\right| \\
&\qquad + \left|\Lambda^2 u\right|\left|\Lambda^3 d\right|\left|\Lambda^4 d\right| + |\Lambda u|\left|\Lambda^4 d\right|^2 dx \\
&=: J_{31} + J_{32} + J_{33} + J_{34}.
\end{aligned}
\tag{21}
$$

Now we estimate J_{31}, J_{32}, J_{33}, and J_{34} one by one:

$$
\begin{aligned}
|J_{31}| &\le C\left\|\Lambda^4 u\right\|_{L^4}\|\Lambda d\|_{L^4}\left\|\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^2 u\right\|_{L^2}^{1/6}\left\|\Lambda^5 u\right\|_{L^2}^{5/6}\|\Lambda d\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{7/6} \\
&\le \frac{1}{8}\left\|\Lambda^5 u\right\|_{L^2}^2 + C\left\|\Lambda^2 u\right\|_{L^2}^{3/7}\left\|\Lambda^4 d\right\|_{L^2}^2, \\
|J_{32}| &\le C\left\|\Lambda^3 u\right\|_{L^4}\left\|\Lambda^2 d\right\|_{L^4}\left\|\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^2 u\right\|_{L^2}^{1/2}\left\|\Lambda^5 u\right\|_{L^2}^{1/2}\|\Lambda d\|_{L^2}^{1/2}\left\|\Lambda^4 d\right\|_{L^2}^{3/2} \\
&\le \frac{1}{8}\left\|\Lambda^5 u\right\|_{L^2}^2 + C\left\|\Lambda^2 u\right\|_{L^2}^{2/3}\left\|\Lambda^4 d\right\|_{L^2}^2, \\
|J_{33}| &\le C\left\|\Lambda^2 u\right\|_{L^4}\left\|\Lambda^3 d\right\|_{L^4}\left\|\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^3 u\right\|_{L^2}^{5/6}\|u\|_{L^2}^{1/6}\|\Lambda d\|_{L^2}^{1/6}\left\|\Lambda^4 d\right\|_{L^2}^{11/6} \\
&\le C\left\|\Lambda^3 u\right\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{11/6}, \\
|J_{34}| &\le C\|\nabla u\|_{L^\infty}\left\|\Lambda^4 d\right\|_{L^2}^2.
\end{aligned}
\tag{22}
$$

The estimate for J_4 is as follows:

$$
\begin{aligned}
|J_4| &= \left\|\Lambda^4 d\right\|_{L^2}^2 - \int_{\mathbb{R}^2}\Lambda^4\left(|d|^2 d\right)\Lambda^4 d \\
&\le \left\|\Lambda^4 d\right\|_{L^2}^2 - 3\int_{\mathbb{R}^2}|d|^2\left|\Lambda^4 d\right|^2 \\
&\quad + C\int_{\mathbb{R}^2}|d||\Lambda d|\left|\Lambda^3 d\right|\left|\Lambda^4 d\right| \\
&\quad + C\int_{\mathbb{R}^2}|d|\left|\Lambda^2 d\right|^2\left|\Lambda^4 d\right| \\
&\quad + C\int_{\mathbb{R}^2}|\Lambda d|^2\left|\Lambda^2 d\right|\left|\Lambda^4 d\right| \\
&=: \left\|\Lambda^4 d\right\|_{L^2}^2 - 3\left\|d\Lambda^4 d\right\|_{L^2}^2 + J_{41} + J_{42} + J_{43}.
\end{aligned}
\tag{23}
$$

We calculate J_{41}, J_{42}, and J_{43}:

$$
\begin{aligned}
|J_{41}| &\le C\left\|\Lambda^3 d\right\|_{L^4}\|\Lambda d\|_{L^4}\left\|d\Lambda^4 d\right\|_{L^2} \\
&\le C\|\Lambda d\|_{L^2}^{1/6}\left\|\Lambda^4 d\right\|_{L^2}^{5/6}\|\Lambda d\|_{L^2}^{5/6}\left\|\Lambda^4 d\right\|_{L^2}^{1/6}\left\|d\Lambda^4 d\right\|_{L^2} \\
&\le C\left\|\Lambda^4 d\right\|_{L^2}^2 + \frac{3}{2}\left\|d\Lambda^4 d\right\|_{L^2}^2,
\end{aligned}
$$

$$\begin{aligned} |J_{42}| &\le C\|\Lambda^2 d\|_{L^4}^2 \|d\Lambda^4 d\|_{L^2} \\ &\le C\|\Lambda d\|_{L^2}\|\Lambda^4 d\|_{L^2}\|d\Lambda^4 d\|_{L^2} \\ &\le C\|\Lambda^4 d\|_{L^2}^2 + \frac{3}{2}\|d\Lambda^4 d\|_{L^2}^2, \\ |J_{43}| &\le C\|\Lambda^2 d\|_{L^4}\|\Lambda d\|_{L^8}^2\|\Lambda^4 d\|_{L^2} \\ &\le C\|\Lambda d\|_{L^2}^{1/2}\|\Lambda^4 d\|_{L^2}^{1/2}\|\Lambda d\|_{L^2}^{3/2}\|\Lambda^4 d\|_{L^2}^{1/2}\|\Lambda^4 d\|_{L^2} \\ &\le C\|\Lambda^4 d\|_{L^2}^2. \end{aligned} \tag{24}$$

Combining J_{41}, J_{42}, and J_{43}, we get

$$J_4 \le C\|\Lambda^4 d\|_{L^2}^2. \tag{25}$$

Combining the above estimates to (20), we get

$$\begin{aligned} &\frac{d}{dt}\left(\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^4 d\|_{L^2}^2\right) + \|\Lambda^5 u\|_{L^2}^2 \\ &\quad \le C\left(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^2 u\|_{L^2}\right)\|\Lambda^4 d\|_{L^2}^2 \\ &\quad\quad + C\left(1 + \|\Lambda^3 u\|_{L^2}\right)\|\Lambda^4 d\|_{L^2}^{11/6}. \end{aligned} \tag{26}$$

Now we estimate the term $\int_{T_0}^t \|\Lambda^3 u\|_{L^2}$ by applying the Gronwall inequality to (12):

$$\begin{aligned} \int_{T_0}^t \|\Lambda^3 u\|_{L^2}^2(\cdot,\tau)\,d\tau &\le \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \\ &\quad + \int_{T_0}^t \|\Delta\nabla u\|_{L^2}^2(\cdot,\tau)\,d\tau \\ &\le \left(\|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2\right) \\ &\quad \times \exp\left(C\int_{T_0}^t 1 + \|\nabla u\|_{L^\infty}(\cdot,\tau)\,d\tau\right). \end{aligned} \tag{27}$$

Here $T_0 \in (0,T)$ will be fixed later and we denote $\nabla u_0 := \nabla u(\cdot,T_0)$, $\Delta d_0 := \Delta d(\cdot,T_0)$. Set $A(t) := \max_{\tau\in(T_0,t)}(\|u\|_{H^3}^2 + \|d\|_{H^4}^2)(\tau)$. Now applying the logarithmic inequality [14]

$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|\nabla u\|_{\mathrm{BMO}}\left(1 + \ln\left(1 + \|u\|_{H^3}^2\right)\right)\right), \tag{28}$$

we get

$$\begin{aligned} &\int_{T_0}^t \|\Lambda^3 u\|_{L^2}^2(\cdot,\tau)\,d\tau \\ &\quad \le C(T_0)\exp\left(C\int_{T_0}^t \|\nabla u\|_{L^\infty}(\cdot,\tau)\,d\tau\right) \\ &\quad \le C(T_0)\exp\left(C\int_{T_0}^t 1 + \|\nabla u\|_{\mathrm{BMO}} \times \left(1 + \ln\left(1 + \|u\|_{H^3}^2\right)\right)(\cdot,\tau)\,d\tau\right) \\ &\quad \le C(T_0)\exp\left(C\int_{T_0}^t \|\nabla u\|_{\mathrm{BMO}}(\cdot,\tau) \times (1 + \ln(1 + A(t)))\,d\tau\right) \\ &\quad \le C(T_0)\exp\left(C\int_{T_0}^t \|\nabla u\|_{\mathrm{BMO}}(\cdot,\tau)\,d\tau \times (1 + \ln(1 + A(t)))\right). \end{aligned} \tag{29}$$

Since $\|\nabla u\|_{\mathrm{BMO}} \in L^1(T_0,T)$, we can take T_0 close enough to T, so that

$$C\int_{T_0}^t \|\nabla u\|_{\mathrm{BMO}}(\cdot,\tau)\,d\tau \le 2\delta \tag{30}$$

for some small positive number δ to be fixed later. With such choice of T_0 we have

$$\int_{T_0}^t \|\Lambda^3 u(\cdot,\tau)\|_{L^2}^2 \tau \le C(T_0)(1 + A(t))^{2\delta}. \tag{31}$$

Hölder's inequality gives

$$\int_{T_0}^t \|\Lambda^3 u(\cdot,\tau)\|_{L^2} \tau \le C(T_0)(1 + A(t))^{\delta}. \tag{32}$$

Fix T_0 satisfying

$$C\int_{T_0}^t \|\nabla u(\cdot,\tau)\|_{\mathrm{BMO}}\tau \le 2\delta, \qquad \ln(1 + A(T_0)) > 1. \tag{33}$$

Combining the above estimates together, we get

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_{H^3}^2 + \|d\|_{H^4}^2\right) \\ &\quad \le C\left(1 + \|\Lambda^3 u\|_{L^2}\right)A(t)^{11/12} \\ &\quad\quad + \left(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^2} + 1\right)A(t) \\ &\quad \le C\left[1 + \|\nabla u\|_{\mathrm{BMO}}(1 + \ln(1 + A(t)))\right. \\ &\quad\quad \left. + \|\nabla^2 u\|_{L^2}\right]A(t) + C\left(1 + \|\Lambda^3 u\|_{L^2}\right)A(t)^{11/12} \\ &\quad \le C(T_0)\left[\left(\|\nabla u\|_{\mathrm{BMO}} + \|\nabla^2 u\|_{L^2} + 1\right)A(t)\right. \\ &\quad\quad \left. \times \ln(1 + A(t)) + \left(1 + \|\Lambda^3 u\|_{L^2}\right)A(t)^{11/12}\right]. \end{aligned} \tag{34}$$

Integrating the above inequality, we have

$$A(t) \leq C(T_0) A_0 + C(T_0) \int_{T_0}^{t} 1 + \left\|\Lambda^3 u\right\|_{L^2}(\cdot,\tau)\, d\tau A(t)^{11/12} + C(T_0) \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\Lambda^2 u\right\|_{L^2}\right) A(t) \times \ln(1 + A(t))\, d\tau, \tag{35}$$

where $A_0 := \|u\|_{H^3}^2(T_0) + \|d\|_{H^4}^2(T_0)$.

Taking $\delta = 1/24$, we have

$$\int_{T_0}^{t} 1 + \left\|\Lambda^3 u\right\|_{L^2} d\tau \leq C(T_0)(1 + A(t))^{1/24}. \tag{36}$$

Thus (35) tells us that

$$A(t) \leq C(T_0) A_0 + C(T_0)(A(t) + 1)^{1/24} A(t)^{11/12} + C(T_0) \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\Lambda^2 u\right\|_{L^2}\right) A(t) \times \ln(1 + A(t))\, d\tau. \tag{37}$$

This in turn gives

$$1 + A(t) \leq C(T_0)(1 + A_0) + C(T_0)(A(t) + 1)^{23/24} + C(T_0) \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\Lambda^2 u\right\|_{L^2}\right) \times (A(t) + 1) \ln(1 + A(t))\, d\tau. \tag{38}$$

We set $B(t) := (1 + A(t))^{1/24}$, $B_0 := (1 + A_0)^{1/24}$ and divide the above inequality by $(1 + A(t))^{23/24}$; using the monotonicity of $A(t)$ we reach

$$B(t) \leq C(T_0)\left[B_0 + 1 + \int_{T_0}^{t} \left(1 + \|\nabla u\|_{\mathrm{BMO}} + \left\|\nabla^2 u\right\|_{L^2}\right) \times B(t) \ln B(t)\, d\tau\right]. \tag{39}$$

The standard Gronwall's inequality now gives

$$B(t) \leq \left[C(T_0)(1 + B_0)\right]^{\exp[C(T_0) \int_{T_0}^{t} 1 + \|\nabla u\|_{\mathrm{BMO}} + \|\Lambda^2 u\|_{L^2} d\tau]}, \tag{40}$$

which leads to

$$A(t) \leq \left[C(T_0)(1 + B_0)\right]^{24 \exp[C(T_0) \int_{T_0}^{t} 1 + \|\nabla u\|_{\mathrm{BMO}} + \|\Lambda^2 u\|_{L^2} d\tau]}. \tag{41}$$

As $\int_{T_0}^{t} \|\nabla u\|_{\mathrm{BMO}} + \|\Lambda^2 u\|_{L^2} d\tau$ remains bounded as $t \nearrow T$, the above inequality contradicts that $A(t) \nearrow \infty$ as $t \nearrow T$, so we complete our proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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