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Research Article

Observability Estimate for the Fractional Order Parabolic Equations on Measurable Sets

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We establish an observability estimate for the fractional order parabolic equations evolved in a bounded domain Ω of $\mathbb{R}^n$. The observation region is $F\times\omega$, where ω and F are measurable subsets of Ω and $(0,T)$, respectively, with positive measure. This inequality is equivalent to the null controllable property for a linear controlled fractional order parabolic equation. The building of this estimate is based on the Lebeau-Robbiano strategy and a delicate result in measure theory provided in Phung and Wang (2013).

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with real analytic boundary. Let $\omega \subset \Omega$ be a Lebesgue measurable subset with positive measure, and denote the characteristic function of ω by χ_ω. Let $T > 0$. Let $F \subset (0,T)$ be a Lebesgue measurable subset with positive measure, and denote the characteristic function of F by χ_F. Now, we define an unbounded operator A in $L^2(\Omega)$ as follows:

$$\begin{aligned} &D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ &Av = -\Delta v, \quad \text{for any } v \in D(A). \end{aligned} \tag{1}$$

Let $\{\lambda_i\}_{i=1}^\infty$, $0 < \lambda_1 < \lambda_2 \leq \cdots$, be the eigenvalues of $A = -\Delta$, and let $\{e_i\}_{i=1}^\infty$ be the corresponding eigenfunctions satisfied that $\|e_i(x)\|_{L^2(\Omega)} = 1$, $i = 1,2,3,\ldots$, which constitutes an orthonormal basis of $L^2(\Omega)$. It is well known that we can define a class of operator A^α $(\alpha \geq 0)$ in $L^2(\Omega)$ as follows:

$$\begin{aligned} &D(A^\alpha) = \left\{ v \in L^2(\Omega) \mid v = \sum_{i=1}^\infty v_i e_i, \sum_{i=1}^\infty \lambda_i^{2\alpha} |v_i|^2 < \infty \right\}, \\ &A^\alpha v = \sum_{i=1}^\infty \lambda_i^\alpha v_i e_i, \quad \text{where } v = \sum_{i=1}^\infty v_i e_i. \end{aligned} \tag{2}$$

Moreover, the operator A^α is a self-adjoint operator and $-A^\alpha$ is an infinitesimal generator of a strong continuous semigroup $\{S_\alpha(t)\}_{t\geq 0}$. Now, we consider the following linear controlled fractional order parabolic equation:

$$\begin{aligned} &\partial_t y(x,t) + A^\alpha y(x,t) = Bu(x,t), \quad (x,t) \in \Omega \times (0,T], \\ &y(x,0) = y_0(x), \quad x \in \Omega, \end{aligned} \tag{3}$$

where $\alpha > 1/2$, B is a linear bounded operator in $L^2(\Omega)$ defined by $Bu = \chi_F(t) \cdot \chi_\omega(x) u$, $y_0 \in L^2(\Omega)$, and $u(\cdot,t)$ is a control function taken from the space $L^2(0,T;L^2(\Omega))$. We denote $y(\cdot; y_0, u)$ to be the unique solution of (3) corresponding to the control u and the initial value y_0. We denote $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to be the usual norm and the inner product in $L^2(\Omega)$, respectively.

In recent years, extensive research has been devoted to the study of differential equations with fractional orders due to their importance for applications in various branches of applied sciences and engineering. Many important phenomena in signal processing, electromagnetics, crowded systems, and fluid mechanics are well described by fractional differential equation (see [1]). In this paper, we always discuss the fractional Laplacian. The fractional Laplacian $-A^\alpha$, with

$\alpha \in (0,1]$, generates the rotationally invariant 2α stable Lévy process. For $\alpha = 1$, this process is the normal Brownian motion B_t on $\mathbb{R}^n$ (see [2]).

Now, we will focus on the issue of what the controllable property is for the controlled system (3). System (3) is said to be null controllable in time T if for any $y_0 \in L^2(\Omega)$, there exists a control function $u \in L^2(0,T;L^2(\Omega))$, such that the solutions of (3) matches

$$y(T;y_0,u) = 0. \tag{4}$$

The problem of null controllability of parabolic equations has also been the object of numerous studies. Extensive related references can be found in [3–7] and the rich works cited therein. Especially, we refer to [5] for a null controllability result for the parabolic equations which plays a crucial role in establishing the main result in our paper. In the above works, the control region ω is always assumed to contain an open ball. The reason is that the main technique used in the argument, Carleman inequality, is required to construct weight functions. The construction of such functions seems to be not possible, when ω do not contain a ball. Recently, the null controllability for the parabolic equations with ω that is a measurable subset of positive measure has been established in [8], where an inequality involving measurable sets for a class of real analytic functions was set up in a skillful way. On the other hand, the classical null controllability for some fractional order parabolic equation was studied in [9, 10]. In particular, in [9] the authors proved that one-dimension problem is not controllable from the boundary for $\alpha \in (0,1/2]$.

By the classical duality argument [11], the controllable properties can be transformed into observability problems on the adjoint system. The adjoint system for (3) may be described as follows:

$$\begin{gathered}\partial_t \varphi(x,t) = A^\alpha \varphi(x,t) \quad (x,t) \in \Omega \times [0,T), \\ \varphi(x,T) = \varphi_0(x), \quad x \in \Omega.\end{gathered} \tag{5}$$

Thus, the exact null controllability property is equivalent to the existence of a constant $C = C(T) > 0$ such that the following inequality holds for every solution of (5):

$$\|\varphi(x,0)\|^2_{L^2(\Omega)} \le C(T) \int_F \int_\omega |\varphi|^2 dx\, dt. \tag{6}$$

Inequality (6) is called observability inequality, and the best constant $C(T)$ in inequality (6) will be referred as the observability constant. In this work, we discuss the internal observability estimate for the adjoint system (5) when ω and F are measurable subsets of Ω and $(0,T)$, respectively, with positive measure. To the best of our knowledge, this observability estimate has not been studied in the past publications.

The main result of the paper is presented as follows.

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is a bounded domain with a real analytic boundary and $\omega \subset \Omega$ is a Lebesgue measurable set with positive measure. Let $T > 0$, and let $F \subset (0,T)$ be a Lebesgue measurable set with positive measure. Let $\alpha > 1/2$. Then, there exists a constant $C = C(\Omega,T,\omega,F,\alpha)$ such that, for any data $\varphi_0 \in L^2(\Omega)$, the solution of* (5) *satisfied*

$$\|\varphi(x,0)\|^2_{L^2(\Omega)} \le C(\Omega,T,\omega,F,\alpha) \int_F \int_\omega |\varphi(x,t)|^2 dx\, dt. \tag{7}$$

Observability inequality (7) in Theorem 1 allows for estimating the total energy of the solutions of (5) at time 0 in terms of the partial energy localized in the observation region $F \times \omega$, where ω and F are measurable subsets of Ω and $(0,T)$, respectively, with positive measure. This inequality is equivalent to the null controllability property for the controlled system (3).

We proceed as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the proof of Theorem 1.

2. Preliminary Results

In this section, we will introduce some notions and preliminary results. Based on classical semigroup theory, we see that the operator $-A^\alpha$ is the generator of a semigroup of contraction in $L^2(\Omega)$, which we denote by $S_\alpha(t)$, $\alpha > 1/2$. Indeed, the semigroup can be written as follows:

$$S_\alpha(t)\psi = \sum_{i=1}^{\infty} e^{-\lambda_i^\alpha t} \langle \psi, e_i \rangle_{e_i}, \quad \text{for } \psi \in L^2(\Omega). \tag{8}$$

From this, it follows that

$$\|S_\alpha(t)\psi\|_{L^2(\Omega)} \le e^{-\lambda_1^\alpha t} \|\psi\|_{L^2(\Omega)}, \tag{9}$$

for any $\alpha > 1/2$ (see [12]).

Throughout the rest of the paper, the following notation will be used. For each measurable set $A \subseteq \mathbb{R}$, $|A|$ stands for its Lebesgue measure in $\mathbb{R}$. The following lemma is quoted from [13].

Lemma 2. *Let $E \subseteq [0,T]$ be a measurable set with a positive measure, and let l be a density point for E. Then for each $z > 1$, there exists a $l_1 \in (l,T)$ such that the sequence $\{l_i\}_{i=1}^\infty$, given by*

$$l_{i+1} = l + \frac{1}{z^i}(l_1 - l), \tag{10}$$

satisfies

$$|E \cap (l_{i+1}, l_i)| \ge \frac{1}{2}(l_i - l_{i+1}). \tag{11}$$

Next, we recall the following results, which play a key role in this paper.

Lemma 3. *Let Ω be a bounded domain in R^n, $n \ge 1$. Suppose that Ω have real analytic boundary. Then, for each subset $\omega \subset \Omega$ with positive measure, there exist two positive constants $C_1 > 1$ and $C_2 > 0$, which only depend on Ω, ω, such that*

$$\sum_{\lambda_i \le r} |a_i|^2 \le C_1 e^{C_2\sqrt{r}} \int_\omega \left| \sum_{\lambda_i \le r} a_i e_i(x) \right|^2 dx \tag{12}$$

for each finite $r > 0$ and each choice of the coefficients $\{a_i\}_{\lambda_i \le r}$ with $a_i \in \mathbb{R}$.

This conclusion can be found in the literature [8].

Next, for each $r > 0$, we define $X_r = \operatorname{span}\{e_i(x)\}_{\lambda_i \le r}$ and $X_r^{\perp} = \operatorname{span}\{e_i(x)\}_{\lambda_i > r}$. Indeed, for each $r > 0$, $L^2(\Omega) = X_r \oplus X_r^{\perp}$.

Lemma 4. *For any $\xi_0 \in X_r^{\perp}$, it always holds that*

$$\|S_\alpha(t)\xi_0\|_{L^2(\Omega)} < e^{-r^\alpha t}\|\xi_0\|_{L^2(\Omega)}. \tag{13}$$

This lemma can be easily obtained by (8) and (9).

3. The Proof of Main Result

Proof. Let $E = \{t \mid t = T - s, \text{where } s \in F\}$. Then, $|E| = |F| > 0$. Let l be a density point for E. By Lemma 2, for $z > 1$, there exists a $l_1 \in (l, T)$ and a sequence $\{l_i\}_{i=1}^{\infty}$ satisfied (10) and (11).

We now define a sequence subset of $(0, T)$ as follows:

$$E_i = \left\{t - \frac{l_i - l_{i+1}}{4} \mid t \in E \cap \left(l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_i\right)\right\}. \tag{14}$$

In fact, E_i is a subset of $(l_{i+1}, l_{i+1} + (3/4)(l_i - l_{i+1}))$, and

$$\begin{aligned} |E_i| &= \left|E \cap \left(l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_i\right)\right| \\ &= \left|E \cap \left\{(l_{i+1}, l_i) \setminus \left(l_{i+1}, l_{i+1} + \frac{l_i - l_{i+1}}{4}\right)\right\}\right|. \end{aligned} \tag{15}$$

By (11), it follow that

$$|E_i| \ge \frac{l_i - l_{i+1}}{4}. \tag{16}$$

Let i_0 be the first natural number satisfying $i_0 > 1/(2\alpha - 1)$; namely, $i_0 = [1/(2\alpha - 1)] + 1$. Let $b > 1$ be a positive number satisfying

$$\left(\frac{b^\alpha}{z}\right)^i \left(1 - \frac{1}{z}\right)(l_1 - l) > 4C_2 b^{(i+1)/2} + 4\ln(8C_1 z), \quad \text{for } i = i_0, i_0 + 1, i_0 + 2, \ldots. \tag{17}$$

Taking $r_i = b^i$, by (8), it follows that, for any $\xi \in X_{r_i}$,

$$\begin{aligned} &\int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} \chi_{E_i}(t) \left\|S_\alpha\left(l_{i+1} + \frac{3}{4}(l_i - l_{i+1})\right)\xi\right\|_{L^2(\Omega)} dt \\ &\le \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} \chi_{E_i}(t) \|S_\alpha(t)\xi\|_{L^2(\Omega)} dt. \end{aligned} \tag{18}$$

Combining with (16) and (12), this shows that

$$\begin{aligned} &\frac{l_i - l_{i+1}}{4}\left\|S_\alpha\left(l_{i+1} + \frac{3}{4}(l_i - l_{i+1})\right)\xi\right\|_{L^2(\Omega)} \\ &\le |E_i| \cdot \left\|S_\alpha\left(l_{i+1} + \frac{3}{4}(l_i - l_{i+1})\right)\xi\right\|_{L^2(\Omega)} \\ &\le \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} \chi_{E_i}(t) \|S_\alpha(t)\xi\|_{L^2(\Omega)} dt \\ &\le C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} \chi_{E_i}(t) \|S_\alpha(t)\xi\|_{L^2(\omega)} dt. \end{aligned} \tag{19}$$

For each $\phi \in L^2(\Omega)$, we can write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in X_{r_i}$ and $\phi_2 \in X_{r_i}^{\perp}$. Taking $\xi = S_\alpha((l_i - l_{i+1})/4)\phi_1$ in (19), it follows that

$$\begin{aligned} &\frac{l_i - l_{i+1}}{4}\|S_\alpha(l_i)\phi_1\|_{L^2(\Omega)} \\ &\le \frac{l_i - l_{i+1}}{4}\left\|S_\alpha\left(l_{i+1} + \frac{3}{4}(l_i - l_{i+1})\right) \circ S_\alpha\left(\frac{l_i - l_{i+1}}{4}\right)\phi_1\right\|_{L^2(\Omega)} \\ &\le C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} \chi_{E_i}(t) \\ &\qquad \times \left\|S_\alpha\left(t + \frac{l_i - l_{i+1}}{4}\right)\phi_1\right\|_{L^2(\omega)} dt \\ &\le C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+((l_i - l_{i+1})/4)}^{l_i} \chi_{E_i}\left(t - \frac{l_i - l_{i+1}}{4}\right) \\ &\qquad \times \|S_\alpha(t)\phi_1\|_{L^2(\omega)} dt. \end{aligned} \tag{20}$$

By the definition of E_i, it is easily to see that

$$\chi_{E_i}\left(t - \frac{l_i - l_{i+1}}{4}\right) = \chi_E(t), \quad \text{for any } t \in \left(l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_i\right). \tag{21}$$

This, together with (20), deduces

$$\begin{aligned} &\frac{l_i - l_{i+1}}{4}\|S_\alpha(l_i)\phi_1\|_{L^2(\Omega)} \\ &\le C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+((l_i - l_{i+1})/4)}^{l_i} \chi_E(t) \\ &\qquad \times \left(\|S_\alpha(t)\phi\|_{L^2(\omega)} + \|S_\alpha(t)\phi_2\|_{L^2(\Omega)}\right) dt \\ &\le C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+((l_i - l_{i+1})/4)}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt \\ &\quad + C_1 e^{C_2\sqrt{r_i}}(l_i - l_{i+1}) \left\|S_\alpha\left(l_{i+1} + \frac{l_i - l_{i+1}}{4}\right)\phi_2\right\|_{L^2(\Omega)} dt. \end{aligned} \tag{22}$$

The last step is based on the energy decay property of $S_\alpha(t)$. Along with Lemma 4, we derive that

$$\begin{aligned} &\frac{l_i - l_{i+1}}{4}\|S_\alpha(l_i)\phi_1\|_{L^2(\Omega)} \\ &\le C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+((l_i - l_{i+1})/4)}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt \\ &\quad + C_1 e^{C_2\sqrt{r_i}}(l_i - l_{i+1}) e^{-r_i^\alpha((l_i - l_{i+1})/4)} \|S_\alpha(l_{i+1})\phi_2\|_{L^2(\Omega)} \end{aligned}$$

$$\leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt + C_1 e^{C_2\sqrt{r_i}} (l_i - l_{i+1}) e^{-r_i^\alpha((l_i-l_{i+1})/4)} \|S_\alpha(l_{i+1})\phi_2\|_{L^2(\Omega)}. \tag{23}$$

Therefore,

$$\begin{aligned}\frac{l_i - l_{i+1}}{4}\|S_\alpha(l_i)\phi\|_{L^2(\Omega)} &\leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt \\ &\quad + C_1 e^{C_2\sqrt{r_i}} (l_i - l_{i+1}) e^{-r_i^\alpha((l_i-l_{i+1})/4)} \|S_\alpha(l_{i+1})\phi_2\|_{L^2(\Omega)} \\ &\quad + \frac{l_i - l_{i+1}}{4}\|S_\alpha(l_i)\phi_2\|_{L^2(\Omega)} \\ &\leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt \\ &\quad + C_1 e^{C_2\sqrt{r_i}} (l_i - l_{i+1}) e^{-r_i^\alpha((l_i-l_{i+1})/4)} \|S_\alpha(l_{i+1})\phi_2\|_{L^2(\Omega)} \\ &\quad + \frac{l_i - l_{i+1}}{4} e^{-r_i^\alpha(l_i-l_{i+1})} \|S_\alpha(l_{i+1})\phi_2\|_{L^2(\Omega)}.\end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}&\frac{l_i - l_{i+1}}{4}\|S_\alpha(l_i)\phi\|_{L^2(\Omega)} \\ &\quad \leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt \\ &\quad + (l_i - l_{i+1}) e^{-r_i^\alpha((l_i-l_{i+1})/4)} \left(C_1 e^{C_2\sqrt{r_i}} + 1\right) \|S_\alpha(l_{i+1})\phi\|_{L^2(\Omega)}.\end{aligned} \tag{25}$$

This leads to

$$\begin{aligned}&\frac{l_i - l_{i+1}}{4C_1 e^{C_2\sqrt{r_i}}}\|S_\alpha(l_i)\phi\|_{L^2(\Omega)} \\ &\quad - \frac{C_1 e^{C_2\sqrt{r_i}} + 1}{C_1 e^{C_2\sqrt{r_i}}} (l_i - l_{i+1}) e^{-r_i^\alpha((l_i-l_{i+1})/4)} \|S_\alpha(l_{i+1})\phi\|_{L^2(\Omega)} \\ &\quad \leq \int_{l_{i+1}}^{l_i} \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt.\end{aligned} \tag{26}$$

Summing (26) from i_0 to ∞, it follows that

$$\begin{aligned}&\frac{l_{i_0} - l_{i_0+1}}{4C_1 e^{C_2\sqrt{r_{i_0}}}}\|S_\alpha(l_{i_0})\phi\|_{L^2(\Omega)} + \sum_{i=i_0}^{\infty} k_i \|S_\alpha(l_{i+1})\phi\|_{L^2(\Omega)} \\ &\quad \leq \int_0^T \chi_E(t) \|S_\alpha(t)\phi\|_{L^2(\omega)} dt,\end{aligned} \tag{27}$$

where

$$k_i = \frac{l_{i+1} - l_{i+2}}{4C_1 e^{C_2\sqrt{r_{i+1}}}} - \frac{C_1 e^{C_2\sqrt{r_i}} + 1}{C_1 e^{C_2\sqrt{r_i}}} (l_i - l_{i+1}) e^{-r_i^\alpha((l_i-l_{i+1})/4)}, \quad i = i_0, i_0 + 1, i_0 + 2 \ldots. \tag{28}$$

Actually, by (17), we can derive that

$$k_i > 0, \quad \text{for any } i = i_0, i_0 + 1, i_0 + 2 \ldots. \tag{29}$$

This, together with (27), shows that

$$\begin{aligned}\|S_\alpha(T)\phi\|_{L^2(\Omega)} &\leq \|S_\alpha(l_{i_0})\phi\|_{L^2(\Omega)} \\ &\leq \frac{4C_1 e^{C_2\sqrt{r_{i_0}}}}{l_{i_0} - l_{i_0+1}} \int_E \|S_\alpha(t)\phi\|_{L^2(\omega)} dt.\end{aligned} \tag{30}$$

Now, we are in the position to prove (7). The solution of (5) can be written as follows:

$$\varphi(t) = S_\alpha(T - t)\varphi_0. \tag{31}$$

Along with (30), we can deduce that

$$\|\varphi(x, 0)\|_{L^2(\Omega)} \leq \frac{4C_1 e^{C_2\sqrt{r_{i_0}}}}{l_{i_0} - l_{i_0+1}} \int_F \|\varphi(t)\|_{L^2(x,\omega)} dt. \tag{32}$$

This completes the proof of the main result. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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