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Research Article

Continuous-Time Mean-Variance Portfolio Selection under the CEV Process

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We consider a continuous-time mean-variance portfolio selection model when stock price follows the constant elasticity of variance (CEV) process. The aim of this paper is to derive an optimal portfolio strategy and the efficient frontier. The mean-variance portfolio selection problem is formulated as a linearly constrained convex program problem. By employing the Lagrange multiplier method and stochastic optimal control theory, we obtain the optimal portfolio strategy and mean-variance efficient frontier analytically. The results show that the mean-variance efficient frontier is still a parabola in the mean-variance plane, and the optimal strategies depend not only on the total wealth but also on the stock price. Moreover, some numerical examples are given to analyze the sensitivity of the efficient frontier with respect to the elasticity parameter and to illustrate the results presented in this paper. The numerical results show that the price of risk decreases as the elasticity coefficient increases.

1. Introduction

The classical mean-variance portfolio selection model, which was first proposed by Markowitz [1], is to minimize the variance of the terminal wealth subject to archiving a given mean return level in a single-period investment. This model has been widely used in financial industry and become the foundation of modern finance theory. Since the pioneer work of Markowitz, the mean-variance portfolio selection model has inspired literally hundreds of extensions and applications. One of the mainstreaming researches is extending the standard portfolio selection model to the dynamic case (see, e.g., [2–5] and the references therein). In 2000, by employing the framework of multiobjective optimization and an embedding technique, Li and Ng [6] firstly derive the exact mean-variance efficient frontier in multiperiod investment; Zhou and Li [7] used the embedding technique and linear-quadratic (LQ) optimal control theory to solve the continuous-time mean-variance problem with stocks price described by geometric Brownian motion (GBM). Recently, by exploiting the stochastic control theory, there has been a series of papers discussing the continuous-time mean-variance portfolio selection problem in different markets (see [8–14] and the references therein).

In most of the continuous-time mean-variance models, the volatility of stock price is always assumed to be independent of stock price. However, many empirical evidences (see, e.g., [15–17]) have revealed that the volatility depends on the stock price. The CEV process, which is a stochastic volatility model and a natural extension of GBM, is capable of describing this dependence. This process was originally proposed by Cox [18] to describe the stock price for European option pricing. Afterwards, owning to its ability to capture the implied volatility smile which was observed across a wide range of markets and risky assets, the CEV process was widely used to analyze the options and asset pricing formula, as was done by Beckers [15] and Davydov and Linetsky [19]. Recently, the CEV process has drawn an increasing attention in optimal investment selection. Xiao et al. [20] and Gao [21] studied the DC pension investment with the CEV risk price process and derived the optimal policy under different utility functions. Gu et al. [22] derived the optimal reinsurance-investment decision under the CRRA or

CARRA utility function when stock price follows the CEV process.

Due to the support of many empirical evidences, it is clear that the introduction of CEV process will make the mean-variance model more practical. However, to the best of our knowledge, there is no paper discussing the applications of CEV processes in mean-variance investment. In this paper, we assume that the stock price follows the CEV process and try to find a mean-variance optimal portfolio strategy and efficient frontier. At first, we introduce an unconstrained stochastic control problem and derive the optimal control strategies in closed form by solving Hamilton-Jacobi-Bellman (HJB) equation. In the second, following the ideas of Lim and Zhou [10], we use the Lagrange multiplier technique to derive both the mean-variance optimal policy and the efficient frontier. Finally, we give some numerical examples to analyze the sensitivity of efficient frontier with respect to the elasticity coefficient and to illustrate the results obtained in this paper. The results in this paper show that the efficient frontier is still of a perfect square and the optimal portfolio strategy depends not only on the total wealth but also on the stock price. Moreover, the price of risk decreases as the elasticity coefficient increases.

This paper proceeds as follows. In Section 2, we give some preliminaries, formulate a continuous-time mean-variance portfolio selection model with the CEV process, and introduce an unconstrained problem. In Section 3, we derive the optimal strategies for the unconstrained problem. Section 4 presents the optimal investment strategies and efficient frontier. Finally, Section 5 gives a numerical example to analyze the sensitivity of efficient frontier with respect to the elasticity coefficient and to illustrate the results presented in this paper and Section 6 concludes this paper.

2. Problem Formulation

Throughout this paper, let T be a fixed terminal time and $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{t\geq 0})$ a fixed filtered complete probability space on which a standard $\mathscr{F}_t$-adapted 1-dimensional Brownian motion $W(t)$ is defined. It is assumed that $\mathscr{F}_t = \sigma\{W(s) : s \leq t\}$. We denote by $\mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ the class of all $\mathscr{F}_t$-adapted, $\mathbb{R}^d$-valued stochastic processes such that $E\int_0^T |f(t)|^2 dt < +\infty$ and by $\mathscr{C}^{1,2,2}$ the set of all functions $H(t,s,x) : [0,T] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with continuous partial derivatives H_t, H_s, H_x, H_{ss}, H_{sx} , and H_{xx}.

Now we consider a financial market consisting of two assets, where one is bond and the other is stock (we can also treat it as equity index). The bond price $P(t)$ satisfies the following ordinary differential equation:

$$dP(t) = rP(t)\,dt, \quad t \in [0,T], \qquad P(0) = P_0 > 0, \tag{1}$$

where the interest rate $r > 0$ is a constant. The stock price $S(t)$ is described by the CEV model (see Yuen et al. [17])

$$dS(t) = S(t)\left[u dt + kS(t)^{\beta} dW(t)\right], \qquad S(0) = S_0 > 0, \tag{2}$$

where u is an expected instantaneous rate of return on the stock, $kS(t)^{\beta}$ is the instantaneous volatility, and β is the elasticity parameter. We assume throughout that u and β satisfy the general conditions; that is, $u > r$ and $\beta < 0$.

Remark 1. Note that the CEV process covers a number of well-known processes. When $\beta = 0$, it reduces to the GBM; when $\beta = -1$, it becomes the absolute diffusion process; when $\beta = -1/2$, it becomes the square root process proposed by Cox and Ross [23].

Suppose that the trading of shares takes place continuously in a self-financing fashion (i.e., there is no consumption or income) and that there are no transaction costs. We denote by $X(t)$ the total wealth of an investor at time $t \in [0,T]$ and by $\pi(t)$ the total stock value of the investor. Then

$$dX(t) = [(u-r)\pi(t) + rX(t)]\,dt + kS(t)^{\beta}\pi(t)\,dW(t), \qquad X(0) = X_0. \tag{3}$$

We call $\pi(\cdot)$ a portfolio of the investor. If $\pi(t) < 0$, that means the investor is short selling in stock; otherwise, that means the investor is borrowing the amount $\pi(t) - X(t)$ at rate r. For a prescribed mean terminal wealth $EX(T) = d$, mean-variance portfolio problem is to determine a portfolio $\pi(\cdot)$ minimizing the variance of the terminal wealth $\operatorname{Var} X(T)$, that is, minimizing

$$\operatorname{Var} X(T) = E[X(T) - EX(T)]^2 = E[X(T) - d]^2. \tag{4}$$

For the risk-free portfolio $\pi(\cdot) \equiv 0$, the wealth process $X(\cdot)$ satisfies $dX(t) = rX(t)dt$ and $X(0) = X_0$ and has, for solution, $X(T) = X_0 e^{rT}$. This implies that the investor should expect a mean return above $X_0 e^{rT}$. Thus, we give the following natural assumption:

$$d \geq X_0 e^{rT}. \tag{5}$$

Definition 2. A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R})$. In this case, one calls $(X(\cdot), \pi(\cdot))$ an admissible pair, where $X(\cdot)$ is the solution to (3).

By Definition 2, the mean-variance problem can be formulated as follows:

$$J^* := \min \operatorname{Var} X(T) = E[X(T) - d]^2, \qquad \text{subject to: } EX(T) = d,\ (X(\cdot), \pi(\cdot)) \text{ is admissible}, \tag{6}$$

where $d \geq X_0 e^{rT}$.

An admissible portfolio $\pi(\cdot)$ is said to be a feasible portfolio for (6) if it satisfies the constraints in (6) and then (6) is said to be feasible.

Definition 3. An admissible portfolio $\pi^*(\cdot)$ is called an efficient portfolio if it is the optimal strategy for (6). The pair $(\operatorname{Var} X^*(T), EX^*(T))$ is called an efficient point, where $X^*(\cdot)$ is the wealth process corresponding to the efficient portfolio $\pi^*(\cdot)$, whereas the set of all efficient points is called an efficient frontier.

It is clear that (6) is a linearly constrained convex program problem. Thus, it can be reduced to an unconstrained problem by introducing a Lagrange multiplier. So, in Section 3, we will consider the following unconstrained problem parameterized by $l \in \mathbb{R}$,

$$\begin{aligned} &\min \qquad E[X(T)-l]^2, \\ &\text{subject to}: \quad (X(\cdot),\pi(\cdot)) \text{ is admissible for } (3), \end{aligned} \tag{7}$$

and solve it by using HJB approach. Furthermore, in Section 4, we employ the results in Section 3 and the Lagrange multiplier method to obtain the mean-variance efficient portfolio and efficient frontier.

3. Solutions to the Unconstrained Problem

In this section, we consider deriving the optimal solution for the unconstrained problem (7). In order to solve it, at first, we establish the HJB equation for (7). In the next, we try to solve the HJB equation via power transformation and variable change method. Finally, we derive the optimal strategy and optimal value function in closed form for (7).

Now we define the value function

$$\begin{aligned} &V(t,s,x) \\ &\quad = \min\left\{E_{t,s,x}[X(T)-l]^2 : S(t)=s,\right. \\ &\qquad \left. X(t)=x, (X(\cdot),\pi(\cdot)) \text{ is admissible for } (3)\right\} \end{aligned} \tag{8}$$

and the variational operator

$$\begin{aligned} L^\pi H(t,s,x) &= H_t + usH_s + [(u-r)\pi + rx]H_x \\ &\quad + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} + \frac{1}{2}k^2\pi^2 s^{2\beta} H_{xx} \\ &\quad + k^2\pi s^{2\beta+1} H_{sx}. \end{aligned} \tag{9}$$

By using stochastic optimal control theory, we obtain the following result.

Theorem 4 (verification theorem). *Let $H(t,s,x) \in \mathscr{C}^{1,2,2}$ satisfy the following HJB equation:*

$$\begin{aligned} &\inf_\pi \{L^\pi H(t,s,x)\} = 0, \\ &H(T,s,x) = (x-l)^2. \end{aligned} \tag{10}$$

Then $H(t,s,x) \le V(t,s,x)$. Moreover, if there exists $\pi^*(\cdot)$ such that

$$\pi^*(t) \in \arg\inf_\pi \{L^\pi H(t,S(t),X^*(t))\} \tag{11}$$

for almost $(t,\omega) \in [0,T]\times\Omega$, where $X^*(t)$ is the solution of (3) when $\pi(t)=\pi^*(t)$, then

$$H(t,s,x) = V(t,s,x) \tag{12}$$

and $\pi^*(\cdot)$ is the optimal control of (7).

Proof. By applying Itô's formula, we have

$$\begin{aligned} &dH(t,S(t),X(t)) \\ &\quad = L^\pi H(t,S(t),X(t))\,dt \\ &\qquad + \left[kS(t)^{\beta+1}H_s(t,S(t),X(t))\right. \\ &\qquad\quad \left. + k\pi S(t)^\beta H_x(t,S(t),X(t))\right]dW(t). \end{aligned} \tag{13}$$

Then, the solution of (13) is given by

$$\begin{aligned} &H(T,S(T),X(T)) \\ &\quad = H(t,S(t),X(t)) + \int_t^T L^\pi H(u,S(u),X(u))\,du \\ &\qquad + \int_t^T \left[kS(u)^{\beta+1}H_s(u,S(u),X(u))\right. \\ &\qquad\quad \left. + k\pi S(u)^\beta H_x(u,S(u),X(u))\right]dW(u). \end{aligned} \tag{14}$$

Since the last term of (14) is a martingale with zero expectation, taking expectation on both sides of (14) yields

$$\begin{aligned} &E_{t,s,x}[H(T,S(T),X(T))] \\ &\quad = H(t,s,x) + E_{t,s,x}\left[\int_t^T L^\pi H(u,S(u),X(u))\,du\right]. \end{aligned} \tag{15}$$

For any π, (10) implies

$$L^\pi H(t,s,x) \ge 0 \tag{16}$$

for all $(t,s,x) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}$.

From (10), (15), and (16), we get

$$H(t,s,x) \le E_{t,s,x}[H(T,S(T),X(T))] = E_{t,s,x}[X(T)-l]^2 \tag{17}$$

and so

$$H(t,s,x) \le \inf\left\{E_{t,s,x}[X(T)-l]^2\right\} = V(t,s,x). \tag{18}$$

When $\pi(\cdot) = \pi^*(\cdot)$, (16), (17), and (18) become equalities; that is, $H(t,s,x) = V(t,s,x)$, which means $\pi^*(\cdot)$ is the optimal control of problem (7). This completes the proof. □

By Theorem 4, the HJB equation for the optimal value function V is as follows: for all $t \in [0,T]$,

$$\inf_\pi \{L^\pi V(t,s,x)\} = 0 \tag{19}$$

with the boundary condition

$$V(T,s,x) = (x-l)^2. \tag{20}$$

Thus, we are now in the position to solve the above HJB equation.

Suppose that (19) and (20) have a solution $V \in \mathscr{C}^{1,2,2}$ with $V_{xx} > 0$. Then (19) attains its minimum at

$$\pi^*(t) = -\frac{k^2 s^{2\beta+1} V_{sx} + (u-r) V_x}{k^2 s^{2\beta} V_{xx}}. \tag{21}$$

Putting (21) in (19), we obtain a partial differential equation (PDE) for the value function V as follows:

$$\begin{aligned} &V_t + suV_s + rxV_x + \frac{1}{2}k^2 s^{2\beta+2} V_{ss} \\ &\quad - \frac{\left[k^2 s^{2\beta+1} V_{sx} + (u-r)V_x\right]^2}{2k^2 s^{2\beta} V_{xx}} = 0 \end{aligned} \tag{22}$$

with

$$V(T, s, x) = (x-l)^2. \tag{23}$$

Inspired by Gao [21], we conjecture a solution to (22) with the following form:

$$V(t, s, x) = g(t, s)\left[x - l - a(t)\right]^2, \tag{24}$$

where the boundary conditions are given by $g(T, s) = 1$ and $a(T) = 0$. Thus,

$$\begin{gathered} V_t = g_t (x-l-a)^2 - 2g(x-l-a)\dot{a}, \\ V_s = g_s (x-l-a)^2, \qquad V_x = 2g(x-l-a), \\ V_{ss} = g_{ss}(x-l-a)^2, \\ V_{xx} = 2g, \qquad V_{sx} = 2g_s(x-l-a), \end{gathered} \tag{25}$$

where $\dot{a}$ denotes the derivative of $a(t)$. Submitting these derivatives into (22), we have

$$\begin{aligned} &\left\{ g_t + sug_s + 2rg + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} \right. \\ &\quad \left. - \frac{\left[k^2 s^{2\beta+1} g_s + (u-r)g\right]^2}{k^2 s^{2\beta} g} \right\} \\ &\times (x-l-a)^2 + 2g(-\dot{a} + rl + ra)(x-l-a) = 0. \end{aligned} \tag{26}$$

Therefore, $a(t)$ and $g(t, s)$ should satisfy the following differential equations:

$$\dot{a} - rl - ra = 0, \tag{27}$$

$$\begin{aligned} &g_t + sug_s + 2rg + \frac{1}{2}k^2 s^{2\beta+2} g_{ss} \\ &\quad - \frac{\left[k^2 s^{2\beta+1} g_s + (u-r)g\right]^2}{k^2 s^{2\beta} g} = 0. \end{aligned} \tag{28}$$

Taking into account the boundary condition $a(T) = 0$, the solution of (27) is as follows:

$$a(t) = l\left[e^{-(T-t)r} - 1\right]. \tag{29}$$

In order to solve the nonlinear second-order partial differential equation (28), we employ the power transformation and variable change technique which were proposed by Cox [18]. Let

$$g(t, s) = f(t, y)^{-1}, \qquad y = s^{-2\beta} \tag{30}$$

with the boundary condition $f(T, y) = 1$. Then

$$\begin{aligned} &g_t = -f_t f^{-2}, \qquad g_s = 2\beta s^{-2\beta-1} f_y f^{-2}, \\ &g_{ss} = 2\beta s^{-2\beta-2} f^{-2} \\ &\quad \times \left[-(2\beta+1) f_y + 4\beta s^{-2\beta} f^{-1} f_y^2 - 2\beta s^{-2\beta} f_{yy}\right]. \end{aligned} \tag{31}$$

Putting those partial derivatives in (28), we know that f satisfies the following equation:

$$\begin{aligned} &f_t + \beta\left[(2\beta+1)k^2 - 2(2r-u)y\right] f_y + 2\beta^2 k^2 y f_{yy} \\ &\quad + (u-r)^2 k^{-2} y f - 2rf = 0, \\ &\quad f(T, y) = 1. \end{aligned} \tag{32}$$

Suppose that the solution of (32) is in the following form:

$$f(t, y) = A(t) e^{B(t)y} \tag{33}$$

with the boundary conditions $A(T) = 1$ and $B(T) = 0$. It follows from (32) and (33) that

$$\begin{aligned} &A^{-1}\dot{A} + k^2(2\beta+1)\beta B - 2r \\ &\quad + \left[\dot{B} - 2(2r-u)\beta B + 2\beta^2 k^2 B^2 + (u-r)^2 k^{-2}\right] y = 0. \end{aligned} \tag{34}$$

By matching coefficients, we derive

$$\dot{B} - 2(2r-u)\beta B + 2\beta^2 k^2 B^2 + (u-r)^2 k^{-2} = 0, \tag{35}$$

$$A^{-1}\dot{A} + k^2(2\beta+1)\beta B - 2r = 0. \tag{36}$$

Taking into account the boundary conditions, we get the solutions to (35) and (36) as follows (see the Appendix for proof details):

$$B(t)=\begin{cases} k^{-2}I(t), \\ \sqrt{2}r>u>r, \\ \left(2k^2\right)^{-1}\left[1+(2r-u)\beta(T-t)\right]^{-1} \\ \quad\times(2r-u)^2(T-t), \\ u=\sqrt{2}r, \\ \left(2\beta k^2\right)^{-1}\left\{2r-u-\sqrt{u^2-2r^2}\right. \\ \qquad\times\tan\left[-\beta\sqrt{u^2-2r^2}(T-t)\right. \\ \qquad\left.\left.+\arctan\dfrac{2r-u}{\sqrt{u^2-2r^2}}\right]\right\}, \\ u>\sqrt{2}r, \end{cases} \tag{37}$$

$$A(t)=\begin{cases} e^{[\lambda_1\beta(2\beta+1)-2r](T-t)} \\ \quad\times\left(\dfrac{\lambda_2-\lambda_1}{\lambda_2-\lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}\right)^{(2\beta+1)/2\beta}, \\ \quad\sqrt{2}r>u>r, \\ e^{-[2r-((2\beta+1)(2r-u)/2)](T-t)} \\ \quad\times\left[1+(2r-u)\beta(T-t)\right]^{-(2\beta+1)/2\beta}, \\ u=\sqrt{2}r, \\ e^{[-2r+((2\beta+1)/2)(2r-u)](T-t)} \\ \quad\times\left|\left(\cos\left(\arctan\dfrac{2r-u}{\sqrt{u^2-2r^2}}\right)\right)\right. \\ \quad\times\left(\cos\left[-\beta\sqrt{u^2-2r^2}(T-t)\right.\right. \\ \qquad\left.\left.+\arctan\dfrac{2r-u}{\sqrt{u^2-2r^2}}\right]\right)^{-1}\Bigg|^{(2\beta+1)/2\beta}, \\ u>\sqrt{2}r, \end{cases} \tag{38}$$

where

$$I(t)=\frac{\lambda_1-\lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}{1-(\lambda_1/\lambda_2)e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}, \tag{39}$$

$$\lambda_{1,2}=\frac{(2r-u)\pm\sqrt{2r^2-u^2}}{2\beta}. \tag{40}$$

From the above discussions, we get the solution to the HJB equations (19) and (20).

Proposition 5. *The solution to the HJB equations* (19) *and* (20) *is given as follows:*

$$V(t,s,x)=\frac{1}{A(t)e^{B(t)s^{-2\beta}}}\left(x-le^{-r(T-t)}\right)^2, \tag{41}$$

where $A(t)$ and $B(t)$ are given by (38) *and* (37)*, respectively.*

Proof. We need to verify that the assumption $V_{xx}>0$ is true. Noting that

$$V_{xx}(t,s,x)=2g(t,s)=2f\left(t,s^{-2\beta}\right)^{-1}=2A^{-1}(t)e^{-B(t)s^{-2\beta}}, \tag{42}$$

we only need to verify that $A(t)>0$ holds for all $t\in[0,T]$.

It is obvious that $A(t)>0$ when $u>\sqrt{2}r$. When $\sqrt{2}r>u>r$, let

$$R(t)=\frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}. \tag{43}$$

Then it is easy to know that

$$\lambda_2-\lambda_1=-\frac{\sqrt{2r^2-u^2}}{\beta}>0 \tag{44}$$

and so $R(t)$ is decreasing in t with

$$R(t)\geq R(T)=1. \tag{45}$$

Thus, $A(t)>0$ when $\sqrt{2}r>u>r$. Similarly, $A(t)>0$ holds when $u=\sqrt{2}r$.

Now putting (29), (30), and (33) in (24), it follows that (41) is true. This completes the proof. □

Finally, from Theorem 4, we obtain the optimal strategy and optimal value for (7) in the following result.

Theorem 6. *The optimal value for unconstrained problem* (7) *is given by* (41) *and the optimal policy is*

$$\begin{aligned}\pi^*(t)=&-\frac{1}{k^2S(t)^{2\beta}}\left[1+\frac{2\beta k^2B(t)}{u-r}\right] \\ &\times(u-r)\left(X^*(t)-le^{-r(T-t)}\right),\end{aligned} \tag{46}$$

where $B(t)$ is given by (37) *and $X^*(t)$ is the solution to* (3) *when $\pi(t)=\pi^*(t)$.*

Proof. It follows from Theorem 4 that (41) is the optimal value for (7). From (21), (24), (29), (30), (33), (37), and (38), we know that (46) is the optimal policy of unconstrained problem (7). This completes the proof. □

Remark 7. If we take l as a certain predefined benchmark and measure deviations from l by the squared distance, then (7) becomes a benchmark portfolio selection problem (see [24, 25] and the references therein).

Remark 8. Taking $t=0$, it follows from (41) that

$$V(0,S_0,X_0)=\frac{1}{A(0)e^{B(0)S_0^{-2\beta}}}\left(X_0-le^{-rT}\right)^2, \tag{47}$$

which is the efficient frontier for the benchmark portfolio selection problem. By Theorem 6, we know that $\pi^*(t)$ defined by (46) is the optimal portfolio strategy for the benchmark portfolio selection problem.

4. Efficient Portfolios and Efficient Frontier

In this section, we try to derive the efficient frontier and the optimal portfolio strategy for (6) by using the results in the above section and the Lagrange multiplier method. On the feasibility of (6), we have the following result.

Theorem 9. *For any prescribed mean target d, problem* (6) *is feasible.*

Proof. For the mean target d, we consider the following admissible portfolio:

$$\pi(t) \equiv \pi_0 = \frac{\left(d - X_0 e^{rT}\right) r}{(u-r)\left(e^{rT}-1\right)}. \tag{48}$$

Putting (48) in (3) and taking expectation on both sides of (3), we conclude that $EX(t)$ satisfies the following ordinary differential equation:

$$\begin{aligned} dEX(t) &= \left[(u-r)\pi_0 + rEX(t)\right] dt, \\ EX(0) &= X_0. \end{aligned} \tag{49}$$

Solving (49), we can get the explicit expression of $EX(T)$ as follows:

$$EX(T) = -\frac{u-r}{r}\pi_0 + \left(X_0 + \frac{u-r}{r}\pi_0\right) e^{rT} = d. \tag{50}$$

This completes the proof. □

Theorem 9 implies that the investor can archive any prescribed mean target. The following result will show the risk that the investor has to take.

Theorem 10. *If $f(\cdot,\cdot)$, $A(\cdot)$, and $B(\cdot)$ are given by* (33), (38), *and* (37), *respectively, then*

$$e^{2rT} f_0 > 1, \tag{51}$$

where $f_0 := f(0, S_0^{-2\beta})$. The efficient frontier of problem (6) *exists and is given by*

$$\operatorname{Var} X^*(T) = \frac{1}{e^{2rT} f_0 - 1}\left[EX^*(T) - X_0 e^{rT}\right]^2, \tag{52}$$

where $EX^*(T) \geq X_0 e^{rT}$. The optimal portfolio associated with the mean return $EX^*(T)$ is as follows:

$$\begin{aligned} \pi^*(t) = {} & \frac{1}{\sigma^2(t)}\left[1 + \frac{2\beta k^2 B(t)}{u-r}\right] \\ & \times (u-r)\left(\gamma e^{-r(T-t)} - X^*(t)\right), \end{aligned} \tag{53}$$

where

$$\begin{aligned} \sigma(t) &= kS^{\beta}(t), \\ \gamma &= \frac{e^{2rT} f_0 E\left[X^*(T)\right] - X_0 e^{rT}}{e^{2rT} f_0 - 1}. \end{aligned} \tag{54}$$

Proof. It is easy to verify that the mean-variance problem (6) has a convex cost which is bounded below and a convex constrained set, which means that (6) is a linearly constrained convex problem. Since the problem (6) is feasible, it follows from Luenberger [26] that

$$\begin{aligned} J^* = {} & \max_{\lambda\in\mathbb{R}} \inf_{(X(\cdot),\pi(\cdot)) \text{ is admissible}} E[X(T) - d]^2 \\ & + 2\lambda E[X(T) - d] < +\infty. \end{aligned} \tag{55}$$

For any fixed λ, the unconstrained problem

$$J(\lambda) := \inf_{(X(\cdot),\pi(\cdot)) \text{ is admissible}} E[X(T) - d]^2 + 2\lambda E[X(T) - d] \tag{56}$$

is equivalent to

$$\overline{J}(\lambda) := \inf_{(X(\cdot),\pi(\cdot)) \text{ is admissible}} E[X(T) - d + \lambda]^2 \tag{57}$$

in the sense that π^* solves (56) if and only if π^* solves (57) and $J(\lambda) = \overline{J}(\lambda) - \lambda^2$. Therefore, it follows from Theorem 6 that

$$\begin{aligned} J(\lambda) &= \overline{J}(\lambda) - \lambda^2 \\ &= V(0, S_0, X_0) - \lambda^2 \\ &= \frac{1}{f_0}\left[X_0 - (d-\lambda)e^{-rT}\right]^2 - \lambda^2 \\ &= \left(\frac{1}{e^{2rT} f_0} - 1\right)\lambda^2 + 2\frac{X_0 e^{rT} - d}{e^{2rT} f_0}\lambda \\ &\quad + \frac{1}{e^{2rT} f_0}\left(X_0 e^{rT} - d\right)^2 \end{aligned} \tag{58}$$

and the optimal policy of (57) is

$$\begin{aligned} \pi(t) = {} & -\frac{1}{k^2 S(t)^{2\beta}}\left[1 + \frac{2\beta k^2 B(t)}{u-r}\right] \\ & \times (u-r)\left[X^*(t) - (d-\lambda)e^{-r(T-t)}\right]. \end{aligned} \tag{59}$$

Because $J(\lambda)$ is quadratic in λ and J^* is finite, we can get that $e^{2rT} f_0 > 1$. In fact, if $e^{2rT} f_0 = 1$, then J^* can only be finite when $2((X_0 e^{rT} - d)/e^{2rT} f_0) = 0$ for all d, which is a contradiction. So it must be the case that $e^{2rT} f_0 > 1$.

Thus, we can get the optimal λ^* for (55) as follows:

$$\lambda^* = \frac{X_0 e^{rT} - d}{e^{2rT} f_0 - 1}. \tag{60}$$

Putting λ^* in (58) and (59), we obtain that

$$\begin{aligned}
J^* &= \left(\frac{1}{e^{2rT}f_0} - 1\right)\left[\frac{X_0e^{rT} - d}{e^{2rT}f_0 - 1}\right]^2 \\
&\quad + 2\frac{X_0e^{rT} - d}{e^{2rT}f_0}\frac{X_0e^{rT} - d}{e^{2rT}f_0 - 1} + \frac{1}{e^{2rT}f_0}\left(X_0e^{rT} - d\right)^2 \\
&= \left[\frac{1}{1 - e^{2rT}f_0} + \frac{2}{e^{2rT}f_0 - 1} + 1\right] \\
&\quad \times \frac{1}{e^{2rT}f_0}\left(X_0e^{rT} - d\right)^2 \\
&= \frac{1}{e^{2rT}f_0 - 1}\left(X_0e^{rT} - d\right)^2, \\
\pi^*(t) &= \frac{1}{k^2S(t)^{2\beta}}\left[1 + \frac{2\beta k^2 B(t)}{u - r}\right] \\
&\quad \times (u - r)\left[\frac{e^{2rT}df_0 - X_0e^{rT}}{e^{2rT}f_0 - 1}e^{-r(T-t)} - X^*(t)\right].
\end{aligned} \tag{61}$$

This completes the proof. □

Remark 11. Theorem 10 shows that the efficient frontier (52) is a parabola and tells the risk that the investor has to bear to archive a prescribed mean target. In particular, if the investor does not want to take any risk, namely, $\operatorname{Var} X^*(T) = 0$, then the expectation of his/her terminal wealth must be X_0e^{rT}, which implies that the optimal strategy of the investor is investing his/her total money in the bond.

Let $\sigma_{X^*(T)}$ be the standard deviation of the terminal wealth. It follows from (52) that

$$EX^*(T) = X_0e^{rT} + \sqrt{e^{2rT}f_0 - 1}\,\sigma_{X^*(T)}. \tag{62}$$

Hence, the efficient frontier in the mean-standard deviation plane is a straight line, which is also called the capital market line. The slope of this line, $K = \sqrt{e^{2rT}f_0 - 1}$, is termed the price of risk.

Before obtaining the efficient frontier with $\beta = 0$, we give the following proposition.

Proposition 12. *Consider*

$$\lim_{\beta\to 0^-} B(t) = k^{-2}(u - r)^2(T - t), \tag{63}$$

$$\lim_{\beta\to 0^-} f\left(0, S_0^{-2\beta}\right) = e^{[-2r+(u-r)^2/k^2]T}, \tag{64}$$

where $B(t)$ is given by (37) *and function f is given by* (33).

Proof. In the case of $\sqrt{2}r > u > r$,

$$\begin{aligned}
B(t) &= k^{-2}I(t), \\
f\left(0, S_0^{-2\beta}\right) &= A(0)\,e^{B(0)S_0^{-2\beta}} \\
&= e^{[\lambda_1\beta(2\beta+1)-2r]T}\left[\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)T}}\right]^{(2\beta+1)/2\beta} \\
&\quad \times e^{k^{-2}S_0^{-2\beta}I(0)},
\end{aligned} \tag{65}$$

where $I(\cdot)$ and $\lambda_{1,2}$ are given by (39) and (40), respectively. Putting (40) in (39), we get

$$\begin{aligned}
I(t) &= \frac{\lambda_1 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}{1 - (\lambda_1/\lambda_2)\,e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}} \\
&= \frac{2r - u + \sqrt{2r^2 - u^2}}{2}\frac{1}{\beta}\left[1 - e^{2\sqrt{2r^2-u^2}(T-t)\beta}\right] \\
&\quad \times \left[1 - \frac{2r - u + \sqrt{2r^2 - u^2}}{2r - u - \sqrt{2r^2 - u^2}}e^{2\sqrt{2r^2-u^2}(T-t)\beta}\right]^{-1}.
\end{aligned} \tag{66}$$

From l'Hôspital's rule, we get that

$$\lim_{\beta\to 0^-}\frac{1 - e^{2\sqrt{2r^2-u^2}(T-t)\beta}}{\beta} = -2\sqrt{2r^2 - u^2}\,(T - t). \tag{67}$$

Thus,

$$\begin{aligned}
\lim_{\beta\to 0^-} I(t) &= \frac{2r - u + \sqrt{2r^2 - u^2}}{2}\left[-2\sqrt{2r^2 - u^2}\,(T - t)\right] \\
&\quad \times \left[1 - \frac{2r - u + \sqrt{2r^2 - u^2}}{2r - u - \sqrt{2r^2 - u^2}}\right]^{-1} \\
&= (u - r)^2(T - t),
\end{aligned} \tag{68}$$

$$\lim_{\beta\to 0^-} B(t) = k^{-2}(u - r)^2(T - t), \tag{69}$$

which means that (63) is true when $\sqrt{2}r > u > r$.

Let

$$\begin{aligned}
&\alpha(\beta) \\
&= \frac{\lambda_1\left(e^{2\beta^2(\lambda_1-\lambda_2)T} - 1\right)}{\lambda_2 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)T}} \\
&= \frac{\left(2r - u + \sqrt{2r^2 - u^2}\right)\left(e^{2\sqrt{2r^2-u^2}T\beta} - 1\right)}{2r - u - \sqrt{2r^2 - u^2} - \left(2r - u + \sqrt{2r^2 - u^2}\right)e^{2\sqrt{2r^2-u^2}T\beta}}.
\end{aligned} \tag{70}$$

Then it follows from (70) that

$$\lim_{\beta\to 0^-}\alpha(\beta)=0,$$

$$\lim_{\beta\to 0^-}\alpha(\beta)\frac{2\beta+1}{2\beta}=\lim_{\beta\to 0^-}\frac{\alpha(\beta)}{2\beta}=-\frac{2r-u+\sqrt{2r^2-u^2}}{2}T. \tag{71}$$

By (71), we have

$$\begin{aligned}\lim_{\beta\to 0^-}&\left[\frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)T}}\right]^{(2\beta+1)/2\beta}\\ &=\lim_{\beta\to 0^-}\left[1+\alpha(\beta)\right]^{(1/\alpha(\beta))\alpha(\beta)((2\beta+1)/2\beta)}\\ &=e^{-((2r-u+\sqrt{2r^2-u^2})/2)T}.\end{aligned} \tag{72}$$

Finally, it follows from (68) and (72) that

$$\begin{aligned}\lim_{\beta\to 0^-}&f\left(0,S_0^{-2\beta}\right)\\ &=e^{-((2r-u+\sqrt{2r^2-u^2})/2)T}\\ &\quad\times e^{k^{-2}(u-r)^2T}\lim_{\beta\to 0^-}e^{[\lambda_1\beta(2\beta+1)-2r]T}\\ &=e^{-((2r-u+\sqrt{2r^2-u^2})/2)T}\\ &\quad\times e^{k^{-2}(u-r)^2T}e^{((2r-u+\sqrt{2r^2-u^2})/2-2r)T}\\ &=e^{[-2r+(u-r)^2/k^2]T};\end{aligned} \tag{73}$$

that is, (64) holds when $\sqrt{2}r>u>r$.

We can prove other two cases in a similar way. So, we omit it here.

This completes the proof. □

From Theorem 10 and Proposition 12, it is very easy to get the following result.

Corollary 13. *When $\beta=0$, the mean-variance efficient frontier and efficient portfolio are given by*

$$\operatorname{Var}X^*(T)=\frac{1}{e^{((u-r)^2/k^2)T}-1}\left(EX^*(T)-X_0e^{rT}\right)^2, \tag{74}$$

$$\begin{aligned}\pi^*(t)&=\frac{1}{k^2}(u-r)\\ &\quad\times\left[\frac{e^{k^{-2}(u-r)^2T}EX^*(T)-X_0e^{rT}}{e^{k^{-2}(u-r)^2T}-1}e^{-r(T-t)}-X^*(t)\right],\end{aligned} \tag{75}$$

respectively.

Remark 14. In a financial market consisting of a risk-free bond and a stock, the results of Corollary 13 are the same as that shown in Zhou and Li [7].

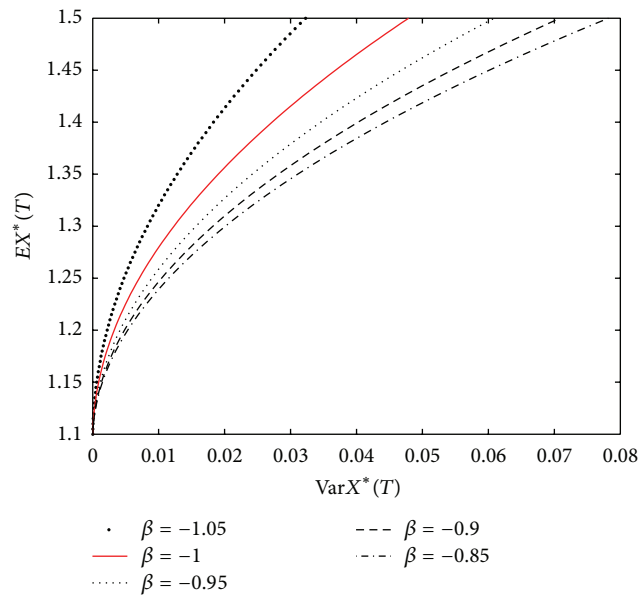

Figure 1: The efficient frontiers in mean-variance plane ($\beta=-1.05,\ldots,-0.85$).

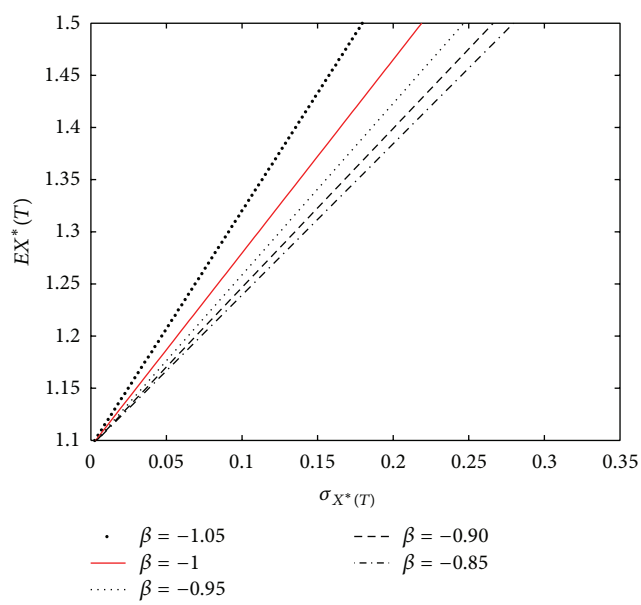

Figure 2: The efficient frontiers in mean-standard deviation plane ($\beta=-1.05,\ldots,-0.85$).

5. Numerical Analysis

In this section, we give some numerical examples to analyze the sensitivity of the mean-variance efficient frontier with respect to the elasticity coefficient and to illustrate the dynamic behaviors of the mean-variance optimal portfolio strategy and the total wealth. The basic parameters in the model are given by $r=0.03$, $u=0.12$, $k=16.16$, $\beta=-1$, and $S_0=67$, which are the same as that in Yuen et al. [17] who estimated the CEV model parameters for Hong Kong stock option market. We consider an investor with an initial

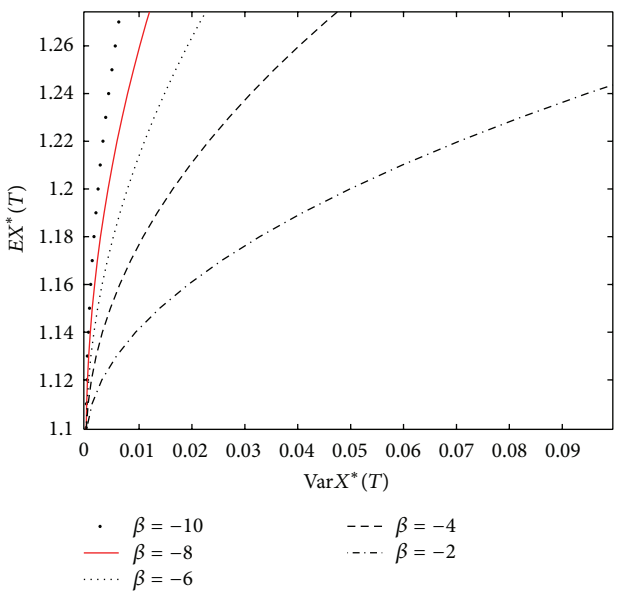

Figure 3: The efficient frontiers in mean-variance plane ($\beta = -10, -8, \ldots, -2$).

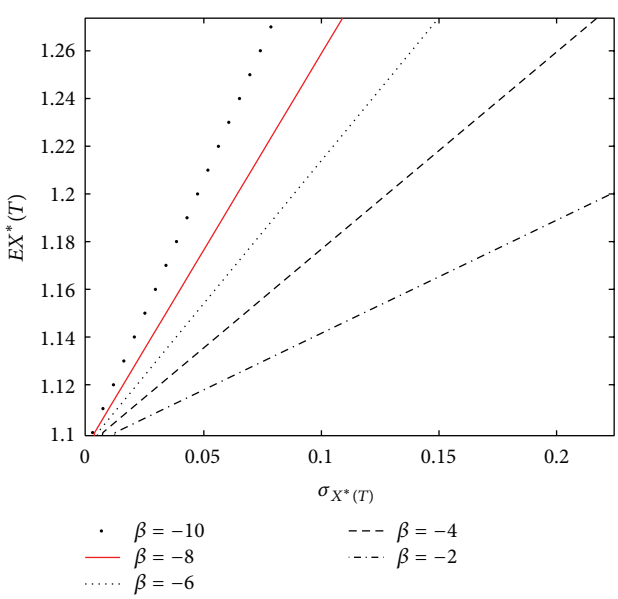

Figure 4: The efficient frontiers in mean-standard deviation plane ($\beta = -10, -8, \ldots, -2$).

endowment $X_0 = HK\$1$ million and wishing to achieve an expected return of 20% in three years; that is, $EX^*(T) = HK\$1.2$ million.

5.1. Sensitivity Analysis of the Efficient Frontier. From the efficient frontier in Theorem 10, it is hard to find what role the elasticity coefficient β plays. Thus, we try to examine the relations between the efficient frontier and the elasticity coefficient β. Now, we consider the following five cases: $\beta = -1.05$, $\beta = -1.00$, $\beta = -0.95$, $\beta = -0.90$, and $\beta = -0.85$.

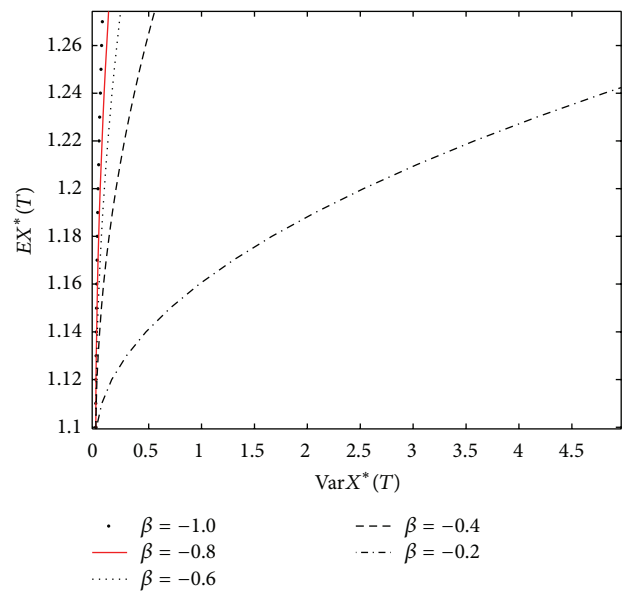

Figure 5: The efficient frontiers in mean-variance plane ($\beta = -1.0, \ldots, -0.2$).

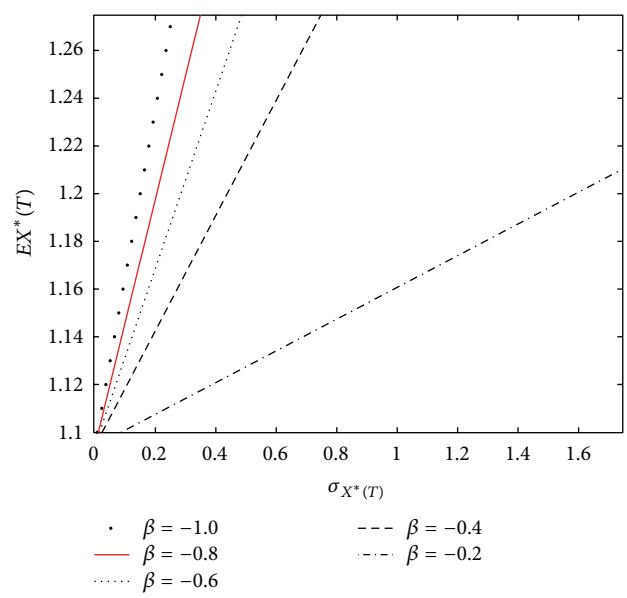

Figure 6: The efficient frontiers in mean-standard deviation plane ($\beta = -1.0, \ldots, -0.2$).

Then, the efficient frontiers under the five cases are given, respectively, by

$$\begin{aligned}
\mathrm{Var}\, X^*(T) &= 0.1960\left(EX^*(T) - 1.0942\right)^2, && \beta = -1.05;\\
\mathrm{Var}\, X^*(T) &= 0.2909\left(EX^*(T) - 1.0942\right)^2, && \beta = -1.00;\\
\mathrm{Var}\, X^*(T) &= 0.3697\left(EX^*(T) - 1.0942\right)^2, && \beta = -0.95;\\
\mathrm{Var}\, X^*(T) &= 0.4303\left(EX^*(T) - 1.0942\right)^2, && \beta = -0.90;\\
\mathrm{Var}\, X^*(T) &= 0.4749\left(EX^*(T) - 1.0942\right)^2, && \beta = -0.85.
\end{aligned} \tag{76}$$

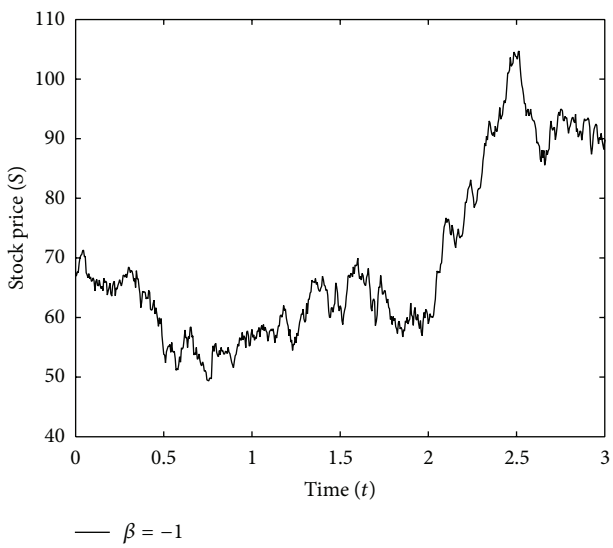

Figure 7: The evolution of stock price over time ($\beta = -1$).

Figures 1 and 2 present the efficient frontiers in mean-variance plane and mean-standard deviation plane, respectively.

Similarly, the efficient frontiers are plotted in Figures 3 and 4 for the case with $k = 100$, $S_0 = 1$, and $\beta = -10, -8, \ldots, -2$ and in Figures 5 and 6 for another case with $k = 1$, $S_0 = 4$, and $\beta = -1.0, -0.8, \ldots, -0.2$.

Clearly, efficient frontiers in mean-variance plane are increasing and concave with respect to the variance of the terminal wealth. Moreover, from Figures 1, 3, and 5, we can also observe that the expected return decreases as the elasticity coefficient increases under the same risk level or the risk increases as the elasticity coefficient increases under the same mean profit level. Figures 2, 4, and 6 show that the efficient frontier in mean-standard deviation is a straight line and the price of risk (which is the slope of this line) increases as the elasticity coefficient β decreases.

5.2. The Optimal Strategy and the Total Wealth. According to Theorem 10, we carry out 600 simulations to get the evolutions of the stock price, the optimal strategy, and total wealth. Figure 7 presents the evolution of stock price.

Putting the basic parameters and $T = 3$ in (37) and (38), respectively, and taking $t = 0$, we get

$$\begin{aligned} A(0) &= e^{-0.03\times 3} \\ &\quad \times \left| \frac{\cos\left(\arctan\left(-2/\sqrt{14}\right)\right)}{\cos\left(0.03\times 3\times\sqrt{14}+\arctan\left(-2/\sqrt{14}\right)\right)} \right|^{1/2} \\ &= 0.8634, \end{aligned}$$

$$\begin{aligned} &B(0)\, S_0^{-2\beta} \\ &= -\frac{67^2}{2\times 16.16^2} \\ &\quad \times \Bigg[-0.06 - 0.03 \\ &\qquad \times \sqrt{14}\tan\left(0.03\times 3\times\sqrt{14}+\arctan\frac{-2}{\sqrt{14}}\right)\Bigg] \\ &= 0.3658. \end{aligned} \tag{77}$$

Thus, $f_0 = A(0)e^{B(0)S_0^{-2\beta}} = 1.2447$ and the price of risk is $K = 0.7002$. From (52), we get that the standard deviation of the investor's goal is $\sigma_{X^*(T)} = \sqrt{\operatorname{Var} X^*(T)} = HK\$0.1511$ million, which implies that the minimized standard deviation is as high as 15.11%.

Then, we get the investor's optimal portfolio. By the definition of γ, we obtain $\gamma = 1.4159$. Thus, the amount that investor should invest in the stock is

$$\begin{aligned} &\pi^*(t) \\ &= \frac{1}{16.16^2}\Bigg\{0.03 - 0.03 \\ &\qquad \times \sqrt{14}\tan\left[0.03\times\sqrt{14}\,(3-t)+\arctan\frac{-2}{\sqrt{14}}\right]\Bigg\} \\ &\quad \times \left[1.4159e^{-0.03(3-t)} - X(t)\right] S^2(t), \end{aligned} \tag{78}$$

which is a function of time t, stock price $S(t)$, and the wealth $X(t)$. In particular, at the initial time $t = 0$, the investor should invest $\pi^*(0) = HK\$0.2398$ million in the stock and the rest of his initial endowment in the bond. The dynamic behaviors of the optimal portfolio strategy and total wealth are plotted in Figure 8.

Similarly, the evolutions of the stock price, the optimal strategy, and total wealth are plotted in Figures 9 and 10 for the case with $k = 100$, $S_0 = 1$, and $\beta = -2$ and in Figures 11 and 12 for the case with $k = 1$, $S_0 = 4$, and $\beta = -0.2$.

6. Conclusion

In this paper, we employ the CEV process to describe the dynamic evolution of the stock price in the mean-variance portfolio selection problem. The results in this paper show that the mean-variance efficient frontier is still a parabola in the mean-variance plane and the optimal strategies are not independent of the stock price. Moreover, under the same risk level, the expected return decreases as the elasticity coefficient increases.

Our results show differences after comparing with some well-known results. Firstly, in most of the papers (see, e.g., [7–9, 12]) where stock price is described by GBM, the optimal mean-variance portfolio is independent of stock price, while our optimal investment strategy depends on the stock price,

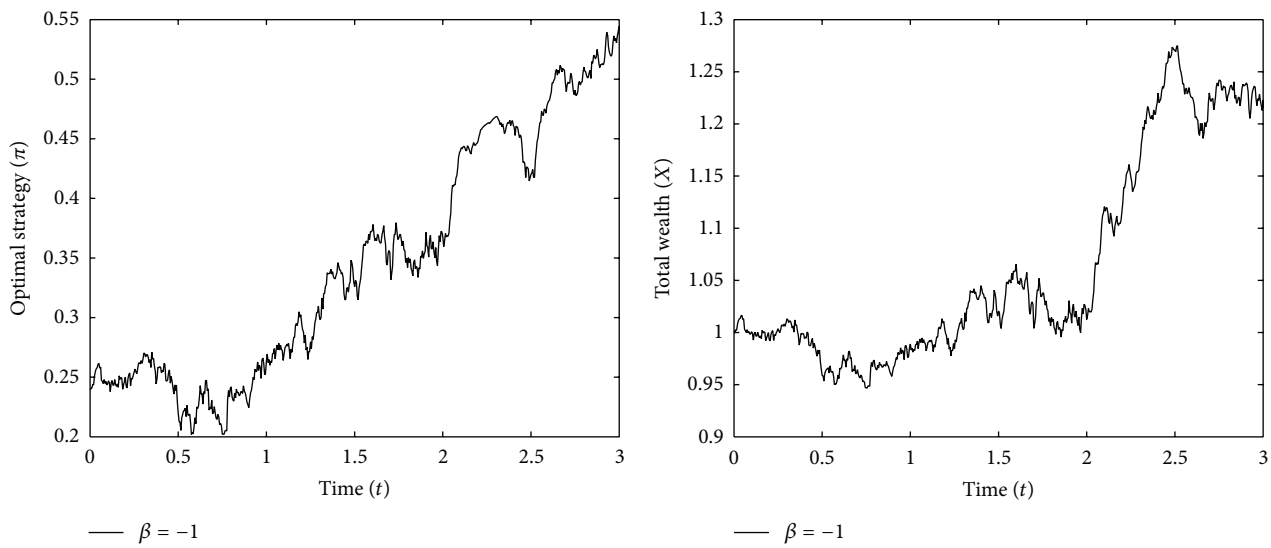

FIGURE 8: The evolutions of optimal strategy and total wealth over time ($\beta = -1$).

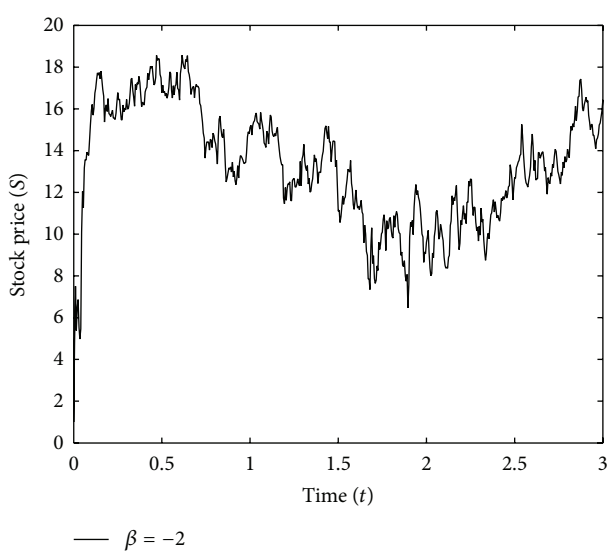

FIGURE 9: The evolution of stock price over time ($\beta = -2$).

which is caused by the fact that the volatility of CEV price process depends on stock price. Secondly, although there are many discussions in mean-variance model with stochastic volatility (see, e.g., [10, 11]), the optimal portfolio is given in terms of the solutions of backward stochastic differential equations, which is difficult to apply in practice. However, our results are completely determined and therefore can be easily applied in practice.

Appendix

Proof Details for Solving (35) and (36)

For the sake of simplicity, we define

$$a = -2\beta^2, \qquad b = 2(2r - u)\beta, \qquad c = -(u - r)^2. \quad \text{(A.1)}$$

Then (35) can be rewritten as

$$\frac{dB(t)}{dt} = ak^2 B^2(t) + bB(t) + \frac{c}{k^2}, \quad \text{with } B(T) = 0. \quad \text{(A.2)}$$

Integrating (A.2) on both sides with respect to the time t, we have

$$\int \frac{1}{ak^2 B^2(t) + bB(t) + (c/k^2)} dB(t) = t + C, \quad \text{(A.3)}$$

where C is a constant.

When $\sqrt{2}r > u > r$, we have

$$\Delta := b^2 - 4ak^2 \frac{c}{k^2} = 4\beta^2 \left(2r^2 - u^2\right) > 0 \quad \text{(A.4)}$$

and so the quadratic equation

$$ak^2 m^2 + bm + \frac{c}{k^2} = 0 \quad \text{(A.5)}$$

has two different real solutions; namely,

$$m_{1,2} = \frac{2r - u \pm \sqrt{2r^2 - u^2}}{2\beta k^2}. \quad \text{(A.6)}$$

Then, we get

$$\int \frac{1}{ak^2 B^2(t) + bB(t) + (c/k^2)} dB(t) = \frac{1}{ak^2 (m_1 - m_2)} \int \frac{1}{B(t) - m_1} - \frac{1}{B(t) - m_2} dB(t). \quad \text{(A.7)}$$

Putting (A.7) in (A.3) and taking into account the boundary condition, we get

$$B(t) = k^{-2} I(t), \quad \text{(A.8)}$$

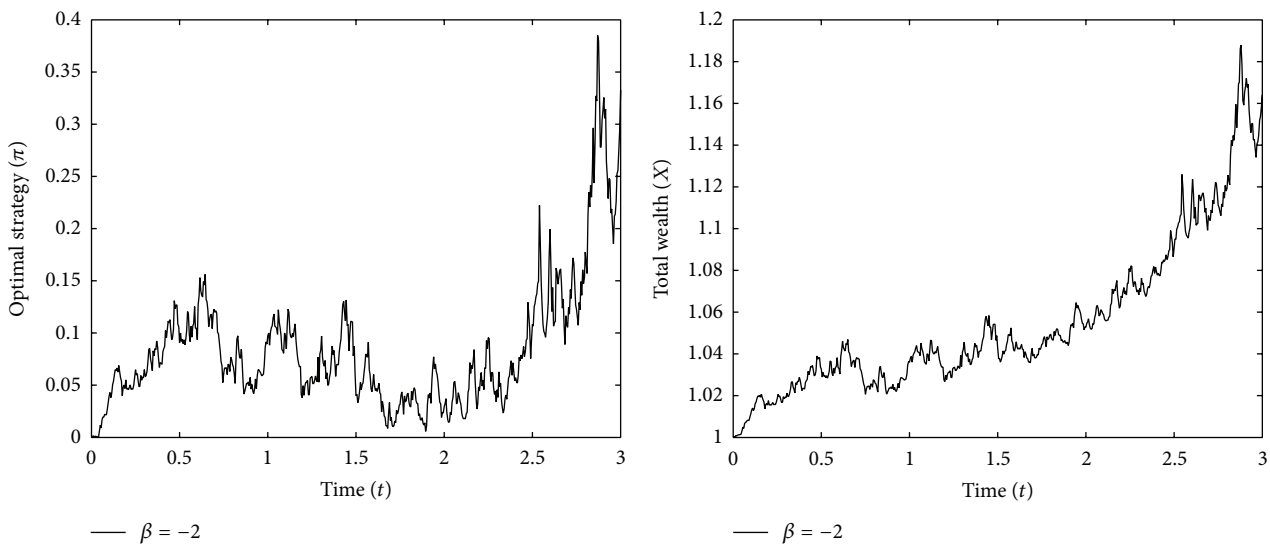

Figure 10: The evolutions of optimal strategy and total wealth over time ($\beta = -2$).

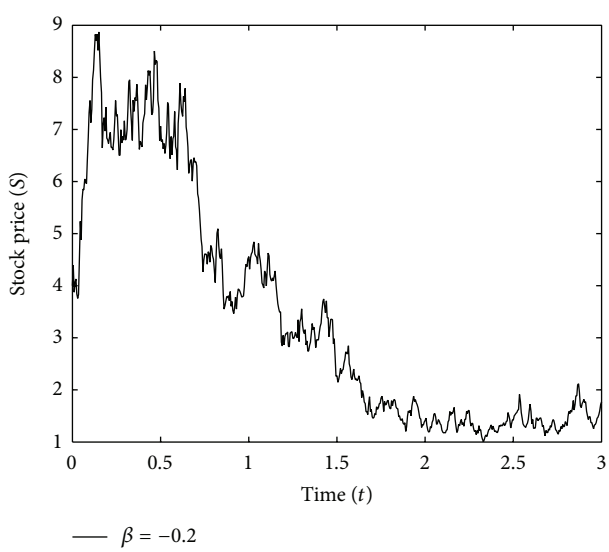

Figure 11: The evolution of stock price over time ($\beta = -0.2$).

where

$$I(t) = \frac{\lambda_1 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}{1 - (\lambda_1/\lambda_2)\, e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}, \tag{A.9}$$

$$\lambda_{1,2} = \frac{(2r-u) \pm \sqrt{2r^2 - u^2}}{2\beta}. \tag{A.10}$$

When $u = \sqrt{2}r$, we have

$$\Delta := b^2 - 4ak^2 \frac{c}{k^2} = 4\beta^2 \left(2r^2 - u^2\right) = 0 \tag{A.11}$$

and so (A.5) has a unique real solution:

$$m = -\frac{b}{2ak^2} = \frac{2r-u}{2\beta k^2}. \tag{A.12}$$

Then, we get

$$\int \frac{1}{ak^2B^2(t) + bB(t) + (c/k^2)} dB(t) = \frac{1}{ak^2} \int \frac{1}{(B(t)-m)^2} dB(t). \tag{A.13}$$

Putting (A.13) in (A.3) and taking into account the boundary condition, we get

$$B(t) = \frac{(2r-u)^2 (T-t)}{2k^2 \left[1 + (2r-u)\beta(T-t)\right]}. \tag{A.14}$$

When $u > \sqrt{2}r$, we have

$$\Delta := b^2 - 4ak^2 \frac{c}{k^2} = 4\beta^2 \left(2r^2 - u^2\right) < 0 \tag{A.15}$$

and so (A.5) has no real solution. Thus, it follows that

$$\int \frac{1}{ak^2B^2(t) + bB(t) + (c/k^2)} dB(t) = \frac{1}{ak^2} \int \frac{1}{\left(B(t) + (b/2ak^2)\right)^2 + \left((4ac - b^2)/4a^2k^4\right)} dB(t). \tag{A.16}$$

Putting (A.16) in (A.3) and taking into account the boundary condition, we get

$$B(t) = \left(2\beta k^2\right)^{-1} \times \left\{2r - u - \sqrt{u^2 - 2r^2} \times \tan\left[-\beta\sqrt{u^2-2r^2}(T-t) + \arctan\frac{2r-u}{\sqrt{u^2-2r^2}}\right]\right\}. \tag{A.17}$$

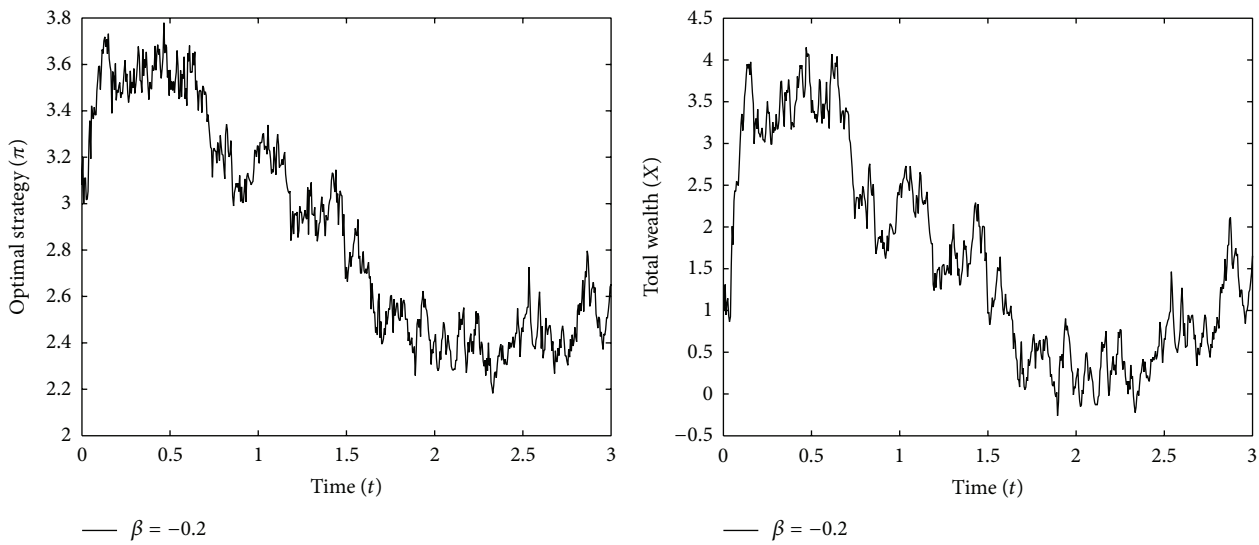

Figure 12: The evolutions of optimal strategy and total wealth over time ($\beta = -0.2$).

Putting (A.8), (A.14), and (A.17) in (36), respectively, together with the boundary condition $A(T) = 1$, we have

$$A(t) = \begin{cases} e^{[\lambda_1\beta(2\beta+1)-2r](T-t)} \\ \quad \times \left[\dfrac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1 e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}\right]^{(2\beta+1)/2\beta}, \\ \qquad \sqrt{2}r > u > r, \\ e^{-[2r-((2\beta+1)(2r-u)/2)](T-t)} \\ \quad \times \left[1 + (2r-u)\beta(T-t)\right]^{-(2\beta+1)/2\beta}, \\ u = \sqrt{2}r, \\ e^{[-2r+((2\beta+1)/2)(2r-u)](T-t)} \\ \quad \times \left| \left(\cos\left(\arctan\dfrac{2r-u}{\sqrt{u^2-2r^2}}\right)\right) \right. \\ \qquad \times \left(\cos\left[-\beta\sqrt{u^2-2r^2}\,(T-t) \right.\right. \\ \qquad\qquad \left.\left. + \arctan\dfrac{2r-u}{\sqrt{u^2-2r^2}}\right]\right)^{-1} \Bigg|^{(2\beta+1)/2\beta}, \\ u > \sqrt{2}r, \end{cases} \tag{A.18}$$

where λ_1 and λ_2 are given by (A.10).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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