Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 375251, 12 pages
http://dx.doi.org/10.1155/2014/375251

Research Article

Circulant Type Matrices with the Sum and Product of Fibonacci and Lucas Numbers

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Received 28 April 2014; Accepted 4 June 2014; Published 19 June 2014

Academic Editor: Tongxing Li

Circulant type matrices have become an important tool in solving differential equations. In this paper, we consider circulant type matrices, including the circulant and left circulant and g-circulant matrices with the sum and product of Fibonacci and Lucas numbers. Firstly, we discuss the invertibility of the circulant matrix and present the determinant and the inverse matrix by constructing the transformation matrices. Furthermore, the invertibility of the left circulant and g-circulant matrices is also discussed. We obtain the determinants and the inverse matrices of the left circulant and g-circulant matrices by utilizing the relation between left circulant, and g-circulant matrices and circulant matrix, respectively.

1. Introduction

Circulant matrices may play a crucial role for solving various differential equations. In [1], Ruiz-Claeyssen and dos Santos Leal introduced factor circulant matrices: matrices with the structure of circulants, but with the entries below the diagonal being multiplied by the same factor. The diagonalization of a circulant matrix and the spectral decomposition are conveniently generalized to block matrices with the structure of factor circulants. Matrix and partial differential equations involving factor circulants are considered. Wu and Zou in [2] discussed the existence and approximation of solutions of asymptotic or periodic boundary value problems of mixed functional differential equations. They focused on (5.13) in [2] with a circulant matrix, whose principal diagonal entries are zeroes. In [3], some Routh-Hurwitz stability conditions are generalized to the fractional order case. The authors considered the 1-system CML (10). They selected a circulant matrix, which reads a tridiagonal matrix. Ahmed and Elgazzar used coupled map lattices (CML) as an alternative approach to include spatial effects in fractional order systems (FOS). Consider the 1-system CML (10) in [4]. They claimed that the system is stable if all the eigenvalues of the circulant matrix satisfy (2) in [4]. Trench considered nonautonomous systems of linear differential equations (1) in [5] with some constraint on the coefficient matrix $A(t)$. One case is that $A(t)$ is a variable block circulant matrix. Kloeden et al. adopted the simplest approximation schemes for (1) in [6] with the Euler method, which reads (5) in [6]. They exploited that the covariance matrix of the increments can be embedded in a circulant matrix. The total loops can be done by fast Fourier transformation, which leads to a total computational cost of $O(m\log m) = O(n\log n)$. Guo et al. concerned on generic Dn-Hopf bifurcation to a delayed Hopfield-Cohen-Grossberg model of neural networks (5.17) in [7], where T denoted an interconnection matrix. They especially assumed T is a symmetric circulant matrix. Lin and Yang discretized the partial integrodifferential equation (PIDE) in pricing options with the preconditioned conjugate gradient (PCG) method, which constructed the circulant preconditioners. By using FFT, the cost for each linear system is $O(n\log n)$, where n is the size of the system in [8]. Lee et al. investigated a high-order compact (HOC) scheme for the general two-dimensional (2D) linear partial differential equation (1.1) in [9] with a mixed derivative. Meanwhile, in order to establish the 2D combined compact difference (CCD2) scheme, they rewrote (1.1) in [9] into (2.1) in [9]. To write the CCD2 system in a concise style, they employed circulant matrix to obtain

the corresponding whole CCD2 linear system (2.10) in [9], whose entries are circulant block.

Circulant type matrices have important applications in various disciplines including image processing, communications, signal processing, encoding, solving Toeplitz matrix problems, and least squares problems. They have been put on firm basis with the work of Davis [10], Jiang and Zhou [11], and Gray [12].

In [13], the authors pointed out the processes based on the eigenvalue of circulant type matrices with i.i.d. entries. There are discussions about the convergence in probability and in distribution of the spectral norm of circulant type matrices in [14]. The g-circulant matrices play an important role in various applications as well. For details, please refer to [15, 16] and the references therein. Ngondiep et al. showed the singular values of g-circulants in [17]. In [18, 19], the authors gave the limiting spectral distributions of left circulant matrices.

The Fibonacci and Lucas sequences are defined by the following recurrence relations [20, 21], respectively:

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n \quad \text{where } F_0 = 0,\ F_1 = 1, \\ L_{n+2} &= L_{n+1} + L_n \quad \text{where } L_0 = 2,\ L_1 = 1. \end{aligned} \tag{1}$$

For $n \geq 0$, the first few values of the sequences are given by the following equation:

n	0	1	2	3	4	5	6	7	8	$\cdots$
F_n	0	1	1	2	3	5	8	13	21	$\cdots$
L_n	2	1	3	4	7	11	18	29	47	$\cdots$

(2)

Let α, β be the roots of characteristic equation $x^2 - x - 1 = 0$; then the Binet formulas of the sequences $\{F_n\}$ and $\{L_n\}$ have the form

$$\begin{aligned} F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \\ \alpha &= \frac{1 + \sqrt{5}}{2}, \qquad \beta = \frac{1 - \sqrt{5}}{2}. \end{aligned} \tag{3}$$

Let $\mathscr{F}_n = F_n \cdot L_n$ and $\mathscr{L}_n = F_n + L_n$, so we can get two new sequences $\mathscr{F}_n$ and $\mathscr{L}_n$ [22]. The two sequences are defined by the following recurrence relations, respectively:

$$\begin{aligned} \mathscr{F}_{n+2} &= 3\mathscr{F}_{n+1} + \mathscr{F}_n, \quad \text{where } \mathscr{F}_0 = 0,\ \mathscr{F}_1 = 1, \\ \mathscr{L}_{n+2} &= \mathscr{L}_{n+1} + \mathscr{L}_n, \quad \text{where } \mathscr{L}_0 = 2,\ \mathscr{L}_1 = 2. \end{aligned} \tag{4}$$

For $n \geq 0$, the first few values of the sequences are given by the following equation:

n	0	1	2	3	4	5	6	7	8	$\cdots$
$\mathscr{F}_n$	0	1	3	8	21	55	144	377	987	$\cdots$
$\mathscr{L}_n$	2	2	4	6	10	16	26	42	68	$\cdots$

(5)

The $\mathscr{F}_n$ is given by the formula $\mathscr{F}_n = (\alpha_1^n - \beta_1^n)/(\alpha_1 - \beta_1)$, where α_1, β_1 are the roots of $x^2 - 3x + 1 = 0$. $\mathscr{L}_n$ is given by the formula $\mathscr{L}_n = F_n + L_n = (\alpha^n - \beta^n)/(\alpha - \beta) + (\alpha^n + \beta^n)$, where α, β are the roots of $x^2 - x - 1 = 0$.

Besides, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices [10, 11]. Unfortunately, the computational complexities of these algorithms are very amazing with the order of matrix increasing. However, some authors gave the explicit determinants and inverse of circulant and skew-circulant involving Fibonacci and Lucas numbers. For example, Dazheng gave the determinant of the Fibonacci-Lucas quasicyclic matrices in [20]. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses by constructing the transformation matrices [21]. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [23]. Lind presented the determinants of circulant and skew-circulant involving Fibonacci numbers [24]. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers [25].

In [22], the authors gave some determinantal and permanental representations of $\mathscr{F}_n$ and $\mathscr{L}_n$ and complex factorization formulas. The purpose of this paper is to obtain the explicit determinants and inverse of circulant type matrices by some perfect properties of $\mathscr{F}_n$ and $\mathscr{L}_n$.

In this paper, we adopt the following two conventions $0^0 = 1$, and for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ in the case $i > n$.

Definition 1 (see [10, 11]). In a circulant matrix (or right circulant matrix [26])

$$\operatorname{Circ}(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}, \tag{6}$$

each row is a cyclic shift of the row above to the right.

Circulant matrix is a special case of a Toeplitz matrix. It is evidently determined by its first row (or column).

Definition 2 (see [11, 26]). In a left circulant matrix (or reverse circulant matrix [13, 14, 18, 19])

$$\operatorname{LCirc}(a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & & \vdots \\ a_n & a_1 & \cdots & a_{n-1} \end{bmatrix}, \tag{7}$$

each row is a cyclic shift of the row above to the left.

Left circulant matrix is a special Hankel matrix.

Definition 3 (see [14, 27]). A g-circulant matrix is an $n \times n$ complex matrix with the following form:

$$A_{g,n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix}, \tag{8}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n.

The first row of $A_{g,n}$ is $(a_1, a_2, \ldots, a_n)$; its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions (equivalently, g mod n positions). Note that $g = 1$ or $g = n+1$ yields the standard *circulant matrix*. If $g = n-1$, then we obtain the *left circulant matrix*.

Lemma 4 (see [21]). *Let $A = Circ(a_1, a_2, \ldots, a_n)$ be a circulant matrix; then one has*

(i) *A is invertible if and only if $f(\omega^k) \neq 0$, $(k = 0, 1, 2, \ldots, n-1)$, where $f(x) = \sum_{j=1}^{n} a_j x^{j-1}$ and $\omega = \exp(2\pi i/n)$;*

(ii) *If A is invertible, then the inverse A^{-1} of A is a circulant matrix.*

Lemma 5. *Define*

$$\Delta := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}; \tag{9}$$

the matrix Δ is an orthogonal cyclic shift matrix (and a left circulant matrix). It holds that $LCirc(a_1, a_2, \ldots, a_n) = \Delta Circ(a_1, a_2, \ldots, a_n)$.

Lemma 6 (see [27]). *The $n \times n$ matrix $\mathbb{Q}_g$ is unitary if and only if $(n, g) = 1$, where $\mathbb{Q}_g$ is a g-circulant matrix with the first row $e^* = [1, 0, \ldots, 0]$.*

Lemma 7 (see [27]). *$A_{g,n}$ is a g-circulant matrix with the first row $[a_1, a_2, \ldots, a_n]$ if and only if $A_{g,n} = \mathbb{Q}_g C$, where $C = Circ(a_1, a_2, \ldots, a_n)$.*

2. Determinant and Inverse of a Circulant Matrix with the Product of the Fibonacci and Lucas Numbers

In this section, let $\mathscr{A}_n = \mathrm{Circ}(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)$ be a circulant matrix. Firstly, we give the determinant equation of matrix $\mathscr{A}_n$. Afterwards, we prove that $\mathscr{A}_n$ is an invertible matrix for $n > 2$, and then we find the inverse of the matrix $\mathscr{A}_n$.

Theorem 8. *Let $\mathscr{A}_n = Circ(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)$ be a circulant matrix; then one has*

$$\det \mathscr{A}_n = \left(1 - \mathscr{F}_{n+1}\right)^{n-1} + \left(-\mathscr{F}_n\right)^{n-2} \sum_{k=1}^{n-1} \left(-\mathscr{F}_k\right) \left(\frac{1 - \mathscr{F}_{n+1}}{-\mathscr{F}_n}\right)^{k-1}, \tag{10}$$

where $\mathscr{F}_n$ is the nth $F_n \cdot L_n$ number.

Proof. Obviously, $\det \mathscr{A}_1 = 1$ satisfies (10). In the case $n > 1$, let

$$\Gamma = \begin{pmatrix} 1 & & & & & & & \\ -3 & & & & & & 1 & \\ 1 & & & & & 1 & -3 & \\ 0 & & & 0 & 1 & -3 & 1 & \\ \vdots & & & \iddots & \iddots & \iddots & & \\ 0 & & & 1 & \iddots & \iddots & & \\ 0 & & 1 & -3 & \iddots & & 0 & \\ 0 & 1 & -3 & 1 & & & & \end{pmatrix}_{n \times n},$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\dfrac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}}\right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\dfrac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}}\right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dfrac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}. \tag{11}$$

We can obtain

$$\Gamma \mathscr{A}_n \Pi_1 = \begin{pmatrix} \mathscr{F}_1 & f'_n & \mathscr{F}_{n-1} & \mathscr{F}_{n-2} & \cdots & \mathscr{F}_2 \\ 0 & f_n & -\mathscr{F}_{n-2} & -\mathscr{F}_{n-3} & \cdots & -\mathscr{F}_1 \\ 0 & 0 & \mathscr{F}_1 - \mathscr{F}_{n+1} & & & \\ 0 & 0 & \mathscr{F}_n & \mathscr{F}_1 - \mathscr{F}_{n+1} & & \\ \vdots & \vdots & & \mathscr{F}_n & & \\ 0 & 0 & & 0 & & \mathscr{F}_1 - \mathscr{F}_{n+1} \end{pmatrix}, \tag{12}$$

where

$$f_n = \mathscr{F}_1 - 3\mathscr{F}_n + \sum_{k=1}^{n-2} \left(-\mathscr{F}_k\right) \left(\frac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}}\right)^{n-(k+1)},$$
$$f'_n = \sum_{k=1}^{n-1} \mathscr{F}_{k+1} \left(\frac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}}\right)^{n-(k+1)}. \tag{13}$$

We obtain

$$\begin{aligned} &\det \Gamma \det \mathscr{A}_n \det \Pi_1 \\ &= \mathscr{F}_1 \left[\mathscr{F}_1 - 3\mathscr{F}_n + \sum_{k=1}^{n-2} \left(-\mathscr{F}_k\right) \left(\frac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}}\right)^{n-(k+1)}\right] \\ &\quad \times \left(\mathscr{F}_1 - \mathscr{F}_{n+1}\right)^{n-2} \\ &= \mathscr{F}_1 \left[\mathscr{F}_1 - \mathscr{F}_{n+1} + \sum_{k=1}^{n-1} \left(-\mathscr{F}_k\right) \left(\frac{-\mathscr{F}_n}{\mathscr{F}_1 - \mathscr{F}_{n+1}}\right)^{n-(k+1)}\right] \end{aligned}$$

$$\begin{aligned}&\times\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{n-2}\\&=\left(1-\mathcal{F}_{n+1}\right)^{n-1}\\&\quad+\left(-\mathcal{F}_n\right)^{n-2}\sum_{k=1}^{n-1}\left(-\mathcal{F}_k\right)\left(\frac{1-\mathcal{F}_{n+1}}{-\mathcal{F}_n}\right)^{k-1},\end{aligned}\tag{14}$$

while

$$\det\Gamma=\det\Pi_1=(-1)^{(n-1)(n-2)/2},\tag{15}$$

we have

$$\begin{aligned}\det\mathcal{A}_n&=\left(1-\mathcal{F}_{n+1}\right)^{n-1}\\&\quad+\left(-\mathcal{F}_n\right)^{n-2}\sum_{k=1}^{n-1}\left(-\mathcal{F}_k\right)\left(\frac{1-\mathcal{F}_{n+1}}{-\mathcal{F}_n}\right)^{k-1}.\end{aligned}\tag{16}$$

Thus, the proof is completed. □

Theorem 9. *Let $\mathcal{A}_n = Circ(\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n)$ be a circulant matrix; if $n>2$, then $\mathcal{A}_n$ is an invertible matrix.*

Proof. When $n=3$, in Theorem 8, we have $\det\mathcal{A}_3=468\neq 0$; hence $\mathcal{A}_3$ is invertible. In the case $n>3$, since $\mathcal{F}_n=(\alpha_1^n-\beta_1^n)/(\alpha_1-\beta_1)$, where $\alpha_1+\beta_1=3$, $\alpha_1\cdot\beta_1=1$.We have

$$\begin{aligned}f\left(\omega^k\right)&=\sum_{j=1}^{n}\mathcal{F}_j\left(\omega^k\right)^{j-1}\\&=\frac{1}{\alpha_1-\beta_1}\sum_{j=1}^{n}\left(\alpha_1^j-\beta_1^j\right)\left(\omega^k\right)^{j-1}\\&=\frac{1}{\alpha_1-\beta_1}\left[\frac{\alpha_1\left(1-\alpha_1^n\right)}{1-\alpha_1\omega^k}-\frac{\beta_1\left(1-\beta_1^n\right)}{1-\beta_1\omega^k}\right]\\&=\frac{1}{\alpha_1-\beta_1}\left[\frac{\left(\alpha_1-\beta_1\right)-\left(\alpha_1^{n+1}-\beta_1^{n+1}\right)}{1-\left(\alpha_1+\beta_1\right)\omega^k+\alpha_1\beta_1\omega^{2k}}\right]\\&\quad+\frac{1}{\alpha_1-\beta_1}\left[\frac{\alpha_1\beta_1\left(\alpha_1^n-\beta_1^n\right)\omega^k}{1-\left(\alpha_1+\beta_1\right)\omega^k+\alpha_1\beta_1\omega^{2k}}\right]\\&=\frac{1-\mathcal{F}_{n+1}+\mathcal{F}_n\omega^k}{1-3\omega^k+\omega^{2k}}\quad(k=1,2,\ldots,n-1).\end{aligned}\tag{17}$$

If there exists ω^l $(l=1,2,\ldots,n-1)$ such that $f(\omega^l)=0$, we obtain $1-\mathcal{F}_{n+1}+\mathcal{F}_n\omega^l=0$ for $1-3\omega^l+\omega^{2l}\neq 0$; thus, $\omega^l=(\mathcal{F}_{n+1}-1)/\mathcal{F}_n$ is a real number. While $\omega^l=\exp(2l\pi i/n)=\cos(2l\pi/n)+i\sin(2l\pi/n)$, hence, $\sin(2l\pi/n)=0$, so we have $\omega^l=-1$ for $0<2l\pi/n<2\pi$. But $x=-1$ is not the root of equation $1-\mathcal{F}_{n+1}+\mathcal{F}_n x=0$ $(n>3)$. We obtain $f(\omega^k)\neq 0$ for any ω^k $(k=1,2,\ldots,n-1)$, while $f(1)=\sum_{j=1}^{n}\mathcal{F}_j=\mathcal{F}_{n+1}-\mathcal{F}_n-1\neq 0$. By Lemma 4, the proof is completed. □

Lemma 10. *Let the matrix $\mathcal{G}=[g_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$g_{ij}=\begin{cases}\mathcal{F}_1-\mathcal{F}_{n+1}, & i=j,\\ \mathcal{F}_n, & i=j+1,\\ 0, & \text{otherwise},\end{cases}\tag{18}$$

and then the inverse $\mathcal{G}^{-1}=[g'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathcal{G}$ is equal to

$$g'_{i,j}=\begin{cases}\dfrac{\left(-\mathcal{F}_n\right)^{i-j}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i-j+1}}, & i\geq j,\\ 0, & i<j.\end{cases}\tag{19}$$

Proof. Let $c_{ij}=\sum_{k=1}^{n-2}g_{ik}g'_{kj}$. Obviously, $c_{i,j}=0$ for $i<j$. In the case $i=j$, we obtain $c_{ii}=g_{ii}g'_{ii}=(\mathcal{F}_1-\mathcal{F}_{n+1})\cdot(1/(\mathcal{F}_1-\mathcal{F}_{n+1}))=1$. For $i\geq j+1$, we obtain

$$\begin{aligned}c_{ij}&=\sum_{k=1}^{n-2}g_{ik}g'_{kj}=g_{i,i-1}g'_{i-1,j}+g_{i,i}g'_{i,j}\\&=\mathcal{F}_n\cdot\frac{\left(-\mathcal{F}_n\right)^{i-j-1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i-j}}\\&\quad+\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)\cdot\frac{\left(-\mathcal{F}_n\right)^{i-j}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i-j+1}}=0.\end{aligned}\tag{20}$$

We verify $\mathcal{G}\mathcal{G}^{-1}=I_{n-2}$, where I_{n-2} is the $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G}=I_{n-2}$. Thus, the proof is completed. □

Theorem 11. *Let $\mathcal{A}_n=Circ(\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n)$ $(n>2)$ be a circulant matrix; then one has*

$$\begin{aligned}\mathcal{A}_n^{-1}=\frac{1}{f_n}Circ\Bigg(&1-\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-i}\left(-\mathcal{F}_n\right)^{i-1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i}},\\&-3+\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}\left(-\mathcal{F}_n\right)^{i-1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i}},\frac{1}{\mathcal{F}_1-\mathcal{F}_{n+1}},\\&\frac{-\mathcal{F}_n}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^2},\frac{\left(-\mathcal{F}_n\right)^2}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^3},\ldots,\\&\frac{\left(-\mathcal{F}_n\right)^{n-3}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{n-2}}\Bigg),\end{aligned}\tag{21}$$

where

$$f_n=\mathcal{F}_1-3\mathcal{F}_n+\sum_{k=1}^{n-2}\left(-\mathcal{F}_k\right)\left(\frac{-\mathcal{F}_n}{\mathcal{F}_1-\mathcal{F}_{n+1}}\right)^{n-(k+1)}.\tag{22}$$

Proof. Let

$$\Pi_2 = \begin{pmatrix} 1 & -f_n' & -\frac{f_n'}{f_n}\mathcal{F}_{n-2} - \mathcal{F}_{n-1} & \cdots & -\frac{f_n'}{f_n}\mathcal{F}_1 \\ 0 & 1 & \frac{\mathcal{F}_{n-2}}{f_n} & \cdots & \frac{\mathcal{F}_1}{f_n} \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{23}$$

where

$$f_n = \mathcal{F}_1 - 3\mathcal{F}_n + \sum_{k=1}^{n-2}(-\mathcal{F}_k)\left(\frac{-\mathcal{F}_n}{\mathcal{F}_1 - \mathcal{F}_{n+1}}\right)^{n-(k+1)},$$
$$f_n' = \sum_{k=1}^{n-1}\mathcal{F}_{k+1}\left(\frac{-\mathcal{F}_n}{\mathcal{F}_1 - \mathcal{F}_{n+1}}\right)^{n-(k+1)}. \tag{24}$$

We have

$$\Gamma\mathcal{A}_n\Pi_1\Pi_2 = \mathcal{D}_1 \oplus \mathcal{G}, \tag{25}$$

where $\mathcal{D}_1 = \operatorname{diag}(\mathcal{F}_1, f_n)$ is a diagonal matrix and $\mathcal{D}_1 \oplus \mathcal{G}$ is the direct sum of $\mathcal{D}_1$ and $\mathcal{G}$. If we denote $\Pi = \Pi_1\Pi_2$, then we obtain

$$\mathcal{A}_n^{-1} = \Pi\left(\mathcal{D}_1^{-1} \oplus \mathcal{G}^{-1}\right)\Gamma. \tag{26}$$

Since the last row elements of the matrix Π are

$$0, 1, \frac{\mathcal{F}_{n-2}}{f_n}, \frac{\mathcal{F}_{n-3}}{f_n}, \ldots, \frac{\mathcal{F}_2}{f_n}, \frac{\mathcal{F}_1}{f_n}. \tag{27}$$

By Lemma 10, if we let $\mathcal{A}_n^{-1} = \operatorname{Circ}(x_1, x_2, \ldots, x_n)$, its last row elements are given by the following equations:

$$\begin{aligned}
x_2 &= -\frac{3}{f_n} + \frac{1}{f_n}\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i}, \\
x_3 &= \frac{\mathcal{F}_1}{f_n(\mathcal{F}_1 - \mathcal{F}_{n+1})}, \\
x_4 &= \frac{1}{f_n}\sum_{i=1}^{2}\frac{\mathcal{F}_{3-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} - \frac{3\mathcal{F}_1}{f_n(\mathcal{F}_1 - \mathcal{F}_{n+1})}, \\
x_5 &= \frac{1}{f_n}\sum_{i=1}^{3}\frac{\mathcal{F}_{4-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} - \frac{3}{f_n}\sum_{i=1}^{2}\frac{\mathcal{F}_{3-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad + \frac{\mathcal{F}_1}{f_n(\mathcal{F}_1 - \mathcal{F}_{n+1})}, \\
&\vdots \\
x_n &= \frac{1}{f_n}\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} - \frac{3}{f_n}\sum_{i=1}^{n-3}\frac{\mathcal{F}_{n-2-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad + \frac{1}{f_n}\sum_{i=1}^{n-4}\frac{\mathcal{F}_{n-3-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i}, \\
x_1 &= \frac{1}{f_n} - \frac{3}{f_n}\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad + \frac{1}{f_n}\sum_{i=1}^{n-3}\frac{\mathcal{F}_{n-2-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i}.
\end{aligned} \tag{28}$$

Let $C_n^{(j)} = \sum_{i=1}^{j}(\mathcal{F}_{j+1-i}(-\mathcal{F}_n)^{i-1}/(\mathcal{F}_1 - \mathcal{F}_{n+1})^i)$ ($j = 1, 2, \ldots, n-2$); we have

$$\begin{aligned}
&C_n^{(2)} - 3C_n^{(1)} \\
&\quad = -\frac{3\mathcal{F}_1}{\mathcal{F}_1 - \mathcal{F}_{n+1}} + \sum_{i=1}^{2}\frac{\mathcal{F}_{3-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad = \frac{-\mathcal{F}_n}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^2}, \\
&-3C_n^{(n-2)} + C_n^{(n-3)} \\
&\quad = -3\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} + \sum_{i=1}^{n-3}\frac{\mathcal{F}_{n-i-2}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad = \frac{(-3\mathcal{F}_1)(-\mathcal{F}_n)^{n-3}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^{n-2}} + \sum_{i=1}^{n-3}\frac{-\mathcal{F}_{n-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad = \sum_{i=1}^{n-2}\frac{-\mathcal{F}_{n-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i}, \\
&C_n^{(j+2)} - 3C_n^{(j+1)} + C_n^{(j)} \\
&\quad = \sum_{i=1}^{j+2}\frac{\mathcal{F}_{j+3-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} - 3\sum_{i=1}^{j+1}\frac{\mathcal{F}_{j+2-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\qquad + \sum_{i=1}^{j}\frac{\mathcal{F}_{j+1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^i} \\
&\quad = \frac{\mathcal{F}_2(-\mathcal{F}_n)^j}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^{j+1}} + \frac{\mathcal{F}_1(-\mathcal{F}_n)^{j+1}}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^{j+2}} \\
&\qquad - \frac{3\mathcal{F}_1(-\mathcal{F}_n)^j}{(\mathcal{F}_1 - \mathcal{F}_{n+1})^{j+1}}
\end{aligned}$$

$$+\sum_{i=1}^{j}\frac{\left(\mathcal{F}_{j+3-i}-3\mathcal{F}_{j+2-i}+\mathcal{F}_{j+1-i}\right)\left(-\mathcal{F}_n\right)^{i-1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{j+1}}$$
$$=\frac{\left(-\mathcal{F}_n\right)^{j+1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{j+2}}\quad\left(j=1,2,\ldots,n-4\right). \tag{29}$$

We obtain

$$\begin{aligned}\mathcal{A}_n^{-1}&=\operatorname{Circ}\left(\frac{1-3C_n^{(n-2)}+C_n^{(n-3)}}{f_n},\frac{C_n^{(n-2)}-3}{f_n},\right.\\&\qquad\frac{C_n^{(1)}}{f_n},\frac{C_n^{(2)}-3C_n^{(1)}}{f_n},\frac{C_n^{(3)}-3C_n^{(2)}+C_n^{(1)}}{f_n},\ldots,\\&\qquad\left.\frac{C_n^{(n-2)}-3C_n^{(n-3)}+C_n^{(n-4)}}{f_n}\right)\\&=\frac{1}{f_n}\operatorname{Circ}\left(1-\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-i}\left(-\mathcal{F}_n\right)^{i-1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i}},\right.\\&\qquad-3+\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}\left(-\mathcal{F}_n\right)^{i-1}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{i}},\frac{1}{\mathcal{F}_1-\mathcal{F}_{n+1}},\\&\qquad\frac{-\mathcal{F}_n}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^2},\frac{\left(-\mathcal{F}_n\right)^2}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^3},\ldots,\\&\qquad\left.\frac{\left(-\mathcal{F}_n\right)^{n-3}}{\left(\mathcal{F}_1-\mathcal{F}_{n+1}\right)^{n-2}}\right).\end{aligned} \tag{30}$$

□

3. Determinant and Inverse of a Circulant Matrix with the Sum of the Fibonacci and Lucas Numbers

In this section, let $\mathfrak{B}_n=\operatorname{Circ}(\mathfrak{L}_1,\mathfrak{L}_2,\ldots,\mathfrak{L}_n)$ be a circulant matrix. Firstly, we give an explicit determinant formula of matrix $\mathfrak{B}_n$. Afterwards, we prove that $\mathfrak{B}_n$ is an invertible matrix for any positive integer n, and then we find its inverse.

Theorem 12. *Let $\mathfrak{B}_n=\operatorname{Circ}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n)$ be a circulant matrix; then one has*

$$\begin{aligned}\det\mathfrak{B}_n=2\Bigg[&\left(2-\mathcal{L}_{n+1}\right)^{n-1}+\left(\mathcal{L}_n-2\right)^{n-2}\\&\times\sum_{k=1}^{n-1}\left(\mathcal{L}_{k+2}-2\mathcal{L}_{k+1}\right)\left(\frac{2-\mathcal{L}_{n+1}}{\mathcal{L}_n-2}\right)^{k-1}\Bigg],\end{aligned} \tag{31}$$

where $\mathcal{L}_n$ is the nth F_n+L_n number.

Proof. Obviously, $\mathfrak{B}_1=2$ satisfies (31), when $n>1$. Let

$$\Sigma=\begin{pmatrix}1&&&&&&&\\-2&&&&&&&1\\-1&&&&&&1&-1\\0&&&0&&1&-1&-1\\\vdots&&&&\iddots&\iddots&\iddots&\\0&&&1&\iddots&\iddots&&\\0&&1&-1&\iddots&&0&\\0&1&-1&-1&&&&\end{pmatrix}_{n\times n},$$
$$\Omega_1=\begin{pmatrix}1&0&0&\cdots&0&0\\0&\left(\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}\right)^{n-2}&0&\cdots&0&1\\0&\left(\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}\right)^{n-3}&0&\cdots&1&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}&1&\cdots&0&0\\0&1&0&\cdots&0&0\end{pmatrix}_{n\times n}. \tag{32}$$

Then

$$\begin{aligned}&\Sigma\mathfrak{B}_n\Omega_1\\&=\begin{pmatrix}\mathcal{L}_1&l_n'&\mathcal{L}_{n-1}&\cdots&\mathcal{L}_3&\mathcal{L}_2\\0&l_n&-2\mathcal{L}_{n-1}+\mathcal{L}_n&\cdots&\cdots&-2\mathcal{L}_2+\mathcal{L}_3\\0&0&\mathcal{L}_1-\mathcal{L}_{n+1}&&&\\0&0&2-\mathcal{L}_n&&&\\\vdots&\vdots&&\ddots&&\\0&0&&&2-\mathcal{L}_n&\mathcal{L}_1-\mathcal{L}_{n+1}\end{pmatrix},\end{aligned} \tag{33}$$

where

$$l_n=\mathcal{L}_1-2\mathcal{L}_n+\sum_{k=1}^{n-2}\left(\mathcal{L}_{k+2}-2\mathcal{L}_{k+1}\right)\left(\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}\right)^{n-(k+1)}, \tag{34}$$

$$l_n'=\sum_{k=1}^{n-1}\mathcal{L}_{k+1}\left(\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}\right)^{n-(k+1)}. \tag{35}$$

We can obtain

$$\begin{aligned}&\det\Sigma\det\mathfrak{B}_n\det\Omega_1\\&=\mathcal{L}_1\Bigg[\mathcal{L}_1-2\mathcal{L}_n\\&\qquad+\sum_{k=1}^{n-2}\left(\mathcal{L}_{k+2}-2\mathcal{L}_{k+1}\right)\left(\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}\right)^{n-(k+1)}\Bigg]\\&\quad\times\left(\mathcal{L}_1-\mathcal{L}_{n+1}\right)^{n-2}\end{aligned}$$

$$= \mathscr{L}_1 \left[\mathscr{L}_1 - \mathscr{L}_{n+1} + \sum_{k=1}^{n-1} \left(\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}\right) \left(\frac{\mathscr{L}_n - 2}{\mathscr{L}_1 - \mathscr{L}_{n+1}} \right)^{n-(k+1)} \right] \times \left(\mathscr{L}_1 - \mathscr{L}_{n+1}\right)^{n-2}$$

$$= 2 \left[\left(2 - \mathscr{L}_{n+1}\right)^{n-1} + \left(\mathscr{L}_n - 2\right)^{n-2} \times \sum_{k=1}^{n-1} \left(\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}\right) \left(\frac{2 - \mathscr{L}_{n+1}}{\mathscr{L}_n - 2} \right)^{k-1} \right], \tag{36}$$

while

$$\det \Sigma = \det \Omega_1 = (-1)^{(n-1)(n-2)/2}. \tag{37}$$

We have

$$\det \mathfrak{B}_n = 2 \left[\left(2 - \mathscr{L}_{n+1}\right)^{n-1} + \left(\mathscr{L}_n - 2\right)^{n-2} \times \sum_{k=1}^{n-1} \left(\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}\right) \left(\frac{2 - \mathscr{L}_{n+1}}{\mathscr{L}_n - 2} \right)^{k-1} \right]. \tag{38}$$

□

Theorem 13. *Let $\mathfrak{B}_n = Circ(\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n)$ be a circulant matrix; then $\mathfrak{B}_n$ is invertible for any positive integer n.*

Proof. Since $\mathscr{L}_n = (\alpha^n - \beta^n)/(\alpha - \beta) + \alpha^n + \beta^n$, where $\alpha + \beta = 1$, $\alpha \cdot \beta = -1$. We have

$$\begin{aligned} f\left(\omega^k\right) &= \sum_{j=1}^{n} \mathscr{L}_j \left(\omega^k\right)^{j-1} \\ &= \sum_{j=1}^{n} \left(\frac{\alpha^j - \beta^j}{\alpha - \beta} + \alpha^j + \beta^j \right) \left(\omega^k\right)^{j-1} \\ &= \sum_{j=1}^{n} \left(\frac{\alpha^j - \beta^j}{\alpha - \beta} \right) \left(\omega^k\right)^{j-1} + \sum_{j=1}^{n} \left(\alpha^j + \beta^j\right) \left(\omega^k\right)^{j-1} \\ &= \frac{1}{\alpha - \beta} \left[\frac{\alpha\left(1 - \alpha^n\right)}{1 - \alpha\omega^k} - \frac{\beta\left(1 - \beta^n\right)}{1 - \beta\omega^k} \right] + \frac{\alpha\left(1 - \alpha^n\right)}{1 - \alpha\omega^k} + \frac{\beta\left(1 - \beta^n\right)}{1 - \beta\omega^k} \\ &= \frac{1 - F_{n+1} - F_n\omega^k}{1 - \omega^k - \omega^{2k}} + \frac{1 - L_{n+1} + \left(2 - L_n\right)\omega^k}{1 - \omega^k - \omega^{2k}} \\ &= \frac{2 - \left(F_{n+1} + L_{n+1}\right) - \left(F_n + L_n - 2\right)\omega^k}{1 - \omega^k - \omega^{2k}} \\ &= \frac{2 - \mathscr{L}_{n+1} - \left(\mathscr{L}_n - 2\right)\omega^k}{1 - \omega^k - \omega^{2k}} \quad (k = 1, 2, \ldots, n-1). \end{aligned} \tag{39}$$

If there exist ω^l $(l = 1, 2, \ldots, n-1)$ such that $f(\omega^l) = 0$, we obtain $2 - \mathscr{L}_{n+1} - (\mathscr{L}_n - 2)\omega^l = 0$ for $1 - \omega^l - \omega^{2l} \neq 0$; $\omega^l = (2 - \mathscr{L}_{n+1})/(\mathscr{L}_n - 2)$ is a real number, while $\omega^l = \exp(2l\pi i/n) = \cos(2l\pi/n) + i\sin(2l\pi/n)$.

Hence, $\sin(2l\pi/n) = 0$, so we have $\omega^l = -1$ for $0 < 2l\pi/n < 2\pi$. But $x = -1$ is not the root of the equation $2 - \mathscr{L}_{n+1} - (\mathscr{L}_n - 2)x = 0$ for any positive integer n. We obtain $f(\omega^k) \neq 0$ for any ω^k $(k = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} \mathscr{L}_j = \mathscr{L}_{n+1} + \mathscr{L}_n - 4 \neq 0$. By Lemma 4, the proof is completed. □

Lemma 14. *Let matrix $\mathscr{H} = [h_{ij}]_{i,j=1}^{n-2}$ be of the form*

$$h_{ij} = \begin{cases} \mathscr{L}_1 - \mathscr{L}_{n+1}, & i = j, \\ 2 - \mathscr{L}_n, & i = j + 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{40}$$

and then inverse $\mathscr{H}^{-1} = [h'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathscr{H}$ is equal to

$$h'_{ij} = \begin{cases} \dfrac{\left(\mathscr{L}_n - 2\right)^{i-j}}{\left(\mathscr{L}_1 - \mathscr{L}_{n+1}\right)^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{41}$$

Proof. Let $r_{ij} = \sum_{k=1}^{n-2} h_{ik} h'_{kj}$. Obviously, $r_{ij} = 0$ for $i < j$. In the case $i = j$, we obtain

$$r_{ii} = h_{ii} h'_{ii} = \left(\mathscr{L}_1 - \mathscr{L}_{n+1}\right) \cdot \frac{1}{\mathscr{L}_1 - \mathscr{L}_{n+1}} = 1. \tag{42}$$

For $i \geq j + 1$, we obtain

$$\begin{aligned} r_{ij} &= \sum_{k=1}^{n-2} h_{ik} h'_{kj} = h_{i,i-1} h'_{i-1,j} + h_{ii} h'_{ij} \\ &= \left(2 - \mathscr{L}_n\right) \cdot \frac{\left(\mathscr{L}_n - 2\right)^{i-j-1}}{\left(\mathscr{L}_1 - \mathscr{L}_{n+1}\right)^{i-j}} + \left(\mathscr{L}_1 - \mathscr{L}_{n+1}\right) \cdot \frac{\left(\mathscr{L}_n - 2\right)^{i-j}}{\left(\mathscr{L}_1 - \mathscr{L}_{n+1}\right)^{i-j+1}} \\ &= 0. \end{aligned} \tag{43}$$

We verify $\mathscr{H}\mathscr{H}^{-1} = I_{n-2}$, where I_{n-2} is the $(n-2) \times (n-2)$ identity matrix. Similarly, we can verify $\mathscr{H}^{-1}\mathscr{H} = I_{n-2}$. Thus, the proof is completed. □

Theorem 15. *Let $\mathfrak{B}_n = Circ(\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n)$ be a circulant matrix; then one has*

$$\begin{aligned}\mathfrak{B}_n^{-1} = \frac{1}{l_n} Circ\Bigg(& 1 - \sum_{i=1}^{n-2} \frac{\mathscr{L}_{n-i-1}(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ & -2 - \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ & \frac{2}{\mathscr{L}_1 - \mathscr{L}_{n+1}}, \frac{2(\mathscr{L}_n - 2)}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^2}, \\ & \frac{2(\mathscr{L}_n - 2)^2}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^3}, \ldots, \frac{2(\mathscr{L}_n - 2)^{n-3}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{n-2}}\Bigg),\end{aligned} \tag{44}$$

where

$$\begin{aligned} l_n &= \mathscr{L}_1 - 2\mathscr{L}_n \\ &\quad + \sum_{k=1}^{n-2} (\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}) \left(\frac{\mathscr{L}_n - 2}{\mathscr{L}_1 - \mathscr{L}_{n+1}}\right)^{n-(k+1)}. \end{aligned} \tag{45}$$

Proof. Let

$$\Omega_2 = \begin{pmatrix} 1 & -\frac{l_n'}{2} & \omega_{13} & \omega_{14} & \cdots & \omega_{1n} \\ 0 & 1 & \omega_{23} & \omega_{24} & \cdots & \omega_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{46}$$

where

$$\begin{aligned} \omega_{1i} &= \frac{1}{2}\left[\frac{l_n'(\mathscr{L}_{n+3-i} - 2\mathscr{L}_{n+2-i})}{l_n} - \mathscr{L}_{n+2-i}\right], \\ \omega_{2i} &= \frac{2\mathscr{L}_{n+2-i} - \mathscr{L}_{n+3-i}}{l_n}, \quad i = 3, 4, \ldots, n, \\ l_n &= \mathscr{L}_1 - 2\mathscr{L}_n \\ &\quad + \sum_{k=1}^{n-2} (\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}) \left(\frac{\mathscr{L}_n - 2}{\mathscr{L}_1 - \mathscr{L}_{n+1}}\right)^{n-(k+1)}, \\ l_n' &= \sum_{k=1}^{n-1} \mathscr{L}_{k+1} \left(\frac{\mathscr{L}_n - 2}{\mathscr{L}_1 - \mathscr{L}_{n+1}}\right)^{n-(k+1)}. \end{aligned} \tag{47}$$

We have

$$\Sigma \mathfrak{B}_n \Omega_1 \Omega_2 = \mathscr{D}_2 \oplus \mathscr{H}, \tag{48}$$

where $\mathscr{D}_2 = \mathrm{diag}(\mathscr{L}_1, l_n)$ is a diagonal matrix and $\mathscr{D}_2 \oplus \mathscr{H}$ is the direct sum of $\mathscr{D}_2$ and $\mathscr{H}$. If we denote $\Omega = \Omega_1 \Omega_2$, then we obtain

$$\mathfrak{B}_n^{-1} = \Omega \left(\mathscr{D}_2^{-1} \oplus \mathscr{H}^{-1}\right) \Sigma. \tag{49}$$

Since the last row elements of the matrix Ω are

$$0, 1, \frac{2\mathscr{L}_{n-1} - \mathscr{L}_n}{l_n}, \frac{2\mathscr{L}_{n-2} - \mathscr{L}_{n-1}}{l_n}, \ldots, \frac{2\mathscr{L}_2 - \mathscr{L}_3}{l_n}. \tag{50}$$

By Lemma 14, if we let $\mathfrak{B}_n^{-1} = \mathrm{Circ}(y_1, y_2, \ldots, y_n)$, then its last row elements are given by the following equations:

$$\begin{aligned} y_2 &= -\frac{2}{l_n} - \frac{1}{l_n} \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ y_3 &= \frac{1}{l_n} \frac{2\mathscr{L}_2 - \mathscr{L}_3}{(\mathscr{L}_1 - \mathscr{L}_{n+1})}, \\ y_4 &= -\frac{1}{l_n} \frac{2\mathscr{L}_2 - \mathscr{L}_3}{(\mathscr{L}_1 - \mathscr{L}_{n+1})} \\ &\quad + \frac{1}{l_n} \sum_{i=1}^{2} \frac{(2\mathscr{L}_{4-i} - \mathscr{L}_{5-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ y_5 &= -\frac{1}{l_n} \frac{2\mathscr{L}_2 - \mathscr{L}_3}{(\mathscr{L}_1 - \mathscr{L}_{n+1})} \\ &\quad - \frac{1}{l_n} \sum_{i=1}^{2} \frac{(2\mathscr{L}_{4-i} - \mathscr{L}_{5-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i} \\ &\quad + \frac{1}{l_n} \sum_{i=1}^{3} \frac{(2\mathscr{L}_{5-i} - \mathscr{L}_{6-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ &\vdots \\ y_n &= \frac{1}{l_n} \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i} \\ &\quad - \frac{1}{l_n} \sum_{i=1}^{n-3} \frac{(2\mathscr{L}_{n-i-1} - \mathscr{L}_{n-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i} \\ &\quad - \frac{1}{l_n} \sum_{i=1}^{n-4} \frac{(2\mathscr{L}_{n-i-2} - \mathscr{L}_{n-i-1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ y_1 &= \frac{1}{l_n}\Bigg[1 - \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i} \\ &\quad - \sum_{i=1}^{n-3} \frac{(2\mathscr{L}_{n-i-1} - \mathscr{L}_{n-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}\Bigg]. \end{aligned} \tag{51}$$

Let $D_n^{(j)} = \sum_{i=1}^{j}((2\mathscr{L}_{j+2-i} - \mathscr{L}_{j+3-i})(\mathscr{L}_n - 2)^{i-1}/(\mathscr{L}_1 - \mathscr{L}_{n+1})^i)$ $(j = 1, 2, \ldots, n-2)$; we have

$$\begin{aligned} & D_n^{(2)} - D_n^{(1)} \\ &\quad = -\frac{2\mathscr{L}_2 - \mathscr{L}_3}{\mathscr{L}_1 - \mathscr{L}_{n+1}} + \sum_{i=1}^{2} \frac{(2\mathscr{L}_{4-i} - \mathscr{L}_{5-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i} \end{aligned}$$

$$= \frac{2(\mathscr{L}_n - 2)}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^2},$$

$$D_n^{(n-3)} + D_n^{(n-2)}$$

$$= \sum_{i=1}^{n-3} \frac{(2\mathscr{L}_{n-i-1} - \mathscr{L}_{n-i})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}$$

$$+ \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}$$

$$= \frac{(2\mathscr{L}_2 - \mathscr{L}_3)(\mathscr{L}_n - 2)^{n-3}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{n-2}} + \sum_{i=1}^{n-3} \frac{\mathscr{L}_{n-i-1}(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}$$

$$= \sum_{i=1}^{n-2} \frac{\mathscr{L}_{n-i-1}(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i},$$

$$D_n^{(j+2)} - D_n^{(j+1)} - D_n^{(j)}$$

$$= \sum_{i=1}^{j+2} \frac{(2\mathscr{L}_{j-i+4} - \mathscr{L}_{j-i+5})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}$$

$$- \sum_{i=1}^{j+1} \frac{(2\mathscr{L}_{j-i+3} - \mathscr{L}_{j-i+4})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}$$

$$- \sum_{i=1}^{j} \frac{(2\mathscr{L}_{j-i+2} - \mathscr{L}_{j-i+3})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}$$

$$= \frac{(2\mathscr{L}_2 - \mathscr{L}_3)(\mathscr{L}_n - 2)^{j+1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{j+2}}$$

$$+ \frac{(2\mathscr{L}_3 - \mathscr{L}_4)(\mathscr{L}_n - 2)^{j}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{j+1}} - \frac{(2\mathscr{L}_2 - \mathscr{L}_3)(\mathscr{L}_n - 2)^{j}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{j+1}}$$

$$= \frac{2(\mathscr{L}_n - 2)^{j+1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{j+2}} \quad (j = 1, 2, \ldots, n-4). \tag{52}$$

We obtain

$$\mathfrak{B}_n^{-1}$$

$$= \mathrm{Circ}\left(\frac{1 - D_n^{(n-2)} - D_n^{(n-3)}}{l_n}, \frac{-2 - D_n^{(n-2)}}{l_n}, \frac{D_n^{(1)}}{l_n}, \frac{D_n^{(2)} - D_n^{(1)}}{l_n}, \frac{D_n^{(3)} - D_n^{(2)} - D_n^{(1)}}{l_n}, \ldots, \frac{D_n^{(n-2)} - D_n^{(n-3)} - D_n^{(n-4)}}{l_n} \right)$$

$$= \frac{1}{l_n} \mathrm{Circ}\left(1 - \sum_{i=1}^{n-2} \frac{\mathscr{L}_{n-i-1}(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, -2 - \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \frac{2}{\mathscr{L}_1 - \mathscr{L}_{n+1}}, \frac{2(\mathscr{L}_n - 2)}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^2}, \frac{2(\mathscr{L}_n - 2)^2}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^3}, \ldots, \frac{2(\mathscr{L}_n - 2)^{n-3}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{n-2}} \right). \tag{53}$$

□

4. Determinant and Inverse of a Left Circulant Matrix with $\mathscr{F}_n$ and $\mathscr{L}_n$ Numbers

In this section, let $\mathscr{A}'_n = \mathrm{LCirc}(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)$ and $\mathfrak{B}'_n = \mathrm{LCirc}(\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_n)$ be left circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrix $\mathscr{A}'_n$ and $\mathfrak{B}'_n$. Afterwards, we prove that $\mathscr{A}'_n$ is an invertible matrix for $n > 2$ and $\mathfrak{B}'_n$ is an invertible matrix for any positive integer n. The inverses of the matrices $\mathscr{A}'_n$ and $\mathfrak{B}'_n$ are also presented.

According to Lemma 5 and Theorems 8, 9, and 11, we can obtain the following theorems.

Theorem 16. *Let $\mathscr{A}'_n = LCirc(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)$ be a left circulant matrix; then one has*

$$\det \mathscr{A}'_n = (-1)^{(n-1)(n-2)/2} \times \left[(1 + \mathscr{F}_{n+1})^{n-1} + (-\mathscr{F}_n)^{n-2} \sum_{k=1}^{n-1} (-\mathscr{F}_k) \left(\frac{1 + \mathscr{F}_{n+1}}{-\mathscr{F}_n} \right)^{k-1} \right], \tag{54}$$

where $\mathscr{F}_n$ is the nth $F_n \cdot L_n$ number.

Theorem 17. *Let $\mathscr{A}'_n = LCirc(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)$ be a left circulant matrix; if $n > 2$, then $\mathscr{A}'_n$ is an invertible matrix.*

Theorem 18. *Let $\mathscr{A}'_n = LCirc(\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n)$ $(n > 2)$ be a left circulant matrix; then one has*

$$\mathscr{A}'^{-1}_n = \frac{1}{f_n} LCirc\left(1 - \sum_{i=1}^{n-2} \frac{\mathscr{F}_{n-i}(-\mathscr{F}_n)^{i-1}}{(\mathscr{F}_1 - \mathscr{F}_{n+1})^i}, \frac{(-\mathscr{F}_n)^{n-3}}{(\mathscr{F}_1 - \mathscr{F}_{n+1})^{n-2}}, \ldots, \frac{(-\mathscr{F}_n)^2}{(\mathscr{F}_1 - \mathscr{F}_{n+1})^3}, \right.$$

$$\frac{-\mathcal{F}_n}{(\mathcal{F}_1-\mathcal{F}_{n+1})^2}, \frac{1}{\mathcal{F}_1-\mathcal{F}_{n+1}}, -3+\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1-\mathcal{F}_{n+1})^i}\Bigg), \tag{55}$$

where

$$f_n=\mathcal{F}_1-3\mathcal{F}_n+\sum_{k=1}^{n-2}(-\mathcal{F}_k)\left(\frac{-\mathcal{F}_n}{\mathcal{F}_1-\mathcal{F}_{n+1}}\right)^{n-(k+1)}. \tag{56}$$

By Lemma 5 and Theorems 12, 13, and 15, the following conclusions can be attained.

Theorem 19. *Let $\mathfrak{B}'_n=LCirc(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n)$ be a left circulant matrix; then one has*

$$\det\mathfrak{B}'_n=2(-1)^{(n-1)(n-2)/2}\times\left[(2-\mathcal{L}_{n+1})^{n-1}+(\mathcal{L}_n-2)^{n-2}\times\sum_{k=1}^{n-1}(\mathcal{L}_{k+2}-2\mathcal{L}_{k+1})\left(\frac{2-\mathcal{L}_{n+1}}{\mathcal{L}_n-2}\right)^{k-1}\right], \tag{57}$$

where $\mathcal{L}_n$ is the nth F_n+L_n number.

Theorem 20. *Let $\mathfrak{B}'_n=LCirc(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n)$ be a left circulant matrix; then $\mathfrak{B}'_n$ is invertible for any positive integer n.*

Theorem 21. *Let $\mathfrak{B}'_n=LCirc(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n)$ be a left circulant matrix; then one can obtain*

$$\mathfrak{B}'^{-1}_n=\frac{1}{l_n}LCirc\Bigg(1-\sum_{i=1}^{n-2}\frac{\mathcal{L}_{n-i-1}(\mathcal{L}_n-2)^{i-1}}{(\mathcal{L}_1-\mathcal{L}_{n+1})^i}, \frac{2(\mathcal{L}_n-2)^{n-3}}{(\mathcal{L}_1-\mathcal{L}_{n+1})^{n-2}},\ldots,\frac{2(\mathcal{L}_n-2)^2}{(\mathcal{L}_1-\mathcal{L}_{n+1})^3}, \frac{2(\mathcal{L}_n-2)}{(\mathcal{L}_1-\mathcal{L}_{n+1})^2}, \frac{2}{\mathcal{L}_1-\mathcal{L}_{n+1}}, -2-\sum_{i=1}^{n-2}\frac{(2\mathcal{L}_{n-i}-\mathcal{L}_{n-i+1})(\mathcal{L}_n-2)^{i-1}}{(\mathcal{L}_1-\mathcal{L}_{n+1})^i}\Bigg), \tag{58}$$

where

$$l_n=\mathcal{L}_1-2\mathcal{L}_n+\sum_{k=1}^{n-2}(\mathcal{L}_{k+2}-2\mathcal{L}_{k+1})\left(\frac{\mathcal{L}_n-2}{\mathcal{L}_1-\mathcal{L}_{n+1}}\right)^{n-(k+1)}. \tag{59}$$

5. Determinant and Inverse of g-Circulant Matrix with $\mathcal{F}_n$ and $\mathcal{L}_n$ Numbers

In this section, let $\mathcal{A}_{g,n}=g\text{-Circ}(\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n)$ and $\mathfrak{B}_{g,n}=g\text{-Circ}(\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n)$ be g-circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $\mathcal{A}_{g,n}$ and $\mathfrak{B}_{g,n}$. Afterwards, we prove that $\mathcal{A}_{g,n}$ is an invertible matrix for $n>2$ and $\mathfrak{B}_{g,n}$ is an invertible matrix if $(n,g)=1$. The inverse of the matrices $\mathcal{A}_{g,n}$ and $\mathfrak{B}_{g,n}$ are also presented.

From Lemmas 6 and 7 and Theorems 8, 9, and 11, we deduce the following results.

Theorem 22. *Let $\mathcal{A}_{g,n}=g\text{-Circ}(\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n)$ be a g-circulant matrix; then one has*

$$\det\mathcal{A}_{g,n}=\det\mathbb{Q}_g\left[(1+\mathcal{F}_{n+1})^{n-1}+(-\mathcal{F}_n)^{n-2}\times\sum_{k=1}^{n-1}(-\mathcal{F}_k)\left(\frac{1+\mathcal{F}_{n+1}}{-\mathcal{F}_n}\right)^{k-1}\right], \tag{60}$$

where $\mathcal{F}_n$ is the nth $F_n\cdot L_n$ number.

Theorem 23. *Let $\mathcal{A}_{g,n}=g\text{-Circ}(\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n)$ be a g-circulant matrix and $(g,n)=1$; if $n>2$, then $\mathcal{A}_{g,n}$ is an invertible matrix.*

Theorem 24. *Let $\mathcal{A}_{g,n}=g\text{-Circ}(\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_n)$ $(n>2)$ be a g-circulant matrix and $(g,n)=1$; then*

$$\mathcal{A}^{-1}_{g,n}=\left[\frac{1}{f_n}Circ\Bigg(1-\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1-\mathcal{F}_{n+1})^i}, -3+\sum_{i=1}^{n-2}\frac{\mathcal{F}_{n-1-i}(-\mathcal{F}_n)^{i-1}}{(\mathcal{F}_1-\mathcal{F}_{n+1})^i}, \frac{1}{\mathcal{F}_1-\mathcal{F}_{n+1}}, \frac{-\mathcal{F}_n}{(\mathcal{F}_1-\mathcal{F}_{n+1})^2}, \frac{(-\mathcal{F}_n)^2}{(\mathcal{F}_1-\mathcal{F}_{n+1})^3},\ldots,\frac{(-\mathcal{F}_n)^{n-3}}{(\mathcal{F}_1-\mathcal{F}_{n+1})^{n-2}}\Bigg)\right]\mathbb{Q}^T_g, \tag{61}$$

where

$$f_n=\mathcal{F}_1-3\mathcal{F}_n+\sum_{k=1}^{n-2}(-\mathcal{F}_k)\left(\frac{-\mathcal{F}_n}{\mathcal{F}_1-\mathcal{F}_{n+1}}\right)^{n-(k+1)}. \tag{62}$$

Taking Lemmas 6 and 7 and Theorems 12, 13, and 15 into account, one has the following theorems.

Theorem 25. *Let $\mathfrak{B}_{g,n} = g\text{-}Circ(\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n)$ be a g-circulant matrix; then one has*

$$\det \mathfrak{B}_{g,n} = 2 \det \mathbb{Q}_g \times \left[(2 - \mathscr{L}_{n+1})^{n-1} + (\mathscr{L}_n - 2)^{n-2} \times \sum_{k=1}^{n-1} (\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}) \left(\frac{2 - \mathscr{L}_{n+1}}{\mathscr{L}_n - 2} \right)^{k-1} \right], \tag{63}$$

where $\mathscr{L}_n$ is the n $F_n + L_n$ number.

Theorem 26. *Let $\mathfrak{B}_{g,n} = g\text{-}Circ(\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n)$ be a g-circulant matrix and $(g, n) = 1$; then $\mathfrak{B}_{g,n}$ is invertible for any positive integer n.*

Theorem 27. *Let $\mathfrak{B}_{g,n} = g\text{-}Circ(\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n)$ be a g-circulant matrix and $(g, n) = 1$; then*

$$\begin{aligned} \mathfrak{B}_{g,n}^{-1} &= \left[\frac{1}{l_n} Circ \left(1 - \sum_{i=1}^{n-2} \frac{\mathscr{L}_{n-i-1} (\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \right. \right. \\ &\quad - 2 - \sum_{i=1}^{n-2} \frac{(2\mathscr{L}_{n-i} - \mathscr{L}_{n-i+1})(\mathscr{L}_n - 2)^{i-1}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^i}, \\ &\quad \frac{2}{\mathscr{L}_1 - \mathscr{L}_{n+1}}, \frac{2(\mathscr{L}_n - 2)}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^2}, \\ &\quad \left. \left. \frac{2(\mathscr{L}_n - 2)^2}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^3}, \ldots, \frac{2(\mathscr{L}_n - 2)^{n-3}}{(\mathscr{L}_1 - \mathscr{L}_{n+1})^{n-2}} \right) \right] \mathbb{Q}_g^T, \end{aligned} \tag{64}$$

where

$$l_n = \mathscr{L}_1 - 2\mathscr{L}_n + \sum_{k=1}^{n-2} (\mathscr{L}_{k+2} - 2\mathscr{L}_{k+1}) \left(\frac{\mathscr{L}_n - 2}{\mathscr{L}_1 - \mathscr{L}_{n+1}} \right)^{n-(k+1)}. \tag{65}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research was supported by the Development Project of Science & Technology of Shandong Province (Grant no. 2012GGX10115) and NSFC (Grant no. 11301252) and the AMEP of Linyi University, China.

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