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*Research Article*

# Pricing of Equity Indexed Annuity under Fractional Brownian Motion Model

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Fractional Brownian motion with Hurst exponent $H \in (1/2, 1)$ is a good candidate for modeling financial time series with long-range dependence and self-similarity. The main purpose of this paper is to address the valuation of equity indexed annuity (EIA) designs under the market driven by fractional Brownian motion. As a result, this paper presents an explicit pricing expression for point-to-point EIA design and bounds for the pricing of high-water-marked EIA design. Some numerical examples are given to illustrate the impact of the parameters involved in the pricing problems.

## 1. Introduction

An indexed annuity is a type of tax-deferred annuity whose credited interest is linked to an equity index—typically the S&P 500 or international index. It guarantees a minimum interest rate (typically between 1% and 3%) if held to the end of the surrender term and protects against a loss of principal. An equity indexed annuity (EIA for short) is a contract with an insurance or annuity company. The returns may be higher than fixed instruments such as CDs, money market accounts, and bonds but not as high as market returns. The guarantees in the contract are backed by the relative strength of the insurer. The contracts may be suitable for a portion of the asset portfolio for those who want to avoid risk and are in retirement or nearing retirement age. The objective of purchasing an equity index annuity is to realize greater gains than those provided by CDs, money markets, or bonds, while still protecting principal.

Because they provide minimum guarantee investment return and lock-in credit rate, EIA designs have been attractive products to customers. Sales have grown dramatically since their introduction in 1995. Indeed, EIA sales for 2007 were nearly 25.2 billion, a 380% increase over their 2000 level of 5.25 billion (see Marrion [1]). According to an industry survey, indexed annuities enjoyed record sales growth in 2010; thirty-nine indexed annuity carriers participated in the 54th edition of AnnuitySpecs.com's Indexed Sales & Market Report, representing 99% of indexed annuity production. Total fourth quarter sales were 8.3 billion, up to 19% from the same period of last year.

The mechanics of EIA are often complex and the returns can vary greatly depending on the month and year that the annuity is purchased. One main method for valuing (pricing) of EIA is the so-called structural method. The idea of the method is to introduce a stochastic process for modeling the behavior of equity involved in the EIA products. Original works on this aspect can be found in Tiong [2]. Stepped works can be found in Boyle and Tian [3], Gerber and Shiu [4], Hardy [5], Jaimungal [6], Kijima and Wong [7], Lee [8], Lin and Tan [9], and Moore [10]. These authors studied the pricing, hedging, and risk management of various features of EIA. There also exist a variety of literatures on EIA valuation by incorporating stochastic interest rate or stochastic mortality into model; for details, see the work of Biffis [11], Biffis et al. [12], Hainaut and Devolder [13], and Jalen and Mamon [14] and references therein.

In aforementioned papers, the dynamic of equity is assumed to be driven by drifted Brownian motion, Brownian motion with Poisson jumps, or Lévy process. All these processes are independent in increments and thus they are semimartingales, which enable the classical Itô calculus for semimartingales to be applied for modeling the equity

market. Another popular tool for the valuation of EIA is Esscher transform; see the work of Lin and Tan [9], Qian et al. [15], and Tiong [2] and references therein. The purpose of Esscher transform is to obtain a Radon-Nikodym derivative for constructing a new measure. Under the new measure, underlying risky asset is embedded into a risk-neutral world. This is also the idea of famous Black-Scholes-Merton option pricing framework. Esscher transform requires that the return process of risky asset (also named by the logarithm of asset price process) should be independent in increments. However, behavioral finance and econometrics as well as empirical studies found that not all financial data are consistent with this basic characteristic. Over the past decades, vast literatures show that many financial market time series display long-range dependence and momentum and self-similar properties; see the work of Aít-Sahalia [16], Andrew [17], and Granger [18], for example. In this case, the classical financial model with independent increments is invalid, consequently, the approach of deriving risk neutral measure by Esscher transform is not available. Thus, it is natural to propose new model for describing such new kinds of financial time series. By theoretical analysis and empirical test, it turns out that fractional Brownian motion (fBm for short) model is good candidate and consequently, the fractional Black-Scholes models are brought forward. The fractional Black-Scholes model is a generalization of the Black-Scholes model, which is based on replacing the standard Brownian motion by a fractional Brownian motion in financial model.

In this paper we introduce fBm into EIA valuing framework and obtain pricing expressions for two main kinds of EIA designs. This paper is organized as follows. In Section 2, we quickly cite some preliminary results which show why fBm captures the financial data with long-range dependence and self-similarity. Since fBm is not a semimartingale, the traditional Itô calculus method is not valid here. Thus, Section 2 also presents some existing basic calculus results on fBm which will be of great importance in our later discussion. In Section 3, an explicit pricing formula for point-to-point EIA design and bound estimation for high-water-marked EIA design are presented. Some numerical examples are presented for comparing our results with the classical model in this section. In Section 4, we present our conclusions and some remaining problems.

## 2. Preliminaries: Long-Range Dependence and fBm

This section just serves as a quick survey of the content presented in Sottinen and Valkeila [19] and references therein.

*Definition 1* (long-range dependence). A stationary sequence $\mathbf{X} = \{X_k, k \in \mathbb{N}\}$ is said to exhibit the statistical long-range dependence, if its autocorrelation function $\rho(k) \widehat{=} \mathrm{Cov}(X_n, X_{k+n})$ satisfies

$$\lim_{k\to\infty} \frac{\rho(k)}{c_\rho k^{-\alpha}} = 1, \tag{1}$$

for some $c_\rho$ and $\alpha \in (0,1)$. This is to say that the dependence between $X_n$ and $X_{k+n}$ decays slowly as $k \to \infty$. In particular,

$$\sum_{k=0}^{\infty} \rho(k) = \infty. \tag{2}$$

In some literatures, (2) is also referred to as the definition of long-range dependence.

*Definition 2* (self-similarity). A centered stochastic process $\mathbf{X} = \{X_t, t \in (0,T)\}$ is said to be statistically self-similar with Hurst exponent $H$, if

$$\{X_t, t \in (0,T)\} \overset{d}{=} \left\{\alpha^{-H} X_{\alpha t}, t \in (0,T)\right\}, \tag{3}$$

for all $\alpha > 0$. Here, $\overset{d}{=}$ denotes the equivalence in distribution.

Following Lemmas 3 and 4 demonstrates the importance of fractional Brownian motion in modeling financial data with long-range dependence; for details we refer to Sottinen and Valkeila [19] and references therein.

**Lemma 3.** *A square integrable self-similar process $X = \{X_t, 0 \le t \le T\}$ with stationary increments admits that*

$$\mathrm{Cov}(X_t, X_s) = \frac{1}{2}\mathrm{Var}(X_1)\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{4}$$

*By (4), one can easily find that it is a necessary condition that $H \in (0,1)$ for the process to be well defined.*

**Lemma 4.** *Suppose that self-similar process $X = \{X_k, k \in \mathbb{N}^+\}$ is centered square integrable process with stationary increments; then the increments*

$$Y_k \widehat{=} X_k - X_{k-1} \tag{5}$$

*are stationary with autocorrelation function $\rho(k) = (1/2)((k+1)^{2H} - 2k^{2H} + (k-1)^{2H})$. Then, when $k \to \infty$, one has*

$$\rho(k) \approx H(2H-1)k^{2H-2}. \tag{6}$$

*Thus, only if $H \in (1/2, 1)$, the increments $\{Y_k, k \in \mathbb{N}^+\}$ exhibit long-range dependence.*

Let $(\Omega, \mathfrak{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space, a fBm $B_t^H$ with Hurst exponent $H \in (0,1)$ is a continuous, centered Gaussian process with covariance function

$$\mathrm{Cov}\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in (0,T), \tag{7}$$

where we assumed that $\mathfrak{F} = \mathcal{F}_T$, for some $T \in (0,\infty)$, and $\mathcal{F}_t = \sigma(B_s^H, 0 \le s \le t \le T)$ satisfy the usual condition and $\mathbb{P}$ is the real world probability.

If $H = 1/2$, then the corresponding fBm is the usual standard Brownian motion. Furthermore, $B_t^H$ has stationary increments and is H self-similar. Valuations of EIA under

model driven by standard Brownian motion have been extensively studied, as we mentioned in introduction. If $H > 1/2$, the process $B_t^H$, $t \in (0,T)$ exhibits a long-range dependence. As it was mentioned in Willinger et al. [20], data with long-range dependence are widely spread in economics and finance and thus are an active research topic (e.g., see [10] for details). Long-range dependence seems also an important feature that explains the well-documented evidence of volatility persistence and momentum effects. Hereafter, we will only consider the case $H > 1/2$ in the rest of this paper, which is most frequently encountered in the real financial data.

To define a fractional analogue of the classical Black-Scholes Pricing model, we need to know how to integrate with respect to (w.r.t. for short) fBm as this is connected to hedging. There are two main ways in defining stochastic integration for fBm. One is path wise, that is, $\omega - \omega$ stochastic integrals w.r.t. fBm as a refinement of Riemann-Stieltjes integrals by using $p$-variation; for details, see Dudley and Norvaisa [21]. Another one is based on white noise analysis and Wick products; for details, see Duncan et al. [22] and, for applications, see Hu and Øksendal [23]. Suppose that $f \in C^1([0,T])$; the following change of variables formula for the path wise integration with respect to the fBm plays important role in what follows:

$$f\left(B_t^H\right) - f\left(B_s^H\right) = \int_s^t f'\left(B_r^H\right) dB_r^H. \tag{8}$$

In option pricing or EIA valuation, the Girsanov theorem for semimartingales is of great importance for which transforms the underlying asset price process into an risk-neutral world; besides Girsanov theorem, Esscher transform is also one main method for obtaining the risk neutral measure. Since Esscher transform is invalid for fBm, to continue our discussion, analogues of Girsanov theorem for fBm is necessary. We start our discussion by defining the so-called fundamental martingale $M = M^H$, which is given by

$$M_t^H = c_1 \int_0^t r^{1/2-H}(t-r)^{1/2-H} dB_r^H; \tag{9}$$

Now, $M^H$ is a martingale with quadratic process

$$\left\langle M^H \right\rangle_t = c_2 t^{2-2H}, \tag{10}$$

where $c_1$ and $c_2$ are certain constants depending on Hurst Exponent $H$. The following two theorems present the Girsanov transform for fBM derived process and martingale representation for fBM; readers are referred to Norros et al. [24] for detailed discussion.

**Lemma 5** (Girsanov theorem). *Let $\mu(r)$ be a deterministic function and define a measure $\mathbb{Q} = \mathbb{Q}^\mu$ by*

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|\mathscr{F}_t &= \exp\left\{\int_0^t \mu(r)\, dM_r^H - \frac{1}{2}\int_0^t \mu^2(r)\, d\langle M\rangle_r^H\right\} \\ &= \exp\left\{\int_0^t \mu(r)\, dM_r^H - \frac{1}{2}c_2(2-2H)\int_0^t \mu^2(r)\, r^{1-2H} dr\right\}. \end{aligned} \tag{11}$$

*Then the process*

$$B_t^H - \int_0^t \mu(r)\, dr, \quad 0 \le t \le T \tag{12}$$

*is an fBm, if and only if under measure $\mathbb{Q}$.*

**Lemma 6** (martingale representation for fractional Brownian motion). *Define $\widetilde{B}_t^H \widehat{=} \mathbb{E}[B_T^H | \mathscr{F}_t^{B^H}]$; then*

$$\widetilde{B}_t^H = B_t^H + \int_0^t g_T(t,s)\, dB_s^H, \tag{13}$$

*where*

$$g_T(t,s) = \frac{\sin(\pi(H-1/2))}{\pi} s^{1/2-H} \int_t^T \frac{r^{H-1/2}(r-t)^{H-1/2}}{r-s} dr. \tag{14}$$

## 3. Pricing EIA

*3.1. EIA Designs and Model.* There are variable EIAs products; in this paper, we discuss the pricing of some most commonly available EIA policies in the market. The simplest one is a point-to-point design, where the policy earns the realized return on the index (or some other risky asset) over a certain period of time at a prescribed participation rate, but with a minimum guarantee. For example, if the realized 5-year return on the S&P 500 is 100% and we assume a participation rate of 80%, then the actual interest credited to the policy will be 80% instead. Even when the market performs poorly, the policy still earns a return of at least the guaranteed rate. Another type of design that we will also touch on in this paper is the look-back or the high-water-marked. This is also referred to as the "no-regret" type of policy, which earns the highest return on the index attained during the life of the policy. The most favorable type of EIA product design seems to be the annual reset. However, due to technical difficulties, we cannot present satisfied framework and results for valuing the annual reset EIA products.

To price an EIA design, one has to bring forward the dynamic of indexed equity. To our knowledge, the risky assets are mostly driven by semimartingales. Due the properties of long-dependence and self-similarity of some financial time series, it is natural and urgent to introduce proper process for modeling such financial data. As it was shown in Section 2, fBm is the ideal continuous time candidate. Thus, we replace

the source of randomness of Brownian motion $W_t$ by fBm $B_t^H$ with index $H \in (1/2, 1)$ and then the dynamic of our risky asset is given by

$$dS_t = S_t \left(\mu dt + \sigma dB_t^H\right). \tag{15}$$

The solution to (15) is called the *geometric fractional Brownian motion*. Similar to the assumption in classical model, we assume that the parameters $r$, $\mu$, $\sigma$, and $H$ are constants.

*Remark 7.* Since our idea is to apply the option pricing formula for valuing EIA designs, an inevitable problem is whether the pricing model is free of arbitrage and whether the pricing model is complete. Under the Wick product framework of Duncan et al. [22], the solution to (15) is

$$S_t = S_0 \exp\left\{\mu t - \frac{\sigma^2}{2} t^{2H} + \sigma B_t^H\right\} \tag{16}$$

and the pricing model is free of arbitrage and complete. However, under the path wise integration framework, the solution to (15) is

$$S_t = S_0 \exp\left\{\mu t + \sigma B_t^H\right\} \tag{17}$$

and there exists arbitrage for the pricing model. In 1997, Rogers [7] showed that the fractional Brownian motion could not be used as a price process for a risky security without introducing arbitrage opportunities.

Although under the framework of classical "fundamental asset pricing theorem," fBm is not good candidate for driving process of financial market; it is still meaningful to consider pricing problem under the market driven by fBm. Our reasons are listed as follows.

(i) (Nonarbitrage is possible in a fBm model.) Thanks to the work of Jarrow et al. [25], it is possible to find a general class of processes, which need not be semimartingales that do not permit arbitrage. The idea is to disallow continuous trading and, moreover, to require a minimal fixed time between successive trades. The fixed time can be as small as one likes, but once chosen, it cannot be changed. This disallows a clustering of trades around a fleeting arbitrage opportunity. One should note that continuous trading strategies generate infinite transaction costs under reasonable models of such costs, and those of unbounded variation generate infinite liquidity costs in any finite time interval (c.f. Soner et al. [26]). As such, these trading strategies could never be used in practice. Thus, the method of avoiding arbitrage proposed by Jarrow et al. [25] is naturally accepted in real world practice.

(ii) (Practical needed) Price processes which are not semimartingales are appearing more regularly in the empirical literature estimating stock price processes (see Aít-Sahalia [16], Andrew [17], Granger [18] Dudley and Norvaisa [21], and references therein); so it necessary to introduce fBm in modeling financial time series.

*Remark 8.* One can compute the price of EIA in path wise fractional model using a weak pricing principle, which coincides with the ones obtained in the generalized pricing model brought forward by Hu and Øksendal [23]. Although the path wise integration theory is not as satisfactory as the Wick product way, the former one takes its own advantages. Firstly, the functional analytic approach makes it impossible to consider the integrals as almost sure limits of the paths of the process under certain partitions of the integral. It should be noted that this interpretation is possible in Brownian motion. Secondly, and more subjectively, one wants to model the paths properties of the price of the indexed equity. The Wick products integration does not fit well to this aim since the path properties play no central role in the integrals.

*3.2. Point-to-Point Design.* Here, we shall discuss only the plain point-to-point design, where the index level at maturity is taken simply as the ending index. In all cases the starting index is the prevailing index level when the policy is issued. Let $S_t$ be the value (or price) of an asset at time $t$ that takes interest force $r$; that is, $S_t$ are paid dividends $rS(t)dt$ between time $t$ and time $t + dt$, where $r$ is nonnegative. Let $\alpha$ be the participation rate, which, in practice, is almost always less than or equal to 1. Suppose that at time $T$, $T > 0$, given an initial premium of 1, we have a policy that pays $e^{\alpha \ln S_T}$, $\alpha > 0$ or a fixed exercise price $K$ ($=e^{\ln K}$), whichever is higher. Therefore, at maturity, the policy earns a percentage of the realized return on the asset over $T$ periods (the term of the policy) which is $\alpha \ln S_t$, with the provision of a minimum guaranteed rate of return, $\ln K$. Thus, we can express the valuation of this policy under fractal model by

$$e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[\max\left(S_t, K\right)\right]. \tag{18}$$

In classical pricing formula, if the market is free of arbitrage, under the risk neutral measure $\mathbb{Q}$, the discounted price process $e^{-rtS_t}$ is a martingale with respect to the natural sigma filtration generated by Brownian motion $\{B_t, 0 \le t \le T\}$. Particularly, we have

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-rT} S_T\right] = S_0. \tag{19}$$

In factional setting, we cannot have the martingale property. However, the Girsanov theorem for fraction Brownian motion provides a unique probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ such that (19) holds. By simple calculation, one can find that (19) holds, if process

$$B_t^H + \frac{\mu - r}{\sigma} t + \frac{\sigma}{2} t^{2H} \tag{20}$$

is the fBm under measure $\mathbb{Q}$. By the Grisanov theorem, $\mathbb{Q}$ is defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\frac{\mu - r}{\sigma}M_t + \sigma H\int_0^t s^{2H-1}dM_s - \frac{1}{2}c_2(2-2H) \times \left(\left(\frac{\mu - r}{\sigma}\right)^2 t^{2-2H} + (\mu - r)t + \sigma^2 t^{2H}\right)\right\}. \tag{21}$$

Under measure $\mathbb{Q}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[f(S_T)] &= \mathbb{E}^{\mathbb{Q}}\left[f\left(S_0\exp\left(\mu T + \sigma B_T^H\right)\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[f\left(S_0\exp\left(\sigma\left[B_T^H + \frac{\mu - r}{\sigma}T + \frac{\sigma}{2}T^{2H}\right] + rT - \frac{\sigma^2}{2}T^{2H}\right)\right)\right].\end{aligned} \tag{22}$$

Note that, under measure $\mathbb{Q}$, $\{B_t^H + ((\mu - r)/\sigma)t + (\sigma/2)t^{2H},\ 0 \le t \le T\}$ is a fractional Brownian motion; integrating its self-similarity property yields

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}[f(S_T)] \\ &\quad = e^{-rT}\frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{\infty} f\left(S_0\exp\left(\sigma y T^H + rT - \frac{\sigma^2}{2}T^{2H}\right)\right)e^{-y^2/2}dy.\end{aligned} \tag{23}$$

Given $r$, $T$, $\mu$, and $\sigma$, denote by $P_\alpha$ the valuation of EIA with participation rate $\alpha$. Without loss of generality, suppose that $S_0 = 1$. Then, we obtain the valuation formula for EIA under point-to-point design directly by (23). Consider

$$\begin{aligned}P_\alpha &= \mathbb{E}^{\mathbb{Q}}\left[\max\left(e^{\alpha\ln S_T}, K\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\max\left(e^{\alpha(\mu T + \sigma B_T^H)}, K\right)\right] \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{\infty}\max\left\{\exp\left(\alpha\left(\sigma y T^H + rT - \frac{\sigma^2}{2}T^{2H}\right)\right), K\right\} \times e^{-y^2/2}dy.\end{aligned} \tag{24}$$

Note that

$$\begin{aligned}&e^{\alpha(\sigma y T^H + rT - (\sigma^2/2)T^{2H})} \le K \iff \alpha\left(\sigma y T^H + rT - \frac{\sigma^2}{2}T^{2H}\right) \le \ln K \\ &\qquad \iff y \le \frac{\ln K/\alpha}{\sigma T^H} - \frac{r}{\sigma}T^{1-H} + \frac{\sigma}{2}T^H.\end{aligned} \tag{25}$$

Thus, (24) can be reformulated as

$$\begin{aligned}&Ke^{-rT}\Phi\left(\frac{\ln K/\alpha}{\sigma T^H} - \frac{r}{\sigma}T^{1-H} + \frac{\sigma}{2}T^H\right) \\ &\quad + e^{(\alpha-1)rT + (\sigma^2/2)T^{2H}\alpha(\alpha-1)} \times \int_{\ln K/\alpha\sigma T^H - (r/\sigma)T^{1-H} + (\sigma/2)T^H}^{\infty} e^{-(y-\alpha\sigma T^H)^2/2}dy \\ &= Ke^{-rT}\Phi\left(\frac{\ln K/\alpha}{\sigma T^H} - \frac{r}{\sigma}T^{1-H} + \frac{\sigma}{2}T^H\right) \\ &\quad + e^{(\alpha-1)rT + (\sigma^2/2)T^{2H}\alpha(\alpha-1)} \times \left[1 - \Phi\left(\frac{\ln K/\alpha}{\sigma T^H} - \frac{r}{\sigma}T^{1-H} + \frac{\sigma}{2}T^H - \alpha\sigma T^H\right)\right].\end{aligned} \tag{26}$$

*Remark 9.* If we take the dividend payment into account, we let $\delta$ be the nonnegative dividend yield rate, that is, the assets pay out dividend $\delta S(t)dt$ between very small time interval $[t, t + dt]$, then the "pure" riskless income of the asset is $(r - \delta)S(t)dt$. One can easily find that, in this case, the pricing formula for point-to-point design is revived as

$$\begin{aligned}&Ke^{-rT}\Phi\left(\frac{\ln K/\alpha}{\sigma T^H} - \frac{(r-\delta)}{\sigma}T^{1-H} + \frac{\sigma}{2}T^H\right) \\ &\quad + e^{[(\alpha-1)r - \alpha\delta]T + (\sigma^2/2)T^{2H}\alpha(\alpha-1)} \times \left[1 - \Phi\left(\frac{\ln K/\alpha}{\sigma T^H} - \frac{r-\delta}{\sigma}T^{1-H} + \frac{\sigma}{2}T^H - \alpha\sigma T^H\right)\right].\end{aligned} \tag{27}$$

By putting $H = 1/2$, then our model is the famous Black-Scholes pricing model, which is also the indexed equity model in Tiong [2] and (3.4) of Tiong [2] is a special case of (27) by putting $H = 1/2$.

Let us now assume that the customer is guaranteed a minimum maturity value, which is a percentage $\beta$, $0 < \beta \le 1$, of the original premium compounded at a minimum guaranteed rate of return $g$, $g > 0$, for the duration of the policy. We can incorporate this feature into the policy by setting the exercise price to be

$$K = \beta e^{gT}, \tag{28}$$

assuming an initial premium of 1. For most of the EIA designs sold by the insurance companies, $\beta$ is set at around 90% and $g$ at 3%. Now, we are going to make some observations about $P_\alpha$.

Generally speaking, the price of design $P_\alpha$ is increasing with respect to parameter $K$, $\sigma$, and $\alpha$. As for $r$ and $T$, the situations are not clear cut. $P_\alpha$ can be increasing or/and decreasing with respect to $r$ and $T$, depending on other parameters. Although we can give some intuitive understanding of these relations by numerical examples, just as what has been done in Tiong [2], here, we want to emphasize on the difference of the EIA valuation under the model driven by Brownian motion and the one driven by fBm. Thus, we take the same parameters as the one taken in Figures 1 and 2 of Tiong [2], respectively.

In Figures 1 and 2, the red curves represent the pricing curves under fBm model and blue curves represent the ones under Brownian motion model. From these two figures, it follows easily that when the participation rate $\alpha$ puts small value, for example 0.2 or 0.3, there is no obvious difference between two pricing models. While the value of $\alpha$ is relatively large, for example 0.6 and 0.9, then, two pricing models are quite different. Generally speaking, at the same surrender term, the valuation of fBm model is higher than the one of Brownian motion.

*3.3. High-Water-Marked Design.* Another popular design among the EIA products is the look-back or high-water-marked method. The idea is that, at maturity, the interest earned on the policy will be based on the growth rate of the highest index value attained during the life of the policy over the index value at the start of the term, which we assume to be one in our calculations. In practice, the method usually looks at the index level at each policy anniversary, and the highest of these is then taken and figured as the index level on the maturity date. Here, we consider the continuous look-back case. Let

$$Y_T = \max_{0\le t\le T}\left(\mu t + \sigma B_t^H\right) = \max_{0\le t\le T}\log\left(\frac{S_t}{S_0}\right) \tag{29}$$

be the maximum rate of return on the index attained over the time interval $[0,T]$. We assume that the payoff of the policy at time $T$ is $e^{\alpha Y_T}$, $0<\alpha<1$ or a fixed minimum guaranteed amount, $K$, whichever is larger. In this case, at time 0, denote by $P_h$ the value of this policy; then

$$\begin{aligned}
P_h &= \mathbb{E}^{\mathbb{Q}}\left[\max\left(e^{\alpha Y_T}, K\right)\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\max(\alpha Y_T, \ln K)}\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\max[\alpha \max_{0\le t\le T}(\mu t+\sigma B_t^H), \ln K]}\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\max[\alpha \max_{0\le t\le T}(\sigma(B_t^H+((\mu-r)/\sigma)t+(\sigma/2)t^{2H})+rt-(\sigma^2/2)t^{2H}), \ln K]}\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\max[\alpha\sigma \max_{0\le t\le T}(\widetilde{B}_t^H+(r/\sigma)t-(\sigma/2)t^{2H}), \ln K]}\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\alpha\sigma \max[\max_{0\le t\le T}(\widetilde{B}_t^H+(r/\sigma)t-(\sigma/2)t^{2H}), \ln K/\alpha\sigma]}\right],
\end{aligned} \tag{30}$$

where $\widetilde{B}_t^H \hat{=} B_t^H+((\mu-r)/\sigma)t+(\sigma/2)t^{2H}$ is a fBm under measure $\mathbb{Q}$. Thus, to derive the pricing formula for high-water-marked design, it is sufficient to get the distribution of the maximum of an fBm with nonlinear drift $rt-(\sigma^2/2)t^{2H}$. Unfortunately, to our knowledge, there are no existing results on the topic and it is also not an easy job for us. Alternatively, we dedicate to derive a lower bound and upper bound for the pricing of EIA under high-water-marked design.

Let $Z(T)\hat{=}\max_{0\le t\le T}(\widetilde{B}_t^H+(r/\sigma)t-(\sigma/2)t^{2H})$. Denote by $F(\cdot)$ the distribution of $Z(T)$ under measure $\mathbb{Q}$ and by $\overline{F}(\cdot)$ the tail distribution of $F(\cdot)$; that is, $\overline{F}(\cdot)=1-F(\cdot)$. Then (30) is reformulated as

$$\begin{aligned}
&K\mathbb{Q}\left(Z(T)\le\frac{\ln K}{\sigma\alpha}\right)+\int_{\ln K/\sigma\alpha}^{\infty}e^{\sigma\alpha y}dF(y)\\
&\quad= K\mathbb{Q}\left(Z(T)\le\frac{\ln K}{\sigma\alpha}\right)-\int_{\ln K/\sigma\alpha}^{\infty}e^{\sigma\alpha y}d\overline{F}(y)\\
&\quad= K\mathbb{Q}\left(Z(T)\le\frac{\ln K}{\sigma\alpha}\right)-K\overline{F}\left(\frac{\ln K}{\sigma\alpha}\right)\\
&\qquad+\sigma\alpha\int_{\ln K/\sigma\alpha}^{\infty}e^{\sigma\alpha y}\overline{F}(y)\,dy\\
&\quad= 2KF\left(\frac{\ln K}{\sigma\alpha}\right)-K+\sigma\alpha\int_{\ln K/\sigma\alpha}^{\infty}e^{\sigma\alpha y}\overline{F}(y)\,dy.
\end{aligned} \tag{31}$$

Note that

$$\begin{aligned}
&\mathbb{Q}\left(Z(T)\le\frac{\ln K}{\sigma\alpha}\right)\\
&\quad= \mathbb{Q}\left(\max_{0\le t\le T}\left(\widetilde{B}_t^H+\frac{r}{\sigma}t-\frac{\sigma}{2}t^{2H}\right)\le\frac{\ln K}{\sigma\alpha}\right)\\
&\quad\le \min_{0\le t\le T}\mathbb{Q}\left(\widetilde{B}_t^H+\frac{r}{\sigma}t-\frac{\sigma}{2}t^{2H}\le\frac{\ln K}{\sigma\alpha}\right)\\
&\quad= \min_{0\le t\le T}\mathbb{Q}\left(\widetilde{B}_t^H\le\frac{\ln K}{\sigma\alpha}-\frac{r}{\sigma}t+\frac{\sigma}{2}t^{2H}\right)\\
&\quad= \min_{0\le t\le T}\int_{-\infty}^{\ln K/\sigma\alpha+rt/\sigma-(\sigma/2)t^{2H}}\frac{1}{\sqrt{2\pi}t^H}e^{-y^2/2t^{2H}}dy\\
&\quad= \min_{0\le t\le T}\int_{-\infty}^{(\ln K/\sigma\alpha)t^{-H}+rt^{1-H}/\sigma-(\sigma/2)t^H}\frac{1}{\sqrt{2\pi}}e^{-s^2/2}ds\\
&\quad= \min_{0\le t\le T}\Phi\left(\frac{\ln K}{\sigma\alpha}t^{-H}+\frac{rt^{1-H}}{\sigma}-\frac{\sigma}{2}t^H\right)\hat{=}\min_{0\le t\le T}h\left(\frac{\ln K}{\sigma\alpha},t,T\right),
\end{aligned} \tag{32}$$

where $\Phi(\cdot)$ is standard normal distribution. Denote by $t_0(y,T)$ the minimizer of $h(y,t,T)$; then $h(y,t_0(y,T),T)|_{y=\ln K/\sigma\alpha}$ is an upper bound of $\mathbb{Q}(Z(T)\le\ln K/\sigma\alpha)$. This upper bound seems to be quite rough; however, just as it was proved in Narayan [27] and Norros et al. [24], such bound is very tight. Since the power exponent of $t$ is not an integer, it is not easy to get explicit expression of $h(y,t_0(y,T),T)|_{y=\ln K/\sigma\alpha}$, but it is easy to get their numerical solution.

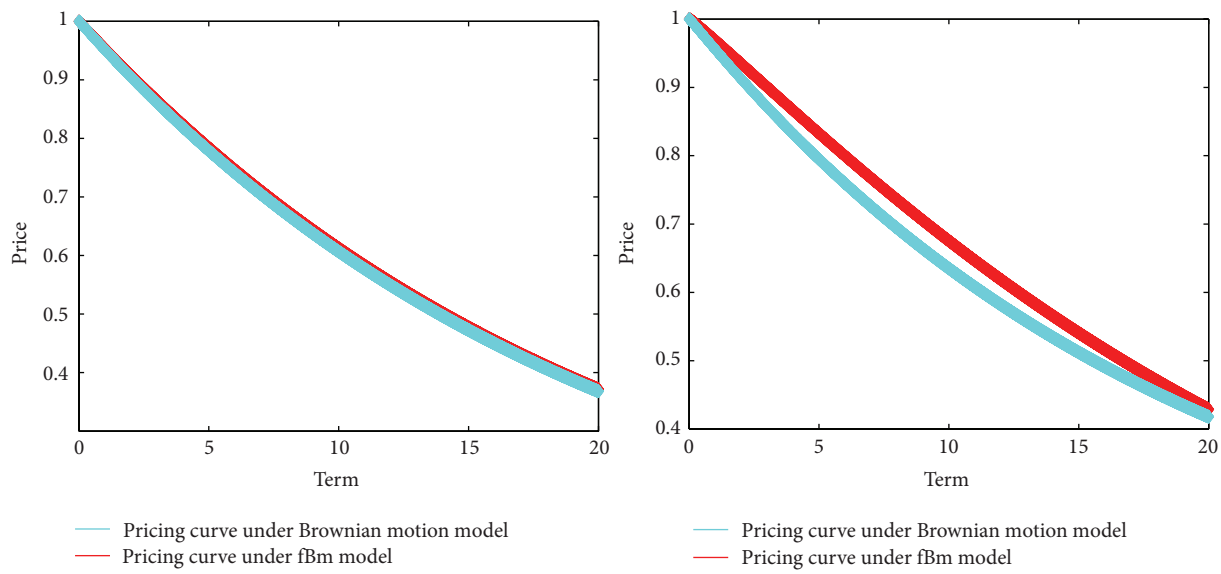


FIGURE 1: Comparison pricing $P_\alpha$ as a function of $T$ given $\sigma = 20\%$, $r = 5\%$, $\delta = 2\%$, and $K = 0.9e^{0.03T}$, for $\alpha = 0.2, 0.3$, respectively.

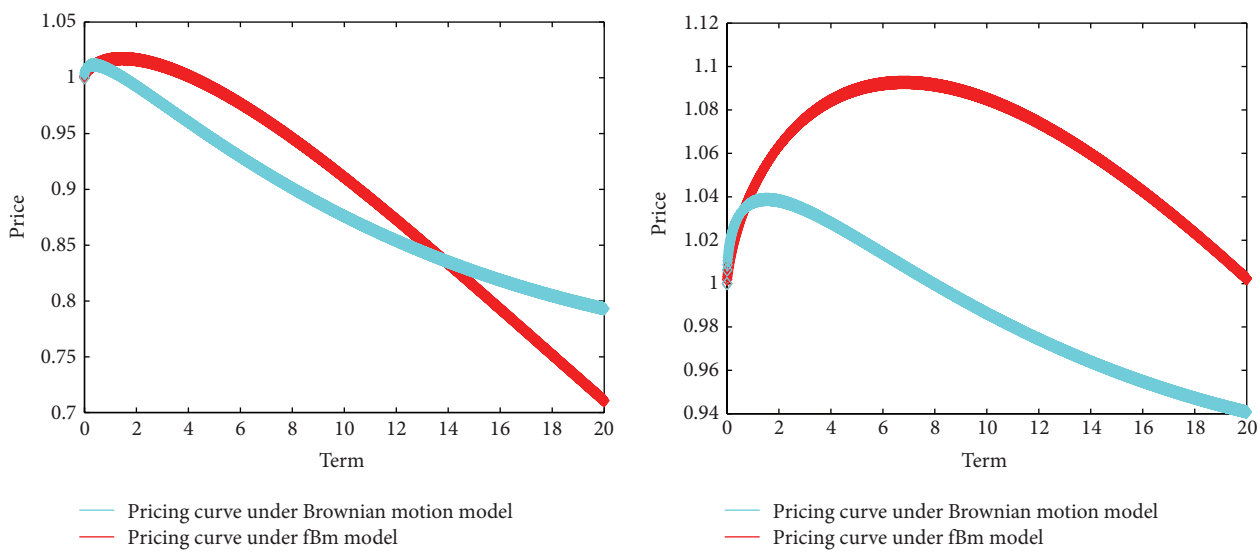


FIGURE 2: Comparison pricing $P_\alpha$ as a function of $T$ given $\sigma = 20\%$, $r = 5\%$, $\delta = 2\%$, and $K = 0.9e^{0.03T}$, for $\alpha = 0.6, 0.9$, respectively.

On the other hand,

$$\begin{aligned}
&\mathbb{Q}\left(Z(T) \le \frac{\ln K}{\sigma\alpha}\right) \\
&\quad = \mathbb{Q}\left(\max_{0\le t\le T}\left(\widetilde{B}_t^H + \frac{r}{\sigma}t - \frac{\sigma}{2}t^{2H}\right) \le \frac{\ln K}{\sigma\alpha}\right) \\
&\quad \ge \mathbb{Q}\left(\max_{0\le t\le T}\left(\widetilde{B}_t^H + \frac{r}{\sigma}t\right) \le \frac{\ln K}{\sigma\alpha}\right) \\
&\quad = 1 - \mathbb{Q}\left(\max_{0\le t\le T}\left(\widetilde{B}_t^H + \frac{r}{\sigma}t\right) > \frac{\ln K}{\sigma\alpha}\right).
\end{aligned} \tag{33}$$

By the Slepian inequality for fBm with drift (e.g., see Michna [28]), it follows that

$$\mathbb{P}\left(\max_{0\le t\le T}\left(B_t^H + \mu t\right) > u\right) \le \mathbb{P}\left(\max_{0\le t\le T}\left(B_{t^{2H}}^{1/2} + \mu t\right) > u\right), \tag{34}$$

where $B_s^{1/2}$ is standard Brownian motion. Denote $\rho(u)$ as

$$\rho(u) = \inf\left\{t > 0 : B_s^{1/2} + \mu t^{1/2H} > u\right\}; \tag{35}$$

then

$$\mathbb{P}\left(\max_{0\le t\le T}\left(B^{1/2}_{s^{2H}}+\mu t\right)>u\right)$$
$$=\mathbb{P}\left(\max_{0\le t\le T^{2H}}\left(B^{1/2}_{s}+\mu t^{1/2H}\right)>u\right)=\mathbb{P}\left(\rho^{1/2H}(u)\le T\right). \tag{36}$$

By using the integral equation for the first passage density for Brownian motion to the barrier $f(s)=u-\mu s^{1/2H}$ (see, e.g., Ferebee [29]), we are able to find a lower density of $\rho(u)$, say $p(s;u)$, which takes the form of

$$p(s;u)=s^{-3/2}\left(u-\left(1-\frac{1}{H}\right)\frac{r}{\sigma}s^{1/2H}\right)\times\phi\left(us^{1/2}-\frac{r}{\sigma}s^{(1-H)/2H}\right), \tag{37}$$

where $\phi(\cdot)$ is the density function of standard normal distribution. By integrating (33), (35), and (37), we have

$$\begin{aligned}\mathbb{Q}\left(Z(T)\le\frac{\ln K}{\sigma\alpha}\right)&\ge 1-\mathbb{Q}\left(\max_{0\le t\le T}\left(\widetilde{B}^{H}_{t}+\frac{r}{\sigma}t\right)>\frac{\ln K}{\sigma\alpha}\right)\\&\ge 1-\mathbb{P}\left(\max_{0\le t\le T}\left(B^{1/2}_{t^{2H}}+\frac{r}{\sigma}t\right)>\frac{\ln K}{\sigma\alpha}\right)\\&=1-\mathbb{P}\left(\rho^{1/2H}\left(\frac{\ln K}{\sigma\alpha}\right)<T\right)\\&\ge 1-\int_0^T p\left(s;\frac{\ln K}{\sigma\alpha}\right)ds.\end{aligned} \tag{38}$$

Together with the definition of $h(y,t_0(y,T),T)|_{y=\ln K/\sigma\alpha}$ and (38), we have

$$\begin{aligned}1-\int_0^T p\left(s;\frac{\ln K}{\sigma\alpha}\right)ds\le\mathbb{Q}\left(Z(T)\le\frac{\ln K}{\sigma\alpha}\right)&\le h(y,t_0(y,T),T)|_{y=\ln K/\sigma\alpha},\\1-h(y,t_0(y,T),T)\le\overline{F}(y)\le\int_0^T p(s;y)\,ds.&\end{aligned} \tag{39}$$

By integrating (30) and (39), we have the following bound estimation for the pricing of high-water-marked design:

$$H_1(K)\le P_h\le H_2(K), \tag{40}$$

where

$$\begin{aligned}H_1(K)&=K\left[1-2\int_0^T p\left(s;\frac{\ln K}{\sigma\alpha}\right)ds\right]+\sigma\alpha\int_{\ln K/\sigma\alpha}^{\infty}\int_0^T p(s;y)\,e^{\sigma\alpha y}ds\,dy,\\H_2(K)&=\left[2h(y,t_0(y,T),T)|_{y=\ln K/\sigma\alpha}-1\right]K+\sigma\alpha\int_{\ln K/\sigma\alpha}^{\infty}\left(1-h(y,t_0(y,T),T)\right)dy.\end{aligned} \tag{41}$$

## 4. Remarks and Conclusion

The valuing of high-water-marked EIA designs are highly depends on the distribution of the supremum of drifted (linear or nonlinear) fBm, although the readers can fall back on the results of Narayan [27] and references therein for the asymptotic distribution of the supremum of fBm and their applications. For example, see Willinger et al. [20] for their applications in performance evaluation and see Michna [28] for their applications in insurance. But, to our knowledge, the distribution of the supremum of fBm with nonlinear drift still remained a problem. Even a good estimation for the bounds of supremum distribution is unsolved. To some extent, our bounds estimation for the pricing of EIA under high-water-marked design is meaningful. In fact, bound estimation is also classical research topic in risk theory and risk management, for example, ruin probability; see Asmussen [30].

Usually, there are three kinds of EIA designs: point-to-point, high-water-marked, and annual reset. In this paper, we just focus on the former two designs. Up to now, we are not able to get an explicit expression for pricing formula or to get any proper bounds estimation for such design and thus it is a remaining problem. It is very important to consider stochastic interest rates when pricing EIA. Lin and Tan [9] considered the model of stochastic interest rates by postulating a Vasicek model which is correlated to the geometric Brownian motion of the risky asset. They argued that the effects of stochastic interest rates are crucial in EIA pricing by simulation results. Kijima and Wong [7] adopted the ordinary arbitrage-free pricing principle to price simple and compound annual reset EIA when the short rate follows the extended Vasicek model. In the above-mentioned literatures, mortality risk is considered to be deterministic or even not included. However, the expected life length has increased considerably in many countries during the past decades with the advances made in the health sciences and medicine; life insurance and annuities are exposed to unanticipated changes over time in the mortality rates of the appropriate reference population, which has forced life insurers to use a stochastic model to describe mortality laws. There exist a variety of literatures on this topic, for example, Biffis [11], Biffis et al. [12], Hainaut and Devolder [13], Jalen and Mamon [14], and Qian et al. [15]. Since up to now, we have no clear idea on the mechanics of both fBm and the stochastic mortality process or interest process, so we just focus on simple case. But these topics are interesting and worthy of further research.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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