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Research Article

The Yang-Laplace Transform for Solving the IVPs with Local Fractional Derivative

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The IVPs with local fractional derivative are considered in this paper. Analytical solutions for the homogeneous and nonhomogeneous local fractional differential equations are discussed by using the Yang-Laplace transform.

1. Introduction

In recent years, the ordinary and partial differential equations have found applications in many problems in mathematical physics [1, 2]. Initial value problems (IVPs) for ordinary and partial differential equations have been developed by some authors in [3–6]. There are analytical methods and numerical methods for solving the differential equations, such as the finite element method [6], the harmonic wavelet method [7–9], the Adomian decomposition method [10–12], the homotopy analysis method [13, 14], the homotopy decomposition method [15, 16], the heat balance integral method [17, 18], the homotopy perturbation method [19], the variational iteration method [20], and other methods [21].

Recently, owing to limit of classical and fractional differential equations, the local fractional differential equations have been applied to describe nondifferentiable problems for the heat and wave in fractal media [22, 23], the structure relation in fractal elasticity [24], and Fokker-Planck equation in fractal media [25]. Some methods were utilized to solve the local fractional differential equations. For example, the local fractional variation iteration method was used to solve the heat conduction in fractal media [26, 27]. The local fractional decomposition method for solving the local fractional diffusion and heat-conduction equations was considered in [28, 29]. The local fractional series expansion method for solving the Schrödinger equation with the local fractional derivative was presented [30]. The Yang-Laplace transform structured in 2011 [22] was suggested to deal with local fractional differential equations [31, 32]. The coupling method for variational iteration method within Yang-Laplace transform for solving the heat conduction in fractal media was proposed in [33].

In this paper, our aim is to use the Yang-Laplace transform to solve IVPs with local fractional derivative. The structure of the paper is as follows. In Section 2, some definitions and properties for the Yang-Laplace transform are given. Section 3 is devoted to the solutions for the homogeneous and nonhomogeneous IVPs with local fractional derivative. Finally, conclusions are presented in Section 4.

2. Yang-Laplace Transform

In this section we show some definitions and properties for the Yang-Laplace transform.

The local fractional integral operator is defined as [22, 23, 26–33]

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)\,(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f\left(t_j\right)\left(\Delta t_j\right)^\alpha, \end{aligned} \tag{1}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_j, \ldots\}$, $[t_j, t_{j+1}]$, $j = 0, \ldots, N-1$, $t_0 = a, t_N = b$, is a partition of the interval $[a, b]$.

As the inverse operator of (1), the local fractional derivative operator is given by [22, 23, 26–33]

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \quad (2)$$

with $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$.

The Yang-Laplace transform is expressed by [22, 31–33]

$$\widetilde{L}_\alpha \{f(x)\} = f_s^{\widetilde{L},\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha, \quad 0 < \alpha \le 1, \quad (3)$$

where $f(x)$ is a local fractional continuous function.

The inverse Yang-Laplace transform reads as [22, 31–33]

$$f(x) = \widetilde{L}_\alpha^{-1}\left\{f_s^{L,\alpha}(s)\right\} = \frac{1}{(2\pi)^\alpha} \times \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{\widetilde{L},\alpha}(s) (ds)^\alpha, \quad (4)$$

where $s^\alpha = \beta^\alpha + i^\alpha \infty^\alpha$ and $\mathrm{Re}(s^\alpha) = \beta^\alpha$.

Some properties for Yang-Laplace transform are presented as follows [21, 22, 22–33]:

$$\widetilde{L}_\alpha \{af(x) + bg(x)\} = a\widetilde{L}_\alpha \{f(x)\} + b\widetilde{L}_\alpha \{g(x)\}, \quad (5)$$

$$\widetilde{L}_\alpha \left\{f^{(n\alpha)}(x)\right\} = s^{n\alpha}\widetilde{L}_\alpha \{f(x)\} - \sum_{k=1}^{n} s^{(k-1)\alpha} f^{(n-k)\alpha}(0), \quad (6)$$

$$\lim_{x \to 0} f(x) = \lim_{s \to \infty} s^\alpha F(s), \quad (7)$$

$$\lim_{x \to \infty} f(x) = \lim_{s \to 0} s^\alpha F(s), \quad (8)$$

$$\widetilde{L}_\alpha \{f(ax)\} = \frac{1}{a^\alpha} f_s^{L,\alpha}\left(\frac{s}{a}\right), \quad a > 0, \quad (9)$$

$$\widetilde{L}_\alpha \left\{x^{k\alpha} f(x)\right\} = (-1)^k \frac{d^{k\alpha} f_s^{L,\alpha}(s)}{ds^{k\alpha}}, \quad (10)$$

$$\widetilde{L}_\alpha \{f(x - c)\} = f_s^{L,\alpha}(s) E_\alpha(-c^\alpha s^\alpha), \quad (11)$$

$$\widetilde{L}_\alpha \{f(x) E_\alpha(c^\alpha x^\alpha)\} = f_s^{L,\alpha}(s - c), \quad (12)$$

$$\widetilde{L}_\alpha \left\{x^{k\alpha} E_\alpha(c^\alpha x^\alpha)\right\} = \frac{\Gamma(1 + k\alpha)}{(s - c)^{(k+1)\alpha}}, \quad (13)$$

$$\widetilde{L}_\alpha \{\sin_\alpha(c^\alpha x^\alpha)\} = \frac{c^\alpha}{s^{2\alpha} + c^{2\alpha}}, \quad (14)$$

$$\widetilde{L}_\alpha \{\cos_\alpha(c^\alpha x^\alpha)\} = \frac{s^\alpha}{s^{2\alpha} + c^{2\alpha}}, \quad (15)$$

$$\widetilde{L}_\alpha \left\{x^{k\alpha}\right\} = \frac{\Gamma(1 + k\alpha)}{s^{(k+1)\alpha}}. \quad (16)$$

3. IVPs with Local Fractional Derivatives

In this section we handle the homogeneous and nonhomogeneous IVPs with local fractional derivative.

3.1. Homogeneous IVPs with Local Fractional Derivative

Example 1. The homogeneous IVPs with local fractional derivative are expressed by

$$\frac{d^{2\alpha} y}{d^{2\alpha} x} - \frac{d^\alpha y}{d^\alpha x} + 2y = 0. \quad (17)$$

The initial boundary conditions are presented as

$$y(0) = 1, \qquad y^{(\alpha)}(0) = 0. \quad (18)$$

From (6) we have

$$\widetilde{L}_\alpha \left\{y^{(\alpha)}(x)\right\} = s^\alpha \widetilde{L}_\alpha \{y(x)\} - y(0), \quad (19)$$

$$\widetilde{L}_\alpha \left\{y^{(2\alpha)}(x)\right\} = s^{2\alpha} \widetilde{L}_\alpha \{y(x)\} - s^\alpha y(0) - f^{(\alpha)}(0). \quad (20)$$

Hence, making use of (19) and (20), (19) can be written as

$$s^{2\alpha} \widetilde{L}_\alpha \{y(x)\} - s^\alpha y(0) - f^{(\alpha)}(0) - \left\{s^\alpha \widetilde{L}_\alpha \{y(x)\} - y(0)\right\} + 2\widetilde{L}_\alpha \{y(x)\} = 0. \quad (21)$$

Hence, we obtain

$$\widetilde{L}_\alpha \{y(x)\} = \frac{1}{s^\alpha + 2} y(0) = \frac{1}{s^\alpha + 2}. \quad (22)$$

So, making use of (13), we get the solution of (17):

$$y(x) = E_\alpha(-2x^\alpha). \quad (23)$$

The solution of (17) for $\alpha = \ln 2/\ln 3$ is shown in Figure 1.

Example 2. Let us consider the homogeneous IVPs with local fractional derivative in the form

$$\frac{d^{4\alpha} y}{d^{4\alpha} x} - y = 0 \quad (24)$$

subject to initial boundary conditions

$$y(0) = 0, \qquad y^{(\alpha)}(0) = 0, \qquad y^{(2\alpha)}(0) = 0, \qquad y^{(3\alpha)}(0) = 1. \quad (25)$$

From (6) we have

$$\widetilde{L}_\alpha \left\{y^{(4\alpha)}(x)\right\} = s^{4\alpha} \widetilde{L}_\alpha \{y(x)\} - s^{3\alpha} y(0) - s^{2\alpha} y^{(\alpha)}(0) - s^\alpha y^{(2\alpha)}(0) - f^{(3\alpha)}(0), \quad (26)$$

so that

$$s^{4\alpha} \widetilde{L}_\alpha \{y(x)\} - s^{3\alpha} y(0) - s^{2\alpha} y^{(\alpha)}(0) - s^\alpha y^{(2\alpha)}(0) - f^{(3\alpha)}(0) - \widetilde{L}_\alpha \{y(x)\} = 0. \quad (27)$$

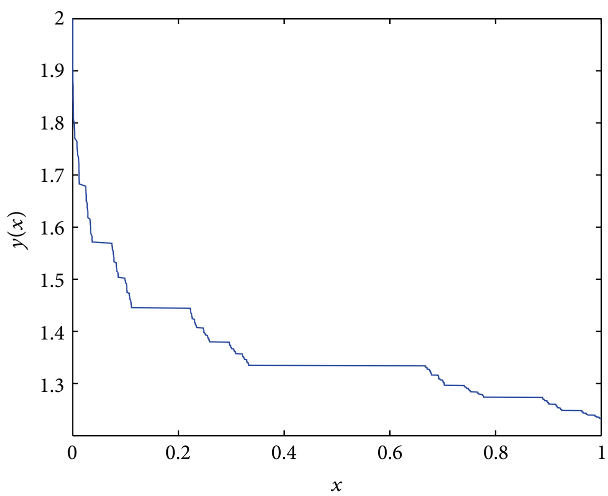

FIGURE 1: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

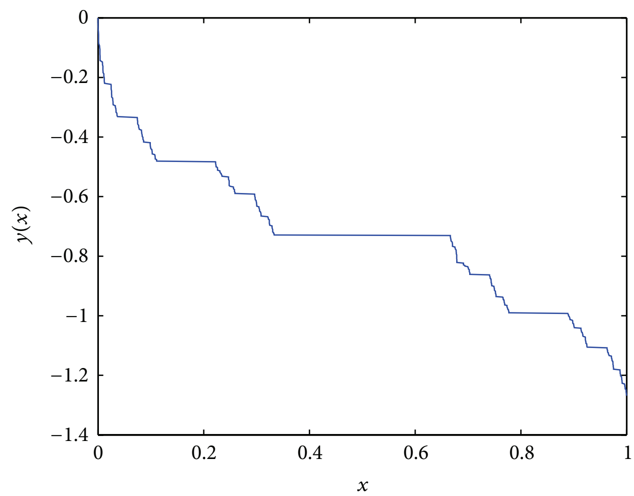

FIGURE 2: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

Hence, (27) can be written as

$$s^{4\alpha}\widetilde{L}_\alpha\{y(x)\} - 1 - \widetilde{L}_\alpha\{y(x)\} = 0, \tag{28}$$

which leads to

$$\widetilde{L}_\alpha\{y(x)\} = \frac{1}{s^{4\alpha}-1}. \tag{29}$$

Therefore, we get

$$\begin{aligned} y(x) &= \widetilde{L}_\alpha^{-1}\left\{\frac{1}{s^{4\alpha}-1}\right\} \\ &= \widetilde{L}_\alpha^{-1}\left\{\frac{1}{2}\left(\frac{1}{2}\frac{1}{s^\alpha-1} - \frac{1}{2}\frac{1}{s^\alpha+1} - \frac{1}{s^{2\alpha}+1}\right)\right\} \\ &= \frac{1}{4}E_\alpha(-x^\alpha) - \frac{1}{4}E_\alpha(x^\alpha) - \frac{1}{2}\sin_\alpha(x^\alpha). \end{aligned} \tag{30}$$

The exact solution of (24) for $\alpha = \ln 2/\ln 3$ is shown in Figure 2.

3.2. Nonhomogeneous IVPs with Local Fractional Derivative

Example 3. We now consider the non-homogeneous IVPs with local fractional derivative

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} - y = \sin_\alpha(x^\alpha) \tag{31}$$

subject to initial boundary conditions

$$y(0) = 0, \qquad y^{(\alpha)}(0) = 1. \tag{32}$$

By using (6), we have

$$\begin{aligned} \widetilde{L}_\alpha\left\{y^{(2\alpha)}(x)\right\} &= s^{2\alpha}\widetilde{L}_\alpha\{y(x)\} - s^\alpha y(0) - f^{(\alpha)}(0), \\ \widetilde{L}_\alpha\{\sin_\alpha(x^\alpha)\} &= \frac{1}{s^{2\alpha}+1} \end{aligned} \tag{33}$$

so that

$$\widetilde{L}_\alpha\{y(x)\} = \frac{3}{4}\left(\frac{1}{s^\alpha-1} - \frac{1}{s^\alpha+1}\right) - \frac{1}{2}\frac{1}{s^{2\alpha}+1}. \tag{34}$$

So,

$$y(x) = \frac{3}{4}E_\alpha(-x^\alpha) - \frac{3}{4}E_\alpha(x^\alpha) - \frac{1}{2}\sin_\alpha(x^\alpha). \tag{35}$$

The exact solution of (31) for $\alpha = \ln 2/\ln 3$ is shown in Figure 3.

Example 4. The non-homogeneous IVPs with local fractional derivative are

$$\frac{d^{2\alpha}y}{d^{2\alpha}x} + y = E_\alpha(x^\alpha). \tag{36}$$

The initial boundary conditions are

$$y(0) = 1, \qquad y^{(\alpha)}(0) = 0. \tag{37}$$

In view of (6), we give

$$\widetilde{L}_\alpha\{y(x)\} = \frac{1}{(s^\alpha+1)(s^{2\alpha}+1)} + \frac{s^\alpha}{s^{2\alpha}+1}. \tag{38}$$

So, we obtain

$$\begin{aligned} y(x) &= \cos_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)}\int_0^x E_\alpha(x-t)^\alpha \sin_\alpha(t^\alpha)(dt)^\alpha \\ &= \cos_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)} \\ &\quad \times \int_0^x E_\alpha(t^\alpha)(\sin_\alpha(x^\alpha)\cos_\alpha(t^\alpha) \\ &\qquad -\cos_\alpha(x^\alpha)\sin_\alpha(t^\alpha))(dt)^\alpha \end{aligned}$$

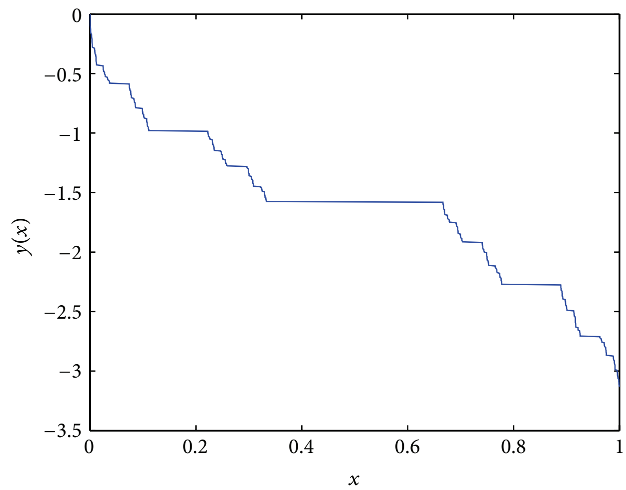

Figure 3: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

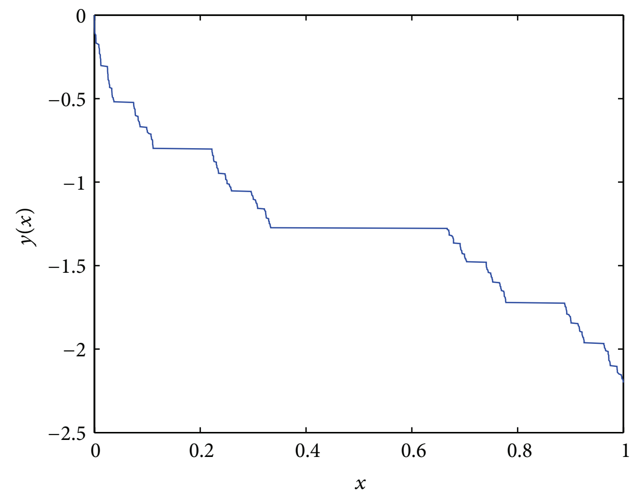

Figure 4: Graph of $y(x)$ for $\alpha = \ln 2/\ln 3$.

$$\begin{aligned}
&= \cos_\alpha\left(x^\alpha\right) \\
&\quad + \sin_\alpha\left(x^\alpha\right)\left\{\frac{1}{\Gamma(1+\alpha)}\int_0^x E_\alpha\left(t^\alpha\right)\cos_\alpha\left(t^\alpha\right)(dt)^\alpha\right\} \\
&\quad - \cos_\alpha\left(x^\alpha\right)\left\{\frac{1}{\Gamma(1+\alpha)}\int_0^x E_\alpha\left(t^\alpha\right)\sin_\alpha\left(t^\alpha\right)(dt)^\alpha\right\} \\
&= \cos_\alpha\left(x^\alpha\right) \\
&\quad + \frac{\sin_\alpha\left(x^\alpha\right)\left\{E_\alpha\left(x^\alpha\right)\left[\cos_\alpha\left(x^\alpha\right)+\sin\left(x^\alpha\right)\right]-1\right\}}{2} \\
&\quad - \frac{\cos_\alpha\left(x^\alpha\right)\left\{E_\alpha\left(x^\alpha\right)\left[\sin_\alpha\left(x^\alpha\right)-\cos_\alpha\left(x^\alpha\right)\right]+1\right\}}{2} \\
&= \frac{1}{2}\left[\cos_\alpha\left(x^\alpha\right)-\sin_\alpha\left(x^\alpha\right)+E_\alpha\left(x^\alpha\right)\right].
\end{aligned} \tag{39}$$

The exact solution of (36) for $\alpha = \ln 2/\ln 3$ is shown in Figure 4.

4. Conclusions

In this work we have used the Yang-Laplace transform to handle the homogeneous and non-homogeneous IVPs with looselocal fractional derivative. Some illustrative examples of approximate solutions for local fractional IVPs are discussed. The nondifferentiable solutions for fractal dimension $\alpha = \ln 2/\ln 3$ are shown graphically. The obtained results illustrate that the Yang-Laplace transform is an efficient mathematical tool to solve the homogeneous and non-homogeneous IVPs with local fractional derivative.

Conflict of Interests

The authors declare that there is no conflicts of interests regarding publication of this paper.

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