# Research Article 

# Bifurcation Structure in a Bimodal Piecewise Linear Business Cycle Model 

Viktor Avrutin, ${ }^{1,2}$ Iryna Sushko, ${ }^{3,4}$ and Fabio Tramontana ${ }^{5}$<br>${ }^{1}$ DESP, University of Urbino, Via Saffi 42, 61029 Urbino, Italy<br>${ }^{2}$ IST, University of Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany<br>${ }^{3}$ Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkovskaya Street 3, Kyiv 01601, Ukraine<br>${ }^{4}$ Kyiv School of Economics, Ivana Mazepy Street 1, Kyiv 01010, Ukraine<br>${ }^{5}$ Department of Economics and Management, University of Pavia, Via S. Felice 5, 27100 Pavia, Italy

Correspondence should be addressed to Fabio Tramontana; fabio.tramontana@unipv.it
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#### Abstract

We study the bifurcation structure of the parameter space of a 1 D continuous piecewise linear bimodal map which describes dynamics of a business cycle model introduced by Day-Shafer. In particular, we obtain the analytical expression of the boundaries of several periodicity regions associated with attracting cycles of the map (principal cycles and related fin structure). By crossing these boundaries the map displays robust chaos.


## 1. Introduction

Applied models defined by piecewise smooth functions appear quite often when one studies a real process characterized by some "nonsmooth" phenomena such as sharp switching between several states, impacts, friction, sliding, and the like. In economic modeling piecewise smooth systems arise, for example, taking into account that the most used economic variables have nonnegativity constraints, or when a process is studied in which the economic agents change their behavior when a relevant indicator reaches certain thresholds, and so forth. The main reason why economists still prefer to build their theoretical models avoiding piecewise smooth functions is related to the lack of knowledge and experience in the investigation of such a kind of models. In fact, a general theory for piecewise smooth dynamical systems, differently from the one for smooth systems, is not yet well established. The studies and results on these systems are growing and rapidly developing nowadays (see, e.g., the books [1, 2] and references therein). During the last decade important results have been obtained in this field, and one of the aims of the present paper is to show that such results can be successfully
applied to investigate a relevant economic model, proposed by Day and Shafer in [3].

We recall that among nonsmooth dynamical systems, those described by piecewise linear maps are the simplest to study due to the linearity of their components, but nevertheless they are quite rich in the outcome of the possible dynamics. In particular, a one-dimensional (1D for short) continuous piecewise linear map with one border point, known as skew tent map, depending on the parameters values can have attracting cycles of any period as well as cyclic chaotic intervals of any period, also called $n$-bands chaotic attractors, which have the relevant property of being robust (as introduced in [4]) with respect to parameter perturbations. The bifurcation structure of the skew tent map has been completely described (see, e.g., [5-8]). Moreover, the skew tent map can be used as a normal form for a socalled border collision bifurcation (BCB for short) which is characteristic in piecewise smooth maps [9, 10]. Recall that a BCB occurs when an invariant set, such as, for example, a fixed point or cycle, collides with a border separating regions of different definition of the map. This bifurcation may lead, for example, from an attracting cycle directly to chaos.

The dynamic behaviors of the skew tent map are used to classify the possible dynamics which may occur after a generic BCB in a 1D continuous piecewise smooth map (see, e.g., [11-13], where the skew tent map is applied to classify BCBs in economic models).

The map considered in the present paper, which represents an economic model, is described by a one-dimensional (1D for short) bimodal piecewise linear map with increasing outermost branches. Clearly, a map with two border points possesses more complicated dynamics, and all the possible outcomes are not yet fully investigated; some results can be found in [14, 15]. In particular, the bifurcation structure of its parameter space includes both regions which belong to the known period adding structure (called also Arnold tongues or mode-locking tongues), which is characteristic for piecewise increasing discontinuous maps and also for certain circle maps (see, e.g., [16-20]). The period adding structure is formed by periodicity regions related to cycles organized according to the Farey summation rule applied to the rotation numbers of the related cycles. Besides this, in bimodal piecewise linear maps the so-called fin structure is also observed (see [15]). We will describe these regions in the parameter space of the considered map, recalling how these two structures are organized and giving formulas of the boundaries of related regions.

The plan of the work is as follows. In Section 2 we recall the Day-Shafer model. Its dynamics are bounded in an absorbing interval, and depending on the parameters values the system may have attracting cycles of any period as well as $n$-band chaotic attractors. In Section 3 we will consider the parameter space of interest, showing how the periodicity regions representing the regions in the parameter space associated with stable attracting cycles may be organized. Moreover, the boundaries of such regions (BCB curves) are obtained analytically (and reported in Appendices). Crossing these boundaries the system may either enter a different periodicity region (via the fin structure mentioned above) or it may become chaotic (and in a regime of robust chaos). Some conclusions are given in Section 4.

## 2. The Day-Shafer Model

The Day-Shafer model we are interested in dates back to 1987. Richard Day has been a pioneer in the application of nonlinear models in economics and finance (see, e.g., [2124]). In particular, in [3] Day and Shafer argued that the trapping set of their nonlinear business cycle model in one of the most interesting cases is well approximated by a piecewise smooth map with two turning points with dynamics bounded in an absorbing interval. Moreover, they explicitly consider a piecewise linear bimodal map as a further approximation. In the particular case in which the dynamics reduces to those on a unimodal piecewise linear map (i.e., a skew tent map), they succeeded in writing the BCB curves associated with the principal (or maximal) cycles. While in the generic case of a piecewise linear bimodal map, only a few numerical results were given. As we will show in the next sections, the results
of the generic case with two turning points can be much improved and many BCB curves can be detected analytically.

Let us briefly recall that in [3] Day and Shafer build a generic map defining a business cycle model of an economy with monetary and real sectors. In the monetary market, demand and supply for money are implicitly given by the interest rate $r$ as a function of the income $Y$ and a money supply parameter $M$. In the real market the interest rate determines the level of investments $I=H(r(Y), Y)=$ $I(Y)$, that is, a component of the income together with consumption $C=C(Y)$. By using the assumption that the level of income at a certain time period depends upon the one-time lagged amounts of consumption and investment, a discrete time equation of the following form is obtained:

$$
\begin{equation*}
Y_{t+1}=F\left(Y_{t}\right)=\lambda I\left(Y_{t}\right)+C\left(Y_{t}\right)+A \tag{1}
\end{equation*}
$$

where the parameter $\lambda \geq 0$ measures how strongly investment translates into new income, and the parameter $A>$ 0 includes all the autonomous components (of investment, consumption, and public expenditure). The function $C(Y)$ is typically monotonically increasing with $C(0)=0$, and the investment function $I(Y)$ usually is increasing for low level of income while it is decreasing when income is high (this is due to the fact that with high income the money market becomes too crowded, inducing the increase of the interest rate that causes a contraction in the investments). Putting together these assumptions a function $F(Y)$ is obtained that is nonlinear and has a bimodal shape. By using explicit functions, Day and Shafer found that for some parameters' configurations the income dynamics are bounded in a trapping set like the one shown in Figure 1. The piecewise linear map shown in Figure 1 represents the function $F(Y)$; it is continuous but not differentiable at the points $Y=Y^{*}$ and $Y=Y^{* *}$. The dynamics are bounded in the interval $\left[F\left(Y^{* *}\right), F\left(Y^{*}\right)\right]=$ [ $Y^{m}, Y^{M}$ ]. For convenience, we can normalize the interval [ $Y^{m}, Y^{M}$ ] to $[0,1]$ and denoting by $x$ the state variable income $Y$ we obtain the map $f: I \rightarrow I, I=[0,1]$ defined as follows:

$$
\begin{align*}
f: x & \longmapsto f(x) \\
& = \begin{cases}f_{L}(x)=a_{L} x+1-a_{L} d_{L} & \text { if } 0 \leq x<d_{L} \\
f_{M}(x)=-\frac{x}{d_{R}-d_{L}}+1+\frac{d_{L}}{d_{R}-d_{L}} & \text { if } d_{L} \leq x<d_{R} \\
f_{R}(x)=a_{R} x-a_{R} d_{R} & \text { if } d_{R} \leq x \leq 1\end{cases} \tag{2}
\end{align*}
$$

where $f_{L}\left(d_{L}\right)=f_{M}\left(d_{L}\right)=1, f_{R}\left(d_{R}\right)=f_{M}\left(d_{R}\right)=0$, and the parameters satisfy

$$
\begin{align*}
a_{L}>0, & a_{R}>0, \quad 0<d_{L}<d_{R}<1,  \tag{3}\\
f(0) & >0, \quad f(1)<1
\end{align*}
$$

so that $f(I)=I$. More correctly, this is a version of the map in which we can investigate the dynamic behavior changing independently the slopes of the external branches and border points. In [3] the authors give the relations between the parameters here considered and those related to the economic model. Moreover, the parameters used above


Figure 1: The function $F(Y)$ in the trapping interval $\left[Y^{m}, Y^{M}\right]$.
(slopes and border points) may all or in part depend on some economic parameters, so that as an economic parameter is varied, it is also possible that it influences several of the above parameters, leading to particular paths in the parameter space.

Clearly, this map (2) has a unique fixed point in the middle branch given by

$$
\begin{equation*}
x_{M}^{*}=\frac{d_{R}}{1+d_{R}-d_{L}} \tag{4}
\end{equation*}
$$

which is always unstable given that the slope of the middle branch $\left(-1 /\left(d_{R}-d_{L}\right)\right)$ is always smaller than -1 . The interval $I$ is trapping so in $I$ we do not have any divergent trajectory, and the attracting set may be periodic (an $n$-cycle with $n>1$ ) or chaotic (a chaotic interval or an $n$-band chaotic attractor with $n>1$ ). Some dynamic behaviors and bifurcation structures of map (2) will be described in the next section.

## 3. Bifurcation Structure of the Parameter Space

The Day-Shafer model presented in the previous section is described by a family of 1D continuous piecewise linear maps $f: I \rightarrow I, I=[0,1]$, as given in (2), where the parameter region of interest, which allows obtaining the required shape of the map, is the region $P$ defined as follows (as $f(0)>0$ leads to $a_{L} d_{R}<1$ while $f(1)<1$ leads to $\left.a_{R}\left(1-d_{R}\right)<1\right)$ :

$$
\begin{gather*}
P=\left\{p: a_{L}>0, a_{R}>0,0<d_{L}<d_{R}<1\right.  \tag{5}\\
\left.a_{L} d_{R}<1, a_{R}\left(1-d_{R}\right)<1\right\}
\end{gather*}
$$

where $p=\left(a_{L}, a_{R}, d_{L}, d_{R}\right)$ denotes a point in the parameter space. For $p \in P$ the left and the right branches of map $f$ are both increasing, while the middle one is decreasing, so


Figure 2: The map $f$ given in (2) for $a_{L}=0.4, a_{R}=0.8, d_{L}=0.3$, and $d_{R}=0.8$.
that $f$ is a bimodal map. An example is shown in Figure 2. As already mentioned in Section 1, the dynamics of 1D bimodal maps have been considered by many researchers (see, e.g., [14, 25-27], etc.). Our aim is to study the bifurcation structure of the region $P$, that is, to describe the possible attractors of map $f$ and the parameter regions corresponding to their existence.

Let us denote by $I_{L}=\left[0, d_{L}\right), I_{M}=\left[d_{L}, d_{R}\right)$, and $I_{R}=\left[d_{R}, 1\right]$ the definition intervals (or, in other words, partitions) of the functions $f_{L}, f_{M}$, and $f_{R}$, respectively. They are separated by the border points $d_{L}$ and $d_{R}$. As already remarked, for $p \in P$, the fixed point is $x_{M}^{*} \in I_{M}$, given in (4), always exists, and is repelling, while the fixed points

$$
\begin{equation*}
x_{L}^{*}=\frac{a_{L} d_{L}-1}{a_{L}-1}, \quad x_{R}^{*}=\frac{a_{R} d_{R}}{a_{R}-1}, \tag{6}
\end{equation*}
$$

associated with the branches $f_{L}$ and $f_{R}$, exist (repelling) for $a_{L}>1$ and $a_{R}>1$. However, for the considered parameter values these fixed points do not belong to the interval $I$ given that $x_{L}^{*}<0$ and $x_{R}^{*}>1$. For our map $f$ the absorbing interval is always associated with all three branches. In fact, given that $f\left(d_{L}\right)=1>d_{R}$ and $f\left(d_{R}\right)=0<d_{L}$ it follows that $I$ is an absorbing invariant interval, $f(I)=I$ (see Figure 2), so that any orbit with an initial value $x_{0} \in I$ is bounded, being trapped in $I$. We denote by $\varphi_{L}$ and $\varphi_{R}$ the boundaries related to the contact of the interval $I$ with the fixed points $x_{L}^{*}$ and $x_{R}^{*}$, given by $x_{L}^{*}=0$ and $x_{R}^{*}=1$, respectively (although not occurring for parameters in $P$ ):

$$
\begin{gather*}
\varphi_{L}: a_{L} d_{L}=1,  \tag{7}\\
\varphi_{R}: a_{R}\left(1-d_{R}\right)=1 . \tag{8}
\end{gather*}
$$

Suppose that $\left\{x_{i}\right\}_{i=1}^{n}$ are the points of an $n$-cycle of map $f$. The symbolic representation of such a cycle is $\sigma=$ $s_{1} s_{2} \cdots s_{n}$, obtained associating to each point $x_{i}$ the symbol $s_{i} \in\{L, M, R\}$ depending on the partition $I_{L}, I_{M}$, or $I_{R}$ which the point $x_{i}$ belongs to. In the following, to denote an $n$ cycle we use its symbolic representation. The region in the parameter space related to the existence and stability of a cycle with symbolic sequence $\sigma$ is denoted by $P_{\sigma}$ and called periodicity region. Clearly, the boundaries of a periodicity region can be related either to the stability loss of the cycle


Figure 3: 2D bifurcation diagram of the map $f$ in the ( $a_{L}, a_{R}$ )-parameter plane in (a), at $d_{L}=0.4, d_{R}=0.8$, and an enlargement of the indicated window in (b). The color bar indicates the correspondence of a color and the period of the related cycle.
due to its eigenvalue crossing $\pm 1$ (recall that for a piecewise linear map such bifurcations are degenerate, see [8]) or to the appearance/disappearance of the cycle due to a border collision bifurcation (see [9]). We recall that if some point of a cycle collides with a border point and neither the period nor the stability of the cycle changes after the collision, we say that this cycle undergoes a persistence border collision, while a border collision bifurcation (BCB) occurs when a qualitative change in the dynamics is observed after the collision.

As noticed in [15], the overall bifurcation structure of the parameter space for a generic 1D bimodal piecewise linear map is characterized by several substructures among which we recall the skew tent map structure, the period adding structure, and a particular one called fin structure (due to the shape of the periodicity regions, as it will be clear below). The simplest one is the skew tent map structure associated with absorbing intervals involving only two adjacent partitions, so that on these absorbing intervals the map is reduced to a skew tent map. However, as already mentioned, by definition for our map $f$ such a possibility is excluded. That is, the parameter space of $f$ does not include the skew tent map structure. The period adding structure is associated with periodicity regions related to attracting cycles whose points belong to the outermost partitions only. Such periodicity regions are ordered according to the Farey summation rule applied to the rotation numbers of the related cycles. This bifurcation structure is observed for our map $f$ and we describe it in detail in Section 3.3. The fin structure which also reveals itself in the parameter space of map $f$ (See Section 3.3
for a deeper analysis) is closely related to the period adding structure being formed by the periodicity regions contiguous to the regions of the period adding structure and related to attracting cycles with just one point belonging to the middle partition and all the other points belonging to the outermost partitions.
3.1. Two-Dimensional Bifurcation Diagrams. We first present a few 2D bifurcation diagrams in various parameter planes to illustrate the overall bifurcation structure of the parameter space of map $f$. In particular, Figure 3 shows such a diagram and its enlargement in the $\left(a_{L}, a_{R}\right)$-parameter plane for $d_{L}=$ $0.4, d_{R}=0.8,0<a_{L}<1 / d_{L}=2.5$, and $0<a_{R}<1 /(1-$ $\left.d_{R}\right)=5$. Here different colors are related to attracting cycles of different periods $n \leq 30$, where the correspondence of a color and the period is indicated in the color bar; white region corresponds either to chaotic attractors or to cycles of higher periodicity. The gray region corresponds to $a_{L}>1, a_{R}>1$, so that all the slopes of $f$ are larger than 1 in modulus; thus, attracting cycles cannot exist, and it is associated with chaotic attractors only (cyclic chaotic intervals).

The choice of the parameters $a_{L}$ and $a_{R}$ to be varied for fixed values of $d_{L}$ and $d_{R}$, as in Figure 3, is not optimal to illustrate the bifurcation structures typical for piecewise linear bimodal maps. To this purpose it is better to fix values for $a_{L}$ and $a_{R}$ and vary $d_{L}$ and $d_{R}$, as it is shown in Figures 4 and 5. In these figures, the characteristic shapes of the periodicity regions belonging to the period adding structure


Figure 4: 2D bifurcation diagram of $f$ in the $\left(d_{L}, d_{R}\right)$-parameter plane for $a_{L}=0.5, a_{R}=1.01$ in (a), and $a_{L}=0.6, a_{R}=1.1$ in (b). The small red region in (b) (see left bottom), defined by $d_{L}<d_{R}<1-1 / a_{R}$, is related to divergent orbits and not involved in the economic model.
are visible. Such regions have one side on the straight line $d_{L}=$ $d_{R}$ and also are observable regions which are contiguous (or attached) to the regions of the period adding structure and are those constituting the "fin structure." A third example in the $\left(d_{L}, d_{R}\right)$-parameter plane is shown in Figure 5 at $a_{L}=0.7$ and $a_{R}=0.8$ fixed. We will see how to get the analytic equations of the BCB curves bounding the periodicity regions evidenced in the figures.
3.2. Period Adding Structure. Let us briefly recall the main elements of the period adding structure. As already mentioned, the periodicity regions constituting this structure are related to attracting $n$-cycles, $n \geq 2$, whose points only belong to the partitions $I_{L}$ and $I_{R}$; that is, their symbolic sequences does not include the symbol $M$.

Following [16] all the cycles associated with the period adding structure are grouped into families according to complexity levels. The complexity level one includes two families, denoted by $\Sigma_{1,1}$ and $\Sigma_{2,1}$, to which the so-called basic cycles belong:

$$
\begin{equation*}
\Sigma_{1,1}=\left\{L R^{n_{1}}\right\}_{n_{1}=1}^{\infty}, \quad \Sigma_{2,1}=\left\{R L^{n_{1}}\right\}_{n_{1}=1}^{\infty} \tag{9}
\end{equation*}
$$

Note that the "central" cycle $L R \equiv R L$ belongs to both families. To get the symbolic sequences of the cycles of families of complexity level two we apply to the families $\Sigma_{1,1}$ and $\Sigma_{2,1}$ the following symbolic replacements:

$$
\kappa_{m}^{L}:=\left\{\begin{array}{l}
L \longrightarrow L R^{m}  \tag{10}\\
R \longrightarrow R L R^{m},
\end{array} \quad \kappa_{m}^{R}:=\left\{\begin{array}{l}
L \longrightarrow L R L^{m} \\
R \longrightarrow R L^{m} .
\end{array}\right.\right.
$$

This method is based on the map replacement technique (see [19, 20]). Namely, at first we substitute in $\Sigma_{1,1}$ each symbol $L$ by $L R^{m}$ and each symbol $R$ by $R L R^{m}$ (replacement $\kappa_{m}^{L}$ ), and then we substitute in $\Sigma_{1,1}$ each symbol $L$ by $L R L^{m}$ and each symbol $R$ by $R L^{m}$ (replacement $\kappa_{m}^{R}$ ). Then the index $m$ is set $m=n_{2}$ in order to write the two families of complexity level two, respectively, as follows:

$$
\begin{align*}
& \Sigma_{1,2}=\left\{L R^{n_{2}}\left(R L R^{n_{2}}\right)^{n_{1}}\right\}_{n_{1}, n_{2}=1}^{\infty},  \tag{11}\\
& \Sigma_{2,2}=\left\{L R L^{n_{2}}\left(R L^{n_{2}}\right)^{n_{1}}\right\}_{n_{1}, n_{2}=1}^{\infty} .
\end{align*}
$$

We notice that the replacement technique is here used to detect the symbolic representation of the existing cycles. However, the same technique is used also to get the equations of the BCB curves of cycles of complexity level higher than one, starting from the equations of those of complexity level one.

Similarly, applying the replacements $\kappa_{m}^{L}$ and $\kappa_{m}^{R}$ to $\Sigma_{2,1}$ we get the symbolic sequences of two more families:

$$
\begin{align*}
& \Sigma_{3,2}=\left\{R L R^{n_{2}}\left(L R^{n_{2}}\right)^{n_{1}}\right\}_{n_{1}, n_{2}=1}^{\infty}, \\
& \Sigma_{4,2}=\left\{R L^{n_{2}}\left(L R L^{n_{2}}\right)^{n_{1}}\right\}_{n_{1}, n_{2}=1}^{\infty} . \tag{12}
\end{align*}
$$

Note that the central cycle $L R R L R \equiv R L R L R$ belongs to both families $\Sigma_{1,2}$ and $\Sigma_{3,2}$, with $n_{1}=n_{2}=1$, while the central cycle $L R L R L \equiv R L L R L$ belongs to both families $\Sigma_{2,2}$ and $\Sigma_{4,2}$, with $n_{1}=n_{2}=1$. All the other cycles in these


Figure 5: 2D bifurcation diagram of $f$ in the $\left(d_{L}, d_{R}\right)$-parameter plane for $a_{L}=0.7, a_{R}=0.8$.
families are distinct. In short this procedure can be written as $\Sigma_{1,2}=\kappa_{n_{2}}^{L}\left(\Sigma_{1,1}\right), \Sigma_{2,2}=\kappa_{n_{2}}^{R}\left(\Sigma_{1,1}\right), \Sigma_{3,2}=\kappa_{n_{2}}^{L}\left(\Sigma_{2,1}\right)$, and $\Sigma_{4,2}=$ $\kappa_{n_{2}}^{R}\left(\Sigma_{2,1}\right)$. So, we get 4 families of complexity level two (one more way to construct the families of the complexity level two consists in consecutive application of the concatenation rule to the "neighbour" symbolic sequences of the first complexity level. Symbolic sequences obtained in such a way are shift invariant to those obtained by symbolic replacements (10) (see $[19,20])$ ). Further, applying the replacements (10) with $m=n_{3}$ to the families of complexity level two we obtain $2^{3}$ families $\Sigma_{j, 3}, j=1, \ldots, 2^{3}$, of complexity level three, and so on. In this way all the symbolic sequences of cycles associated with the period adding structure are obtained.

Now let us turn to the boundaries of the periodicity regions (i.e., the BCB curves) related to the cycles of map $f$ associated with the period adding structure. They can be confined by the boundaries related to their existence. A cycle disappears when are crossed these boundaries, which are related to the BCBs occurring when a point of the cycle close to a border point, $x=d_{L}$ or $x=d_{R}$, collides with it in a saddle-node border collision bifurcation (merging with a companion unstable cycle). For the boundaries of a periodicity region associated with the stability of the cycle, first note that an $n$-cycle whose symbolic sequence $\sigma$ does not include the symbol $M$ has multiplier $\lambda_{\sigma}=a_{L}^{k} a_{R}^{n-k}>0$, where $k$ and $n-k$ are the numbers of symbols $L$ and $R$, respectively, in $\sigma$. Thus, a degenerate flip bifurcation (DFB for short), related to $\lambda_{\sigma}=-1$, cannot occur for such a cycle, so that the periodicity regions of the period adding structure cannot have DFB boundaries. A degenerate +1 bifurcation (DB1 for
short) associated with $\lambda_{\sigma}=1$ occurs if $a_{L}^{k} a_{R}^{n-k}=1$. Obviously, this condition defines a boundary of the periodicity region only if the related cycle exists.

To illustrate how the periodicity regions of the period adding structure are ordered let us consider first the limit case $d_{L}=d_{R} \equiv d$ at which the considered map becomes discontinuous, say $\tilde{f}$,

$$
x \longmapsto \tilde{f}(x)= \begin{cases}\tilde{f}_{L}(x)=a_{L} x+1-a_{L} d & \text { if } 0 \leq x<d  \tag{13}\\ \widetilde{f}_{R}(x)=a_{R} x-a_{R} d & \text { if } d<x \leq 1\end{cases}
$$

In Figures 4 and 5, when the parameters belong to the straight line $d_{L}=d_{R}$ then the continuous bimodal map $f$ reduces to the discontinuous one, $\tilde{f}$.

The dynamics of 1 D discontinuous piecewise monotone maps have been studied by many researchers (see, e.g., [17, 18, 28, 29]). In particular, the piecewise linear case has been recently reconsidered (after [16]) in [19, 20]. In the cited references, it is described as the period adding structure which is characteristic for piecewise increasing maps, when invertible on the absorbing interval. It is easy to check that map $\tilde{f}$ is invertible on the absorbing interval $I$ if $f_{L}(f(d))>$ $f_{R}(f(d))$, that is, if $f_{L}(0)>f_{R}(1)$, in which case $\tilde{f}$ is called gap map. It is noninvertible if $f_{L}(0)<f_{R}(1)$ being also called overlapping map, while for $f_{L}(0)=f_{R}(1)$ map $\tilde{f}$ is called circle map. The boundary defined by $f_{L}(0)=f_{R}(1)$, that holds for

$$
\begin{align*}
\kappa=\{ & \left(a_{L}, a_{R}, d\right): a_{L}>0, a_{R}>0,  \tag{14}\\
& \left.0<d<1,1-d a_{L}=a_{R}(1-d)\right\},
\end{align*}
$$

is related to the changes of invertibility of $\tilde{f}$. In Figure 6 we show the 2D bifurcation diagram of $\tilde{f}$ in the $\left(d, a_{R}\right)$ parameter plane for $a_{L}=0.5$, and the curve $\kappa$ is there plotted. Note that the dynamics along the straight line defined by $a_{R}=1.1$ is related to the dynamics along the straight line $d_{L}=d_{R}$ in Figure 4(a). Below the curve $\kappa$ one can observe the period adding structure. Above the curve $\varphi_{R}$ given in (8) a generic trajectory of $\tilde{f}$ diverges, while in between $\varphi_{R}$ and $\kappa$ the existing attractors of the map are chaotic intervals.

The period adding structure in map $\tilde{f}$ is illustrated by the 1D bifurcation diagram in Figure 7 corresponding to the cross section along the horizontal line with an arrow indicated in Figure 6. Note that for $\varphi_{R}<d<\kappa$, where $\varphi_{R} \approx 0.167, \kappa \approx$ 0.286 , the attractor is chaotic. It can be also clearly seen that the boundaries of the periodicity regions, for example, of the 2 -cycle, are related to the collision of the points of the cycle with the border point $x=d$.

In Appendix A we give the analytic equations of the boundaries of the periodicity regions of the period adding structure for map $f$, which holds also for map $\tilde{f}$ substituting $d_{L}=d_{R}=d$. The periodicity regions of complexity level one and two of map $f$, corresponding to the bifurcation diagram presented in Figure 5, are shown in Figure 8 by light gray and dark gray regions, respectively. In that figure, the boundaries of the gray regions are plotted using the formulas given in Appendix A. For example, after simplifications we get that


Figure 6: 2D bifurcation diagram of map $\tilde{f}$ in the $\left(d, a_{R}\right)$-parameter plane for $a_{L}=0.5$.


Figure 7: 1D bifurcation diagram of map $\tilde{f}$ for $a_{L}=0.5, a_{R}=1.2$, and $d \in\left(\varphi_{R}, 1\right)$, where $\varphi_{R} \approx 0.167$. The related parameter path is indicated in Figure 6 by the horizontal line with an arrow.
periodicity region $P_{R L}$ related to the attracting 2-cycle $R L$ of map $f$ is defined as follows:

$$
\begin{equation*}
P_{R L}=\left\{p \in P: 1-\frac{d_{L}}{a_{R}}<d_{R}<1-a_{L} d_{L}\right\}, \tag{15}
\end{equation*}
$$

and the equations $d_{R}=1-\left(d_{L} / a_{R}\right), d_{R}=1-a_{L} d_{L}$ define the two BCB curves giving the boundaries of $P_{R L}$. For fixed $d_{L}=0.4, d_{R}=0.8$, as in Figure 3, the boundaries of $P_{R L}$ are just segments of the vertical and horizontal straight lines


Figure 8: Periodicity regions of the cycles $L R^{n}$ and $R L^{n}, n=$ $1, \ldots, 11$, of complexity level one are shown in light gray. A few periodicity regions of complexity level two are shown in dark gray. Here $a_{L}=0.7, a_{R}=0.8$ as in Figure 5.
defined by $a_{L}=\left(1-d_{R}\right) / d_{L}=0.5$ and $a_{R}=d_{L} /\left(1-d_{R}\right)=2$, respectively.
3.3. Fin Structure. As already mentioned, the fin structure in the parameter space of a 1D bimodal piecewise linear map consists of periodicity regions which are attached to the regions of the period adding structure described in the previous section. For example, one can clearly see in Figure 5 two $2 \cdot 2$-periodicity regions and two $2 \cdot 3$-periodicity regions attached on both sides to the period-2 region $P_{R L}$, as well as $3 \cdot 2$-, $3 \cdot 3$ - and $3 \cdot 4$-periodicity regions attached on both sides to the period-3 regions $P_{R L^{2}}$ and $P_{L R^{2}}$, and so on. These regions belong to the fin structure which is formed by the periodicity regions called $n \cdot k$-fins, $k \geq 1$, related to attracting cycles having only one point in the interval $I_{M}$ and all the other points are in $I_{L}$ and $I_{R}$. The n-periodicity region of the period adding structure to which a fin is attached is called trunk region, and its fins have the same complexity level as the complexity level of the trunk. In fact, in Figure 5, two 2•1-fins of complexity level one of the trunk regions $P_{R L}$ can also be seen.

A fin cycle can appear if the parameter point crosses one of two BCB boundaries of a trunk, due to which one periodic point enters the partition $I_{M}$. As explained in [15], for an $n$. $k$-cycle whose periodicity region has the common boundary with the region $P_{L R^{n-1}}$, the symbolic sequences of the cycles in the fins are $\left(L R^{n-1}\right)^{k-1} M R^{n-1}, k=1,2, \ldots$, on one side of the region $P_{L R^{n-1}}$, and $\left(L R^{n-1}\right)^{k-1} L R^{n-2} M$ on the other side. Interchanging $L$ and $R$ in these sequences we get the symbolic sequences of the cycles related to $n \cdot k$-fins whose trunks are $P_{R L^{n-1}}$ regions. The number of existing fins depends on the parameters and, in fact, some trunk regions may have fins on one side only, or have no fins at all, as can be seen, for example, in Figure 4(b).

As for the boundaries of a fin, it can be shown that each $n \cdot k$-fin region for $k \geq 2, n \geq 2$, related to a cycle with symbolic
sequence $\sigma$, has at most four boundaries, among which one is the common BCB boundary with the related trunk region, one boundary is related to DFB of the cycle (whose eigenvalue is $\lambda_{\sigma}=a_{L}^{m} a_{R}^{n k-m-1} /\left(d_{L}-d_{R}\right)<0$, where $m$ is the number of symbols $L$ in $\sigma$ ), and two other boundaries are related to two more BCBs of the cycle. Each $n \cdot 1$-fin region has only three boundaries, namely, one DFB boundary and two BCB boundaries. The DFB boundary is defined by the condition $\lambda_{\sigma}=-1$, while the BCB boundaries are obtained using the skew tent map as border collision normal form.

In Appendix B, some basic formulas related to the boundaries of a fin region are given, while below, as an example, we describe the $2 \cdot 1$ - and $2 \cdot 2$-fins of the region $P_{R L}$ shown in Figure 9. There are two $2 \cdot 1$-fins, $P_{M R}$ and $P_{L M}$, contiguous to the trunk regions $P_{R L}$. As we mentioned above, the $2 \cdot 1$-fins are exceptional being confined by three boundaries instead of four. In particular, the fin $P_{M R}$ is confined by the BCB and DFB boundaries

$$
\begin{align*}
& B C_{L R}=\left\{p \in P: d_{R}=1-\frac{d_{L}}{a_{R}}\right\}  \tag{16}\\
& D F_{M R}=\left\{p \in P: d_{R}=d_{L}+a_{R}\right\}
\end{align*}
$$

respectively, and the boundary defined by $d_{L}=0$. The fin $P_{L M}$ is confined by the BCB and DFB boundaries

$$
\begin{align*}
& B C_{R L}=\left\{p \in P: d_{R}=1-a_{L} d_{L}\right\},  \tag{17}\\
& D F_{L M}=\left\{p \in P: d_{R}=d_{L}+a_{L}\right\},
\end{align*}
$$

respectively, and the boundary $d_{R}=1$.
Next, let us consider the $2 \cdot 2$-fins, $P_{L R M R}$ and $P_{L R L M}$, which are also contiguous to the trunk regions $P_{R L}$. Applying the formulas given in Appendix B we get that the boundaries of the fin $P_{L R M R}$ are given by the BCB curves satisfying the equations given below:

$$
\begin{gather*}
B C_{L R}=\left\{p \in P: d_{R}=1-\frac{d_{L}}{a_{R}}\right\}, \\
D F_{L R M R}=\left\{p \in P: d_{R}=d_{L}+a_{R}^{2} a_{L}\right\}, \\
B C_{L R M R}^{1}=\left\{p \in P: d_{R}=d_{L}+a_{R}\right\},  \tag{18}\\
B C_{L R M R}^{2}=\left\{p \in P: d_{L}=\frac{d_{R}^{2}-\left(a_{R}+1\right) d_{R}+a_{R}}{a_{R} a_{L}+d_{R}}\right\} .
\end{gather*}
$$

Note that $B C_{L R M R}^{1}=D F_{M R}$; that is, the DFB of the cycle $L M$ occurs simultaneously with the BCB of the cycle $L R M R$ (see Figure 9). The fin $P_{\text {LRLM }}$ is confined by the following boundaries:

$$
\begin{gather*}
B C_{R L}=\left\{p \in P: d_{R}=1-a_{L} d_{L}\right\}, \\
D F_{L R L M}=\left\{p \in P: d_{R}=d_{L}+a_{R} a_{L}^{2}\right\}, \\
B C_{L R L M}^{1}=\left\{p \in P: d_{R}=d_{L}+a_{L}\right\},  \tag{19}\\
B C_{L R L M}^{2}=\left\{p \in P: d_{R}=\frac{d_{L}^{2}+a_{L}\left(d_{L}-a_{R}\right)-1}{d_{L}-1-a_{L} a_{R}}\right\},
\end{gather*}
$$



Figure 9: The 2•1-fins $P_{M R}$ and $P_{L M}$, and 2•2-fins $P_{L R M R}$ and $P_{L R L M}$, contiguous to the trunk regions $P_{L R}$. Here $a_{L}=0.7, a_{R}=0.8$ as in Figure 5.
and, as already remarked in the previous case, here also we have $B C_{L R L M}^{1}=D F_{L M}$ (see Figure 9).

We note, from the point of view of our specific economic application, the presence of fin cycles is important because it makes it possible to reach intermediate levels of income (those belonging to the middle branch) that are excluded in period adding cycles. Moreover, the fin structure makes the bifurcation diagram even more complicated and as a consequence it becomes quite hard to foresee what would happen when a parameter of the map varies.

## 4. Conclusions

In this work we have considered a pioneering model by Day and Shafer [3] which describes a business cycle by using a bimodal piecewise linear map. Our investigation shows how rich are the dynamic behaviors of the system, going from attracting cycles of any period to robust chaotic intervals, depending on the parameters values. By using recently developed techniques, we have studied a typical bifurcation diagram and obtained analytically the border collision bifurcation curves that separate different periodicity regions and degenerate flip bifurcation curves. The transition to chaos was also studied.

Complex dynamics are usually the consequence of the introduction of some nonlinearity in a system. Moreover, specially in economics, it is quite natural to have a nonlinearity coming simply by some constraint. That is, an economic model is often characterized by different functional definitions depending on some threshold reached by the dynamic variables (income, prices, etc.). In this way, the models are described by piecewise smooth systems (in place of smooth ones), whose theoretical results are still under study among scholars in dynamical systems. Thus, results similar to those of our model can be obtained in the study of other continuous piecewise linear systems.

## Appendices

## A.

Consider a generic family of 1D continuous piecewise linear bimodal maps $g: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$
\begin{align*}
g: & x \longmapsto g(x) \\
& = \begin{cases}g_{L}(x)=a_{L} x+\mu_{L} & \text { if } x<d_{L}, \\
g_{M}(x)=a_{M} x+\mu_{M} & \text { if } d_{L} \leq x \leq d_{R} \\
g_{R}(x)=a_{R} x+\mu_{R} & \text { if } x>d_{R},\end{cases} \tag{A.1}
\end{align*}
$$

where $a_{L}>0, a_{R}>0, a_{M}<0, d_{L}<d_{R}$.
To describe the period adding structure observed in the parameter space of map $g$ consider first the basic cycles $L R^{n_{1}}$ and $R L^{n_{1}}$ belonging to the families $\Sigma_{1,1}$ and $\Sigma_{2,1}$ of complexity level one defined in (9). As stated in [15], the periodicity regions $P_{L R^{n_{1}}}$ and $P_{R L^{n_{1}}}$ of map $g$ are defined as

$$
\begin{gather*}
P_{L R^{n_{1}}}=\left\{p: \Psi_{1,1}\left(a_{L}, a_{R}, \mu_{R}, d_{R}, n_{1}\right)<\mu_{L}\right. \\
\left.\quad<\Phi_{1,1}\left(a_{L}, a_{R}, \mu_{R}, d_{L}, n_{1}\right)\right\},  \tag{A.2}\\
P_{R L^{n_{1}}}=\left\{p: \Psi_{1,1}\left(a_{L}, a_{R}, \mu_{L}, d_{L}, n_{1}\right)>\mu_{R}\right.  \tag{A.3}\\
> \\
\left.\quad \Phi_{1,1}\left(a_{L}, a_{R}, \mu_{L}, d_{R}, n_{1}\right)\right\},
\end{gather*}
$$

where

$$
\begin{align*}
\Phi_{1,1}\left(a_{L}, a_{R}, \mu, d, n_{1}\right)= & -\psi\left(a_{R}, n_{1}\right) \mu \\
& +\varphi\left(a_{R}, a_{L}, n_{1}\right) d \\
\Psi_{1,1}\left(a_{L}, a_{R}, \mu, d, n_{1}\right)= & -\left(a_{L}+\psi\left(a_{R}, n_{1}-1\right)\right) \mu  \tag{A.4}\\
& +a_{R} \varphi\left(a_{R}, a_{L}, n_{1}\right) d
\end{align*}
$$

with

$$
\begin{equation*}
\varphi(a, b, n)=\frac{1-a^{n} b}{a^{n}}, \quad \psi(a, n)=\frac{1-a^{n}}{(1-a) a^{n}} . \tag{A.5}
\end{equation*}
$$

These formulas are valid for the map $f$ given in (2) substituting

$$
\begin{equation*}
\mu_{L}=1-a_{L} d_{L}, \quad \mu_{R}=-a_{R} d_{R} . \tag{A.6}
\end{equation*}
$$

The periodicity regions related to cycles belonging to the families $\Sigma_{1,2}$ and $\Sigma_{2,2}$ of complexity level two given in (11) have the form

$$
\begin{align*}
& P_{\kappa_{n_{2}}^{L}\left(L R^{n_{1}}\right)} \\
& \quad=\left\{p: \Psi_{1,2}\left(a_{L}, a_{R}, \mu_{R}, d_{R}, n_{1}, n_{2}\right)<\mu_{L}\right.  \tag{A.7}\\
& \left.\quad<\Phi_{1,2}\left(a_{L}, a_{R}, \mu_{R}, d_{L}, n_{1}, n_{2}\right)\right\}, \\
& P_{\kappa_{n_{2}}^{R}\left(L R^{n_{1}}\right)} \\
& =\left\{p: \Psi_{2,2}\left(a_{L}, a_{R}, \mu_{R}, d_{R}, n_{1}, n_{2}\right)<\mu_{L}\right.  \tag{A.8}\\
& \left.\quad<\Phi_{2,2}\left(a_{L}, a_{R}, \mu_{R}, d_{L}, n_{1}, n_{2}\right)\right\},
\end{align*}
$$

respectively, where $\kappa_{n_{2}}^{L}$ and $\kappa_{n_{2}}^{R}$ are defined in (10), and $\Phi_{1,2}$, $\Psi_{1,2}, \Phi_{2,2}$, and $\Psi_{2,2}$ are defined as

$$
\begin{align*}
& \Phi_{1,2}\left(a_{L}, a_{R}, \mu, d, n_{1}, n_{2}\right) \\
&=-\psi\left(a_{R}, n_{2}\right) \mu \\
&-\left(a_{L} a_{R}^{n_{2}} \psi\left(a_{L} a_{R}^{n_{2}+1}, n_{1}\right) \mu-\varphi\left(a_{L} a_{R}^{n_{2}+1}, a_{L} a_{R}^{n_{2}}, n_{1}\right) d\right) \\
& \times\left(a_{R}^{n_{2}}\left(1+\psi\left(a_{L} a_{R}^{n_{2}+1}, n_{1}\right)\right)\right)^{-1}, \\
& \Psi_{1,2}( \left(a_{L}, a_{R}, \mu, d, n_{1}, n_{2}\right) \\
&=-\left(a_{L} a_{R}^{n_{2}}\left(a_{L} a_{R}^{n_{2}}+\psi\left(a_{L} a_{R}^{n_{2}+1}, n_{1}-1\right)\right) \mu\right) \\
& \times\left(a_{R}^{n_{2}}\left(1+a_{L} a_{R}^{n_{2}}+\psi\left(a_{L} a_{R}^{n_{2}+1}, n_{1}\right)-1\right)\right)^{-1} \\
&-\psi\left(a_{R}, n_{2}\right) \mu \\
&+\frac{a_{L} a_{R}^{n_{2}+1} \varphi\left(a_{L} a_{R}^{n_{2}+1}, a_{L} a_{R}^{n_{2}}, n_{1}\right) d}{a_{R}^{n_{2}}\left(1+a_{L} a_{R}^{n_{2}}+\psi\left(a_{L} a_{R}^{n_{2}+1}, n_{1}\right)-1\right)}, \\
& \Phi_{2,2}\left(a_{L}, a_{R}, \mu, d, n_{1}, n_{2}\right) \\
&= \varphi\left(a_{L}^{n_{2}} a_{R}, a_{L}^{n_{2}+1} a_{R}, n_{1}\right) d-a_{L}^{n_{2}}\left(1+\psi\left(a_{L}^{n_{2}} a_{R}, n_{1}\right)\right) \mu \\
& a_{L}^{n_{2}}\left(a_{R}+\psi\left(a_{L}, n_{2}\right)\left(1+\psi\left(a_{L}^{n_{2}} a_{R}, n_{1}\right)\right)\right) \\
& \Psi_{2,2}( \left(a_{L}, a_{R}, \mu, d, n_{1}, n_{2}\right) \\
&=\left(a_{R} \varphi\left(a_{L}^{n_{2}} a_{R}, a_{L}^{n_{2}+1} a_{R}, n_{1}\right) d\right. \\
&\left.-\left(1+a_{L}^{n_{2}+1} a_{R}+\psi\left(a_{L}^{n_{2}} a_{R}, n_{1}-1\right)\right) \mu\right)  \tag{A.9}\\
& \times\left(a_{R}+\psi\left(a_{L}, n_{2}\right)\left(1+a_{L}^{n_{2}+1} a_{R}+\psi\left(a_{L}^{n_{2}} a_{R}, n_{1}-1\right)\right)\right)^{-1} .
\end{align*}
$$

To get the periodicity regions of the cycles belonging to the families $\Sigma_{3,2}$ and $\Sigma_{4,2}(12)$ one has to interchange the indices $L$ and $R$, as well as to change the inequality signs to the opposite ones in (A.8) and (A.7), respectively.

The periodicity regions of a complexity level $k$ are obtained from the periodicity regions of the level $k-1$ by a recursive algorithm described in detail in [15].

## B.

Let us recall the analytic representation of the boundaries of $n \cdot k$-fins of map $g$ given in (A.1), which have the common boundary with the trunk region $P_{L R^{n-1}}$. Consider first the fins $P_{\sigma}$ contiguous to $P_{L R^{n-1}}$, corresponding to the attracting cycles $\sigma=\left(L R^{n-1}\right)^{k-1} M R^{n-1}, n \geq 2, k \geq 2$. It is proved in [15] that
any fin $P_{\sigma}$, if it exists (as a fin region may be an empty set), is confined by the boundaries defined as follows:

$$
\begin{align*}
& B C_{L R^{n-1}} \\
&=\left\{p \in P: \mu_{L}=-\frac{1-a_{R}^{n-1}}{\left(1-a_{R}\right) a_{R}^{n-1}} \mu_{R}+\frac{1-a_{R}^{n-1} a_{L}}{a_{R}^{n-1}} d_{L}\right\}, \\
& D F_{\sigma}=\left\{p \in P: a_{M} a_{R}^{k(n-1)} a_{L}^{k-1}=-1\right\}, \\
& B C_{\sigma}^{1}=\left\{p \in P: a_{M} a_{R}^{n-1}=\frac{a_{L} a_{R}^{n-1}\left(\left(a_{L} a_{R}^{n-1}\right)^{-k+1}-1\right)}{a_{L} a_{R}^{n-1}-1}\right\}, \\
& B C_{\sigma}^{2}=\left\{p \in P: a_{R}^{n-1} a_{M} \frac{\left(a_{R}^{n-1} a_{L}\right)^{k-1}-1}{a_{R}^{n-1} a_{L}-1}\right. \\
& \times\left(\left(a_{R}^{n-2} a_{L}+\frac{a_{R}^{n-2}-1}{a_{R}-1}\right) \mu_{R}+a_{R}^{n-2} \mu_{L}\right) \\
&+a_{R}^{(n-1) k} a_{M} a_{L}^{(k-1)} d_{R} \\
&\left.+\left(a_{R}^{n-2} a_{M}+\frac{a_{R}^{n-2}-1}{a_{R}-1}\right) \mu_{R}+a_{R}^{n-2} \mu_{M}=d_{R}\right\}, \tag{B.1}
\end{align*}
$$

where the BCB boundary $B C_{L R^{n-1}}$ which corresponds to $\mu_{L}=$ $\Phi_{1,1}\left(a_{L}, a_{R}, \mu_{R}, d_{L}, n-1\right)$ given in (A.2) is common with the trunk region $P_{L R^{n-1}}$. The boundary $D F_{\sigma}$ corresponds to the degenerate flip bifurcation, and $B C_{\sigma}^{1}$ and $B C_{\sigma}^{2}$ are related to two more BCBs of $\sigma$.

The $n \cdot k$-fin $P_{\rho}$ attached to the trunk region $P_{L R^{n-1}}$ on the other side and related to the cycle $\rho=\left(L R^{n-1}\right)^{k-1} L R^{n-2} M$, $k \geq 2$, is confined by the boundaries defined as follows:

$$
\begin{gathered}
B C_{R L R^{n-2}}=\left\{p \in P: \mu_{L}=-\left(a_{L}+\frac{1-a_{R}^{n-2}}{\left(1-a_{R}\right) a_{R}^{n-2}}\right) \mu_{R}\right. \\
\left.+a_{R} \frac{1-a_{R}^{n-1} a_{L}}{a_{R}^{n-1}} d_{R}\right\}, \\
D F_{\rho}=\left\{p \in P: a_{M} a_{R}^{k(n-1)-1} a_{L}^{k}=-1\right\}, \\
B C_{\rho}^{1}=\left\{p \in P: a_{L} a_{M} a_{R}^{n-2}=\frac{a_{L} a_{R}^{n-1}\left(\left(a_{L} a_{R}^{n-1}\right)^{-k+1}-1\right)}{a_{L} a_{R}^{n-1}-1}\right\},
\end{gathered}
$$

$$
\begin{align*}
B C_{\rho}^{2}= & \left\{p \in P: a_{R}^{n-2} a_{M} a_{L} \frac{\left(a_{R}^{n-1} a_{L}\right)^{k-1}-1}{a_{R}^{n-1} a_{L}-1}\right. \\
& \times\left(\frac{a_{R}^{n-1}-1}{a_{R}-1} \mu_{R}+a_{R}^{n-1} \mu_{L}\right)+a_{R}^{(n-1) k-1} a_{M} a_{L}^{k} d_{L} \\
& \left.+\frac{a_{R}^{n-2}-1}{a_{R}-1} \mu_{R}+a_{M} a_{R}^{n-2} \mu_{L}+\mu_{M}=d_{L}\right\} . \tag{B.2}
\end{align*}
$$

Here the boundary $B C_{R L R^{n-2}}$ corresponds to $\mu_{L}=$ $\Psi_{1,1}\left(a_{L}, a_{R}, \mu_{R}, d_{R}, n-1\right)$ given in (A.2).

The boundaries of the fins contiguous to the trunk regions $P_{R L^{n-1}}$ are obtained interchanging the indexes $L$ and $R$ in the above expressions. The fin regions of the higher complexity levels are obtained using the map replacement technique (see [15] for details).

Applying these results to map $f$ given in (2), we get that the fin region $P_{\sigma}$, associated with the cycles $\sigma=$ $\left(L R^{n-1}\right)^{k-1} M R^{n-1}, n \geq 2, k>1$, is confined by the following boundaries:

$$
\begin{align*}
& B C_{L R^{n-1}}=\left\{p \in P: d_{L}=a_{R}^{n-1}-\frac{\left(1-a_{R}^{n-1}\right) d_{R} a_{R}}{\left(1-a_{R}\right)}\right\}, \\
& D F_{\rho}=\left\{p \in P: d_{L}=d_{R}-a_{R}^{k(n-1)} a_{L}^{k-1}\right\}, \\
& B C_{\sigma}^{1}=\left\{p \in P: d_{L}=d_{R}+\frac{a_{L} a_{R}^{n-1}-1}{a_{L}\left(\left(a_{L} a_{R}^{n-1}\right)^{1-k}-1\right)}\right\}, \\
& B C_{\sigma}^{2}=\left\{p \in P: d_{R}=-a_{R}^{n-1} \frac{\left(a_{R}^{n-1} a_{L}\right)^{k-1}-1}{\left(a_{R}^{n-1} a_{L}-1\right)\left(d_{R}-d_{L}\right)}\right. \\
& \times\left(-\left(a_{R}^{n-2} a_{L}+\frac{a_{R}^{n-2}-1}{a_{R}-1}\right) a_{R} d_{R}\right. \\
&\left.+a_{R}^{n-2}\left(1-a_{L} d_{L}\right)\right) \\
&- \frac{a_{R}^{(n-1) k} a_{L}^{(k-1)} d_{R}}{d_{R}-d_{L}}-\left(-\frac{a_{R}^{n-2}}{d_{R}-d_{L}}+\frac{a_{R}^{n-2}-1}{a_{R}-1}\right) \\
&\left.\times a_{R} d_{R}+\frac{a_{R}^{n-2} d_{R}}{d_{R}-d_{L}}\right\} \tag{B.3}
\end{align*}
$$

Accordingly, the $n \cdot k$-fin $P_{\rho}$ related to the cycle $\rho=$ $\left(L R^{n-1}\right)^{k-1} L R^{n-2} M, k \geq 2$, is confined by the boundaries defined as

$$
\begin{aligned}
& B C_{R L R^{n-2}}=\left\{p \in P: 1-a_{L} d_{L}=\left(a_{L}+\frac{1-a_{R}^{n-2}}{\left(1-a_{R}\right) a_{R}^{n-2}}\right)\right. \\
&\left.\times a_{R} d_{R}+a_{R} \frac{1-a_{R}^{n-1} a_{L}}{a_{R}^{n-1}} d_{R}\right\},
\end{aligned}
$$

$$
\begin{align*}
& D F_{\rho}=\left\{p \in P: d_{L}=d_{R}-a_{R}^{k(n-1)-1} a_{L}^{k}\right\}, \\
& B C_{\rho}^{1}=\left\{p \in P: d_{L}=d_{R}+\frac{a_{L} a_{R}^{n-1}-1}{a_{R}\left(\left(a_{L} a_{R}^{n-1}\right)^{-k+1}-1\right)}\right\}, \\
& B C_{\rho}^{2}=\left\{p \in P: d_{L}=-a_{R}^{n-2} a_{L} \frac{\left(a_{R}^{n-1} a_{L}\right)^{k-1}-1}{\left(a_{R}^{n-1} a_{L}-1\right)\left(d_{R}-d_{L}\right)}\right. \\
& \times\left(-\frac{a_{R}^{n-1}-1}{a_{R}-1} a_{R} d_{R}+a_{R}^{n-1}\left(1-a_{L} d_{L}\right)\right) \\
&-\frac{a_{R}^{k(n-1)-1} a_{L}^{k} d_{L}}{d_{R}-d_{L}}-\frac{a_{R}^{n-2}-1}{a_{R}-1} \\
&\left.\times a_{R} d_{R}-\frac{a_{R}^{n-2}\left(1-a_{L} d_{L}\right)}{d_{R}-d_{L}}+\frac{d_{R}}{d_{R}-d_{L}}\right\} . \tag{B.4}
\end{align*}
$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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