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Research Article

Traveling Wave Solution in a Diffusive Predator-Prey System with Holling Type-IV Functional Response

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We establish the existence of traveling wave solution for a reaction-diffusion predator-prey system with Holling type-IV functional response. For simplicity, only one space dimension will be involved, the traveling solution equivalent to the heteroclinic orbits in R^3. The methods used to prove the result are the shooting argument and the invariant manifold theory.

1. Introduction

The paper will study the traveling wave solution for a diffusive predator-prey system with Holling type-IV functional response, which is as follows:

$$\begin{aligned} u_t &= d_1 u_{xx} + Au\left(1 - \frac{u}{K}\right) - \frac{Buw}{1 + Eu^2}, \\ w_t &= d_2 w_{xx} + w\left(\frac{Du}{1 + Eu^2} - C\right). \end{aligned} \tag{1}$$

All parameters are positive constant. The functions $u(x,t)$ and $w(x,t)$ are the densities of the prey and predator, respectively; d_1 and d_2 are diffusive rates of the prey and predator, respectively; K is the carrying capacity of the prey; C is the death rate of the predator; and A is the growth factor of the prey. We may refer to [1, 2] for more biological implications.

Recently, the system (1) and some related systems have been studied by many researchers for an understanding of the most basic features of a spatially distributed interaction; we can refer to [3–10]. Gardner [8] proved the existence of traveling wave solutions for a diffusive predator-prey system with Holling type-II functional response by using the connection index. Numerical simulation in Owen and Lewis [11] shows that a diffusive predator-prey system with Holling type-II functional response, when the diffusive rates of the prey and the predator are not zero, possesses traveling wave solutions. Huang et al. [12] proved theoretically that the numerical simulation in [11] is true. Huang et al. considered the system

$$\begin{aligned} u_t &= u\left[\alpha(b - u) - \frac{w}{1 + u}\right], \\ w_t &= w_{xx} - w\left(1 - \frac{\beta u}{1 + u}\right), \end{aligned} \tag{2}$$

and they obtained that if $c > \sqrt{4(b\beta - 1 - b)/(1 + b)}$, $(b + 1)/b < \beta < b/(b - 1)$, and $(1 - \alpha)(\beta - 1) \geq (2\beta/(1 + b))\sqrt{(b\beta - 1 - b)/(1 + b)}$, then there are nonnegative solutions of system (2) satisfying $w(+\infty) = \alpha(1/(\beta-1)+1)(b-1/(\beta-1))$, $w(-\infty) = 0$, $u(+\infty) = 1/(\beta - 1)$, $u(-\infty) = b$.

Dunbar [13] studied the following system:

$$\begin{aligned} U_t &= U(1 - W), \\ W_t &= W_{xx} + \rho W(U - 1) \end{aligned} \tag{3}$$

and obtained the following.

(a) If $0 < c < \sqrt{4\alpha(1 - \beta)}$, then there exist traveling wave front solutions of the system (3) satisfying $u(-\infty) = 0$, $w(-\infty) = 0$, $u(+\infty) = \beta$, $w(+\infty) = 1 - \beta$.

(b) If $c \geq \sqrt{4\alpha(1-\beta)}$, then there exist traveling wave front solutions of the system (3) satisfying $u(-\infty) = 1$, $w(-\infty) = 0$, $u(+\infty) = \beta$, $w(+\infty) = 1-\beta$.

Dunbar [14] investigated the system

$$U_t = \alpha U(\gamma - U) - \frac{UW}{1+U}, \qquad W_t = W_{xx} - W + \beta\frac{UW}{1+U} \tag{4}$$

and obtained that if $c^2 > 4(\gamma\beta - \gamma - 1)/(1+\gamma)$ and $(1+\gamma)/\gamma < \beta \leq \gamma/(\gamma-1)$, then there is a bounded solution of (4) satisfying $u(-\infty) = \gamma$, $w(-\infty) = 0$, $u(+\infty) = 1/(\beta-1)$, and $w(+\infty) = \alpha(\gamma - 1/(\beta-1))(1+1/(\beta-1))$.

Li and Wu [15] studied a system with Holling type-III functional response and proved the existence of traveling wave solutions by using the shooting argument in R^3 together with a Lyapunov function [16], LaSalle's invariance principle [17], and the Hopf bifurcation theorem [18]. We may refer to Murray [19], Mischaikow and Reineck [20], and Volpert et al. [21] for more results.

We notice that the Holling type-II and the Holling type-III functional response are monotonic in the first quadrant, while the Holling type-IV functional response considered in this paper is nonmonotonic in the first quadrant. It is an interesting problem to know whether the above results are available for the system (1). We should mention that although the techniques used here are similar to those in [12–15, 22], there are several differences. Firstly, it is a more complex system. The systems studied in [13, 22] are the ones with the Lotka-Volterra functional response. The systems studied in [12, 14, 15] are the ones with the Holling type-II or Holling type-III functional response. Secondly, we construct a different Wazewski set W and a new Lyapunov function. For simplicity, we assume that $d_1 = 0$ can be considered to correspond to a situation in which the prey species is evenly distributed. We should mention that the assumption is not essential.

For further simplification, taking

$$u^* = \sqrt{E}u, \qquad w^* = \frac{B}{C}w, \qquad x' = \sqrt{\frac{C}{d_2}}x,$$
$$t' = Ct, \qquad \alpha = \frac{A}{\sqrt{E}CK}, \qquad b = \sqrt{E}K, \qquad \beta = \frac{D}{\sqrt{E}C}, \tag{5}$$

and dropping the stars on u, w and the primes on x, t for convenience, we obtain

$$u_t = \alpha u(b-u) - \frac{uw}{1+u^2}, \qquad w_t = w_{xx} - w + \frac{\beta uw}{1+u^2}. \tag{6}$$

There are several reasonable parameter restrictions. We assume that $b > 1$ or equivalently that $E > 1/K^2$, so that the satiation effect is great enough. We also assume that $\alpha > 0$ and $\beta > 2$, which ensure that the system (6) has positive equilibrium point corresponding to constant coexistence of the two species. Obviously, the system (6) has four equilibria points: $(0,0)$, $(b,0)$, (u_0, w_0), and (u_1, w_1), which are equilibria of the corresponding ODE system without diffusion, where

$$u_0 = \frac{\beta - \sqrt{\beta^2 - 4}}{2}, \qquad w_0 = \alpha(b-u_0)(1+u_0^2),$$
$$u_1 = \frac{\beta + \sqrt{\beta^2 - 4}}{2}, \qquad w_1 = \alpha(b-u_1)(1+u_1^2). \tag{7}$$

In this paper, we also require that $b < u_1$, which ensures that equations (6) has only a positive equilibrium. We notice that $u_0 < b$, so the system (6) has only one positive equilibrium point. The equilibrium (0, 0), representing the absence of both species, is a saddle point. The equilibrium $(b, 0)$, representing the population of the prey at the environmental carrying capacity in the absence of predators, is unstable. The equilibrium (u_0, w_0), representing the time constant coexistence of both species, is stable. We establish the traveling wave solution connecting the equilibria $(b, 0)$ and (u_0, w_0), which is called the "waves of invasion"; see Chow and Tam [23].

The paper is organized as follows. In the next section, we first recall a lemma which is a variant of Wazewski's Theorem and then we state the result on the existence of traveling wave solution. Section 3 is devoted to prove the result.

2. Main Result

In order to establish the existence of traveling wave solution of the system (6), we assume that the solution has the special form $u(x,t) = u(x+ct)$, $w(x,t) = w(x+ct)$, where the wave speed parameter c is positive. Substituting $u(x,t) = u(s)$, $w(x,t) = w(s)$, $s = x + ct$ into the system (6), the responding system becomes

$$cu' = \alpha u(b-u) - \frac{uw}{1+u^2}, \qquad cw' = w'' - w + \frac{\beta uw}{1+u^2}. \tag{8}$$

Here $'$ denotes the differentiation with respect to the variable s. We require that the traveling wave solutions u and w are nonnegative and satisfy the boundary conditions

$$u(-\infty) = b, \quad u(+\infty) = u_0, \qquad w(-\infty) = 0, \quad w(+\infty) = w_0. \tag{9}$$

We write the system (6) as a first order system in R^3

$$u' = \frac{\alpha}{c}u(b-u) - \frac{uw}{c(1+u^2)}, \qquad w' = z, \qquad z' = cz + w - \frac{\beta uw}{1+u^2}. \tag{10}$$

In this section a variant of Wazewski's Theorem, which is a formalization and extension of the shooting method, is stated.

This proposition recognizes that the flow defined by the solutions of a differential system gives a topological mapping between regions of phase space. The statement and the proof of Wazewski's Theorem are given in [24].

Consider a system

$$y' = f(y), \qquad ' = \frac{d}{ds}, \quad y \in R^n. \tag{$*$}$$

Here $f : R^n \to R^n$ is a continuous function and satisfies the Lipschitz condition. Let $y(s, y_0)$ be the unique solution of $(*)$ satisfying $y(0, y_0) = y_0$. For convenience, we set $y(s, y_0) = y_0 \cdot s$; let $Y \cdot S$ be the set of points $y_0 \cdot s$, where $y_0 \in Y$ and $s \in S$.

Given $W \subseteq R^n$, the notation $\mathrm{cl}(W)$ is used for the closure of W. Define

$$W^- = \{y_0 \in W \mid \forall s > 0, y_0 \cdot [0, s) \not\subseteq W\}; \tag{11}$$

W^- is called the immediate exit set of W. Given $\Sigma \subset W$, let

$$\Sigma^0 = \{y_0 \in \Sigma \mid \exists s_0 = s_0(y_0) \text{ such that } y_0 \cdot s_0 \notin W\}. \tag{12}$$

For $y_0 \in \Sigma^0$, define

$$T(y_0) = \sup\{s \mid y_0 \cdot [0, s] \subseteq W\}; \tag{13}$$

$T(y_0)$ is called an exit time. Note that $y_0 \cdot T(y_0) \in W^-$ and $T(y_0) = 0$ if and only if $y_0 \in W^-$.

Lemma 1. *Suppose that*

(i) *if $y_0 \in \Sigma$ and $y_0 \cdot [0, s] \subseteq \mathrm{cl}(W)$, then $y_0 \cdot [0, s] \subseteq W$;*

(ii) *if $y_0 \in \Sigma$, $y_0 \cdot s \in W$, $y_0 \cdot s \notin W^-$, then there is an open set V_s about $y_0 \cdot s$ disjoint from W^-;*

(iii) *$\Sigma = \Sigma^0$, Σ is a compact set and intersects a trajectory of $y' = f(y)$ only once.*

Then the mapping $F(y_0) = y_0 \cdot T(y_0)$ is a homeomorphism from Σ to its image on W^-. The proof is given in [22]. A set $W \subseteq R^n$ satisfying the conditions (i) and (ii) is called a Wazewski set.

Theorem 2. *(i) If $\beta > (1 + b^2)/b$, and $0 < c < 2\sqrt{(\beta b - 1 - b^2)/(1 + b^2)}$, then there are no nonnegative solutions of the system* (10) *satisfying the boundary conditions* (9).

(ii) If $\beta > (1 + b^2)/b$, $c > \sqrt{2\beta - 4}$, $\alpha(1 + b^2) < c^2$, and $b < 2\sqrt{1 + u_0^2} - u_0$, then there exists nonnegative solution of the system (10) *satisfying the boundary conditions* (9), *which correspond to traveling wave solution of the system* (6).

3. Proofs of the Result

The eigenvalues of the linearization of the system (10) at $(b, 0, 0)$ are

$$\begin{aligned} \lambda_1 &= -\frac{\alpha b}{c}, \\ \lambda_2 &= \frac{c - \sqrt{c^2 - 4(\beta b - 1 - b^2)/(1 + b^2)}}{2}, \\ \lambda_3 &= \frac{c + \sqrt{c^2 - 4(\beta b - 1 - b^2)/(1 + b^2)}}{2}. \end{aligned} \tag{14}$$

If $0 < c < 2\sqrt{(\beta b - 1 - b^2)/(1 + b^2)}$, then λ_2 and λ_3 are a pair of complex conjugate eigenvalues with positive real part. By Theorems 6.1 and 6.2 in [16], there is a two-dimensional unstable manifold base at $(b, 0, 0)$; the critical point is a spiral point on this unstable manifold, so the trajectory approaching $(b, 0, 0)$ as $s \to -\infty$ must have $w(s) < 0$ for some s. It violates the requirement that the solution of the system (10) must be nonnegative. It proves the first part of Theorem 2.

We only need to discuss the case $c \geq 2\sqrt{(\beta b - 1 - b^2)/(1 + b^2)}$. In fact we require the stronger condition $c > \sqrt{2\beta - 4}$ for mathematical simplicity. With the requirement there are three distinct real eigenvalues $\lambda_1 < 0 < \lambda_2 < \lambda_3$. Let the eigenvectors $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ associated with $\lambda_1, \lambda_2, \lambda_3$, respectively, be

$$\begin{gathered} \mathbf{e_1} = (1, 0, 0), \qquad \mathbf{e_2} = (1, p(\lambda_2), \lambda_2 p(\lambda_2)), \\ \mathbf{e_3} = (1, p(\lambda_3), \lambda_3 p(\lambda_3)). \end{gathered} \tag{15}$$

Here $p(\lambda_i) = -(1/b)(c\lambda_i + cb^2\lambda_i + \alpha b + \alpha b^3) < 0$, $i = 2, 3$.

Applying Theorems 6.1 and 6.2 of [16], there exists a one-dimension strongly unstable manifold Ω_1 tangent to $\mathbf{e_3}$ at $(b, 0, 0)$. A parametric representation for the strongly unstable manifold Ω_1 in a small neighborhood of $(b, 0, 0)$ is

$$f_1(m) = (b, 0, 0) + m \cdot \mathbf{e_3} + O(|m|). \tag{16}$$

There exists a two-dimension unstable manifold Ω_2 tangent to the span of $\mathbf{e_2}$ and $\mathbf{e_3}$ at $(b, 0, 0)$. A parametric representation for the two-dimensional unstable manifold Ω_2 in a small neighborhood of $(b, 0, 0)$ is

$$f_2(m, n) = (b, 0, 0) + m \cdot \mathbf{e_3} + n \cdot \mathbf{e_2} + O(|m| + |n|). \tag{17}$$

The idea of constructing the Wazewski set W is similar to that in Dunbar [22]: it will be the complement of three blocks in R^3, two of which are chosen so that z' has the same sign as z so solutions entering these blocks would not have $z \to 0$ as $s \to \infty$. Thus we define the Wazewski set W as follows:

$$W = R^3 \setminus (P \cup Q). \tag{18}$$

Here

$$\begin{gathered} P = \{(u, w, z) \mid u < u_0, w > w_0, z > 0\}, \\ Q = \{(u, w, z) \mid u > u_0, w < w_0, z < 0\}. \end{gathered} \tag{19}$$

Note that W is a closed set. Let

$$\begin{aligned} J = &\{(u,v,w,z) \mid u > u_1, 0 \le w \le w_0, z = 0\} \\ &\cup \{(u,v,w,z) \mid u_0 \le u < u_1, w \le 0, z = 0\} \\ &\cup \{(u,v,w,z) \mid u = u_1, w_1 \le w \le 0, z = 0\}. \end{aligned} \tag{20}$$

By checking the vector field on ∂W, we obtain

$$W^- = \partial W \setminus (J \cup \{(u_0, w_0, 0)\}). \tag{21}$$

Details of proof that W^- is the set described above are tedious. We only examine the part ∂Q of ∂W as an example, which shows why the set J must be excluded from ∂W to obtain W^-. The other proofs are similar. The boundary of ∂Q is $u = u_0$, $w = w_0$, or $z = 0$.

(1) $u = u_0$, $w = w_0$, and $z < 0$.

Since $w' = z < 0$, $w < w_0$, $u' = [(\alpha/c)u(b-u) - uw/c(1+u^2)]_{(u_0,w_0)} = 0$, $u'' = -(u_0 z/c(1+u_0^2)) > 0$, and $u > u_0$, thus the trajectory enters Q.

(2) $u = u_0$, $w < w_0$, and $z = 0$.

Since $u' = [(\alpha/c)u(b-u) - uw/c(1+u^2)]_{u=u_0} > 0$, $z' = [cz + w(1-\beta u/(1+u^2))]_{u=u_0} = 0$, $z'' = \beta u' w(u_0^2 - 1)/(1+u_0^2)^2$, and $u_0 < 1$, thus $u > u_0$ and we obtain the following.

(i) $0 < w < w_0$; then $z'' < 0$, $z < 0$, and the trajectory enters Q.

(ii) $w < 0$; then $z'' > 0$, $z > 0$, and the trajectory does not enter Q.

(iii) $w = 0$, $z = 0$. Consider the system

$$w' = z, \qquad z' = cz + w - \frac{\beta u w}{1+u^2}. \tag{22}$$

We come to the conclusion that the u-axis is an invariant manifold and the trajectory does not enter Q.

(3) $u > u_0$, $w = w_0$, and $z = 0$.

Since $u_1 = (\beta + \sqrt{\beta^2 - 4})/2 > b$ and $z' = w_0(1 - \beta u/(1+u^2))$, we obtain the following.

(i) $u_0 < u < u_1$; then $z' < 0$, $z < 0$, $w'' = z' < 0$, $w < w_0$, and the trajectory enters Q.

(ii) $u > u_1$; then $z' > 0$, $z > 0$, $w'' = z' > 0$, $w > w_0$, and the trajectory does not enter Q.

(iii) $u = u_1$; then $u' = (\alpha/c)u_1(b-u_1) - u_1 w_0/c(1+u_1^2) < 0$, $z' = 0$, and $z'' = [w - \beta u w/(1+u^2)]' = \beta u' w_0(u_1^2 - 1)/(1+u_1^2)^2 < 0$.

That is, $z < 0$, $w < w_0$, and the trajectory enters Q.

(4) $u = u_0$, $w < w_0$, and $z < 0$.

From the proof of (2), we come to the conclusion that $u' > 0$, $u > u_0$, and the trajectory enters Q.

(5) $u > u_0$, $w = w_0$, and $z < 0$.

Since $w' = z < 0$ and $w < w_0$, the trajectory enters Q.

(6) $u_0 < u < u_1$, $w < w_0$, and $z = 0$.

Since $z' = w(1-\beta u/(1+u^2))$, we obtain the following.

(i) $0 < w < w_0$; then $z' < 0$ and $z < 0$, which implies that the trajectory enters Q.

(ii) $w = 0$; similar to the proof of (2iii), the trajectory does not enter Q.

(iii) $w < 0$; then $z' > 0$ and $z > 0$; that is, the trajectory does not enter Q.

(7) $u = u_1$, $w < w_0$, and $z = 0$.

Since $u' = (\alpha/c)u_1(b-u_1) - u_1 w/c(1+u_1^2)$, $z' = [cz + w(1-\beta u/(1+u^2))]_{u=u_1} = 0$, and $z'' = \beta u' w(u_1^2 - 1)/(1+u_1^2)^2$, we obtain the following.

(i) $0 < w < w_0$; then $u' < 0$, $z'' < 0$, $z < 0$, and the trajectory enters Q.

(ii) $w = 0$; similar to the proof of (2iii), the trajectory does not enter Q.

(iii) $w_1 < w < 0$; then $u' < 0$ and $z'' > 0$; that is, $z > 0$, and the trajectory does not enter Q.

(iv) $w = w_1$. It is a singular point $(u_1, w_1, 0)$ and is not in the immediate exit set.

(v) $w < w_1$; then $u' > 0$, $z'' < 0$, and $z < 0$, which implies that the trajectory enters Q.

(8) $u > u_1$, $w < w_0$, and $z = 0$.

Since $z' = w(1 - \beta u/(1+u^2))$, then we obtain the following.

(i) $0 < w < w_0$; then $z' > 0$, $z > 0$, and the trajectory does not enter Q.

(ii) $w = 0$; the trajectory does not enter Q.

(iii) $w < 0$; then $z' < 0$, $z < 0$, and the trajectory enters Q.

In order to use Lemma 1, we construct the set Σ on a sphere surrounding $(b, 0, 0)$ in the two-dimensional unstable manifold Ω_2 by Lemma 3 to Lemma 7. The specification of the arc requires the identification of the endpoints on the circle. One endpoint is the intersection of the circle with the strongly unstable manifold Ω_1 and the other is the intersection of the circle with the plane defined by $z = 0$. Lemmas 3–6 are simple comparison arguments showing that the first endpoint on the strongly unstable manifold is carried by the flow into P and the other is carried into Q. We use the notation $\Lambda_1 = \{(u,w,z) \mid u \le b, w \ge 0, z \ge 0\}$.

Lemma 3. *Let $c > \sqrt{2\beta - 4}$. A solution of the system* (10) *having a point, corresponding to $s = 0$ without loss of generality, such that $u(0) < b$, $w(0) > 0$, and $z(0) > (c/2)w(0)$, will have $w(s) > 0$ and $z(s) > (c/2)w(s)$ for all $s > 0$. In particular, it is true for trajectories on the branch of strongly unstable manifold Ω_1 in the octant Λ_1.*

Proof. Suppose, to the contrary, that there exists an $s > 0$ such that $u(s) < b$, but $z(s) \le (c/2)w(s)$. Let $s_1 = \inf\{s \mid z(s) \le (c/2)w(s), u(s) < b\}$. Since $w(0) > 0$ and for $s \in [0, s_1)$, $w'(s) = z(s) > (c/2)w(s)$, we have $w(s_1) > 0$ and $z'(s_1) - (c/2)w'(s_1) \le 0$. Using that $z(s_1) = (c/2)w(s_1)$, we obtain $(c^2/4) + 1 - \beta u(s_1)/(1+u^2(s_1)) \le 0$. Since $u(s_1)/(1+u^2(s_1)) \le 1/2$, it follows that $c^2/4 + 1 - \beta/2 \le 0$; that is, $c^2 \le 2\beta - 4$; it is a contradiction with $c^2 > 2\beta - 4$. It completes the proof. □

Lemma 4. *A trajectory on the portion of strongly unstable manifold Ω_1 in the octant Λ_1 must satisfy*

$$w(s) \ge -c^2 (u(s) - b) \tag{23}$$

for all s.

Proof. The solution approaches $(b, 0, 0)$ tangent to $\mathbf{e_3}$ and the eigenvector $\mathbf{e_3}$ at $(b, 0, 0)$ has $w = p(\lambda_3)(u - b)$ such that $w(0) > -c^2(u(0)-b)$. Suppose to the contrary that there exists an $s > 0$ such that $w(s) < -c^2(u(s)-b)$. Let $s_1 = \inf\{s \mid w(s) < -c^2(u(s) - b)\}$; then $w'(s_1) < -c^2u'(s_1)$. Since $u' = (\alpha/c)u(b - u) - uw/c(1 + u^2)$, $w' = z$, it follows that $z(s_1) \le (c/2)w(s_1)$; it is a contradiction. It completes the proof. □

Lemma 5. *Let $d > (c + \sqrt{c^2 + 4})/2$ be a fixed number. A solution of the system* (10) *having a point, corresponding to $s = 0$ without loss of generality, such that $0 < u(0) < b$ and $z(0) < dw(0)$, will have $z(s) < dw(s)$ for all $s > 0$ such that $w(s) > 0$. In particular, this is true for trajectories on the branch of strongly unstable manifold Ω_1 in the octant Λ_1.*

Proof. Suppose, to the contrary, that there exists an $s > 0$ such that $w(s) > 0$, but $z(s) \ge dw(s)$. Let $s_1 = \inf\{s > 0 \mid z(s) \ge dw(s)\}$; then $z(s_1) = dw(s_1)$ and $z(s) < dw(s)$ for $0 \le s < s_1 z'(s_1) - dw'(s_1) \ge 0$. Substituting z' and w', we obtain $cz(s_1) - dz(s_1) + w(s_1) - \beta u(s_1)w(s_1)/(1 + u^2(s_1)) \ge 0$; that is, $[-d^2 + cd + 1 - \beta u(s_1)/(1 + u^2(s_1))]w(s_1) \ge 0$. However, the choice of d implies that $-d^2 + cd + 1 < 0$; thus $[-d^2+cd+1-\beta u(s_1)/(1+u^2(s_1))]w(s_1) < 0$. This contradiction shows that $z(s) < dw(s)$ for s such that $w(s) > 0$. □

Lemma 6. *Suppose that a solution of the system* (10) *has a point such that*

$$u(0) < b, \qquad 0 < w(0) < -\frac{(dc + \alpha b)\left(1 + b^2\right)}{u_0}(u(0) - b), \qquad z(0) < dw(0). \tag{24}$$

Then for all $s > 0$, as long as $u(s) > u_0$, $w(s) > 0$, the trajectory must satisfy that

$$w(s) < -\frac{(dc + \alpha b)\left(1 + b^2\right)}{u_0}(u(s) - b). \tag{25}$$

In particular, it is true for trajectories on the branch of strongly unstable manifold Ω_1 in the octant Λ_1.

Proof. We first show that $u(s) < b$ for all $s > 0$ such that $w(s) > 0$. If it is not true, then there is a first s_1 such that $u(s_1) = b$, $u'(s_1) \ge 0$, and $w(s_1) > 0$. However, we have

$$u'(s_1) = \frac{u}{c}\left[\alpha(b - u) - \frac{w}{1 + u^2}\right]_{s=s_1} < 0. \tag{26}$$

It is a contradiction; then $u(s) < b$ for all s such that $w(s) > 0$.

Let $A = (dc + \alpha b)(1 + b^2)/u_0$; suppose that there exists a first time $s_2 > 0$ such that $u(s_2) > u_0$, $w(s_2) > 0$, and $w(s_2) = -A(u(s_2)-b)$; then $w'(s_2) \ge -Au'(s_2)$. By Lemma 5, we obtain

$$dw(s_2) \ge z(s_2) \ge -\frac{Au}{c}\left[\alpha(b - u) - \frac{w}{1 + u^2}\right]_{s=s_2}. \tag{27}$$

From $w(s_2) = -A(u(s_2) - b)$ and $u_0 < u(s_2) < b$, we have

$$d \ge -\frac{1}{c}\left[\alpha u - \frac{Au}{1 + u^2}\right]_{s=s_2} > \frac{1}{c}\left[-\alpha b + \frac{Au_0}{1 + b^2}\right] = d. \tag{28}$$

It is a contradiction, which completes the proof. □

Combining the results of these lemmas, we follow the trajectory of a solution of the system (10) on the strongly unstable manifold Ω_1. Define

$$\begin{aligned}\mathfrak{R} = \Big\{ &(u, w, z) \mid u_0 < u < b, \\ &-c^2(u(s) - b) \le w \\ &\le -\frac{(dc + \alpha b)\left(1 + b^2\right)(u(s) - b)}{u_0}, \\ &\frac{c}{2}w < z < dw \Big\}.\end{aligned} \tag{29}$$

Lemmas 3–6 show that the trajectory of a solution of the system (10) on the strongly unstable manifold Ω_1 is contained in $\mathfrak{R}$. Recall the assumption that $\alpha < c^2/(1 + b^2)$; then

$$w \ge c^2(b - u(s)) > \alpha\left(1 + u^2\right)(b - u(s)), \tag{30}$$

which implies that $u' < 0$ in the region $\mathfrak{R}$. Thus, for a solution of the system (10) on the strongly unstable manifold Ω_1, $u(s)$ decreases until $u(s_1) = u_0$ for some finite s_1; the trajectory of this solution hits ∂P on the face $u = u_0$, $w > w_0$, and $z > 0$. The vector field on this face shows that a solution of the system (10) on Ω_1 enters the region P at some finite time.

Lemma 7. *In a sufficiently small neighborhood of $(b, 0, 0)$, the two-dimensional unstable manifold Ω_2 intersects the plane defined by $z = 0$ in a C^1 curve Γ, given by $w = J(u)$ and $z = 0$.*

Proof. The proof, which is similar to that of Lemma 5 in [13], is therefore omitted. □

We are interested in the portion of the curve Γ in the region $u < b$. The function $J(u)$ can be approximated to the first order by

$$w = J(u) = \frac{p(\lambda_2)\,p(\lambda_3)(\lambda_3 - \lambda_2)}{\lambda_3 p(\lambda_3) - \lambda_2 p(\lambda_2)}(u - b). \tag{31}$$

Thus the w-coordinate of points along the curve Γ will satisfy $w > 0$. From the direction of the vector field on the plane defined by $u_0 < u < b$, $w > 0$, and $z = 0$, any trajectory passing through a point of Γ near $(b, 0, 0)$ will immediately enter the region Q.

Now, we place a sufficiently small circle surrounding $(b, 0, 0)$ on the two-dimensional unstable manifold Ω_2 such that the circle is contained in the neighborhood of $(b, 0, 0)$ given in Lemma 7 and the conditions of Lemmas 3–6 are satisfied. The circle intersects the curve Γ. Define Σ to be the arc of this circle contained in the octant Λ_1, whose endpoints are the intersections of the circle with Ω_1 and the curve Γ.

We now prove part (ii) of Theorem 2, which requires two steps. Firstly, we use Lemma 1 to produce a trajectory which remains in the region W. Secondly, we construct a Lyapunov function to prove the trajectory approaches $(u_0, w_0, 0)$.

Lemma 8. *There exists a point $y^* \in \Sigma$ such that the solution $y(s, y^*)$ of the system* (10) *remains in the region W for all s.*

Proof. The proof, which is similar to the proof of Lemma 3.7 in [15], is therefore omitted. □

Lemma 9. *The solution $y(s, y^*)$ must be in the bounded region*

$$\Psi = \left\{(u, w, z) \mid 0 < u < b, 0 < w < k(u), -\frac{1}{c}w < z < dw\right\} \tag{32}$$

for all s, where

$$k(u) = \begin{cases} -\dfrac{(dc + \alpha b)\left(1 + b^2\right)(u - b)}{u_0}, & u_0 < u < b, \\ -\dfrac{(dc + \alpha b)\left(1 + b^2\right)(u_0 - b)}{u_0}, & 0 < u \le u_0. \end{cases} \tag{33}$$

Proof. Since the plane defined by $u = 0$ is an invariant manifold, the first coordinate of y_1 is strictly positive, and thus $u_1 > 0$ for all s. Suppose $y(s, y^*)$ enters the region $N_1 = \{(u, w, z) \mid w \le 0\}$; let $s_1 = \inf\{s \mid y(s, y^*) \in N_1\}$; then we have $w_1(s_1) = 0$, $w_1'(s_1) \le 0$, and $z_1(s_1) \le 0$. We know that the u-axis is an invariant manifold, $z_1(s_1) < 0$. Since $y(s, y^*)$ does not enter Q then $u_1(s_1) < u_0$. From the system (10), we obtain $u_1'(s_1) > 0$, which must enter the region

$$N_2 = \left\{(u, w, z) \mid u_1(s_1) < u < u_0, w < 0, z < 0\right\}. \tag{34}$$

In the region N_2, $z_1(s)$ and $w_1(s)$ are decreasing, and thus $u_1'(s)$ is bounded below by the positive

$$\frac{\alpha}{c} \min\left\{u_1(s_1)\left(b - u_1(s_1)\right), u_0\left(b - u_0\right)\right\}. \tag{35}$$

Then $u_1(s)$ increases to u_0 in a finite time and $y(s, y^*)$ enters Q. It is a contradiction, so $w_1(s) > 0$ for all s.

From Lemma 6, we obtain that $w_1 < -(dc + \alpha b)(1 + b^2)(u_1 - b)/u_0$ for $u_0 < u_1 \le b$. Since $w_1(s) > 0$, we have $u_1(s_1) < b$ for all s. Suppose that there is an s such that $w_1(s) \ge -A_0(u_0 - b)$ for $0 < u_1 \le u_0$, where $A_0 = (dc + \alpha b)(1 + b^2)/u_0$. Let $s_2 = \inf\{s \mid w_1(s) \ge -A_0(u_0 - b)\}$, so $u_1(s_2) \le u_0$, $w_1(s_2) > w_0$, and $z_1(s_2) = w_1'(s_2) \ge 0$. Then either $y(s_2, y^*) \in P$ or $y(s_2, y^*)$ immediately enters P, which is a contradiction.

Suppose that there exists an s_3 such that $z_1(s_3) < -(1/c)w_1(s_3) < 0$; then $z_1(s) < -(1/c)w_1(s)$ for all $s > s_3$. If it is not true, there exists an $s_4 > s_3$ such that $z_1(s_4) = -(1/c)w_1(s_4)$, and thus $z_1'(s_4) + (1/c)w_1'(s_4) \ge 0$. From the system (10), we have

$$\frac{1}{c^2} + \frac{\beta u}{1 + u^2} \le 0, \tag{36}$$

which is impossible. So if $z_1(s_3) < -(1/c)w_1(s_3) < 0$, then $z_1(s) < -(1/c)w_1(s)$ continues to hold for $s > s_3$. Thus, $z_1' = cz_1 + w_1 - \beta u_1 w_1/(1 + u_1^2) < -\beta u_1 w_1/(1 + u_1^2) < 0$ and $z_1(s) < z_1(s_3)$ for all $s > s_3$ and $w_1'(s) = z_1(s)$ is strictly negative and bounded away from zero by $z_1(s_3)$. Then $w_1(s) < 0$ for some finite s; it is a contradiction. Notice that a trajectory starting on Σ tangent to $\mathbf{e_2}$ or $\mathbf{e_3}$ has $z = \lambda_2 w$ or $z = \lambda_3 w$. Since λ_2 and $\lambda_3 < d$, we have $z_1(s) < dw_1(s)$ for all s, which completes the proof of this lemma. □

Lemma 10. *The trajectory $y(s, y^*) \to (u_0, w_0, 0)$ as $s \to +\infty$.*

Proof. In order to show the trajectory will approach the point $(u_0, w_0, 0)$, we construct a Lyapunov function as follows:

$$\begin{aligned} V(u, w, z) = {} & c\left[\left(1 + u_0^2\right)u - u_0 \ln u - \frac{u_0}{2}u^2\right] \\ & + u_0\left[c\left(w - w_0\right) - z\right] + w_0 u_0\left(\frac{z}{w} - c\ln\frac{w}{w_0}\right). \end{aligned} \tag{37}$$

We obtain that $V(u, w, z)$ is continuous and bounded below on Ψ,

$$\begin{aligned} \frac{dV}{ds} &= \frac{\partial V}{\partial u}u_t + \frac{\partial V}{\partial w}w_t + \frac{\partial V}{\partial z}z_t \\ &= \alpha\left(u - u_0\right)\left(1 - u_0 u\right)(b - u) + \frac{\left(u - u_0\right)\left(u_0 u - 1\right)}{1 + u^2}w \\ &\quad + \left(w_0 - w\right)\frac{\left(u - u_0\right)\left(u_0 u - 1\right)}{1 + u^2} - \frac{w_0 u_0}{w^2}z^2 \\ &= \left(u - u_0\right)\left(u_0 u - 1\right)\left[\frac{w_0}{1 + u^2} - \alpha(b - u)\right] - \frac{w_0 u_0}{w^2}z^2 \\ &= \left(u - u_0\right)\left(u_0 u - 1\right)f(u) - \frac{w_0 u_0}{w^2}z^2. \end{aligned} \tag{38}$$

Here $f(u) = w_0/(1 + u^2) - \alpha(b - u)$. Recall the assumption that $b < 2\sqrt{1 + u_0^2} - u_0$; then

$$\left(u - u_0\right)\left(u_0 u - 1\right)f(u) \le 0. \tag{39}$$

Therefore, dV/ds is always nonpositive in Ψ. Moreover, $dV/ds = 0$ if and only if $z = 0$, $u = u_0$, and the largest invariant subset of this segment is the single point

$(u_0, w_0, 0)$. By LaSalle's invariance principle, it follows that $y(s) \to (u_0, w_0, 0)$ as $s \to +\infty$, which completes the proof of Theorem 2. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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