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Research Article

Solvability for a Fractional Order Three-Point Boundary Value System at Resonance

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A class of fractional order three-point boundary value system with resonance is investigated in this paper. Using some techniques of inequalities, a completely new method is incorporated. We transform the problem into an integral equation with a pair of undetermined parameters. The topological degree theory is applied to determine the particular value of the parameters so that the system has a solution.

1. Introduction

In this paper, we consider the following fractional differential system:

$$\begin{gathered} D_{0+}^{\alpha} X(t) + f(t, X(t)) = 0, \\ 0 < t < 1, \quad X = (x_1(t), x_2(t)), \\ X(0) = Y(0) = 0, \qquad X(1) = \frac{1}{\eta^{\alpha-1}} X(\eta), \quad 0 < \eta < 1, \end{gathered} \tag{1}$$

where D_{0+}^{α} is standard Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$, $0 < \eta < 1$ and $f = (f_1, f_2)$ is a nonlinear two-dimension continuous vector function.

In the last few decades, many authors have focused on the dynamics of differential equations [1–7]; most of them have investigated fractional differential equations which have been applied in many fields such as physics, mechanics, chemistry, and engineering; see [8–13]. In particular, the positive solutions of the boundary value problem have attracted many authors' attention [14–25].

Recently, the existence of solutions of three-point boundary value problem

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \beta u(\eta), \quad 0 < \eta < 1, \end{gathered} \tag{2}$$

where D_{0+}^{α} is standard Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$ has been studied by many authors under the case that $\beta\eta < 1$. They obtained some nice results by using some fixed point theorems; see [26–28].

In [29], Ahmad and Nieto considered the existence results for following three-point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$\begin{gathered} D_{0+}^{\alpha} u(t) = f\left(t, v(t), D_{0+}^{p} v(t)\right) = 0, \quad 0 < t < 1, \\ D_{0+}^{\beta} v(t) = g\left(t, u(t), D_{0+}^{q} u(t)\right) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \gamma u(\eta), \\ v(0) = 0, \qquad v(1) = \gamma v(\eta), \end{gathered} \tag{3}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\alpha - q \ge 1$, $\beta - p \ge 1$, $\gamma\eta^{\alpha-1} < 1$, $\gamma\eta^{\beta-1} < 1$, D_{0+}^{α} is standard Riemann-Liouville fractional derivative and $f, g : [0, 1] \times R \times R \to R$ are given continuous functions. An existence result was proved in their paper by applying the Schauder fixed point theorem.

However, few authors have investigated fractional differential boundary value problems with resonance [1, 2, 30–32].

In this paper, we establish some sufficient conditions for the existence of the boundary value system (1) by using intermediate value theorems. To present the main results, we assume that $f(t,X) = (f_1(t,X), f_2(t,X))$ satisfies the following.

(H) $f(t,X) \in C([0,1]\times R\times R, R\times R)$, $X=(x_1,x_2)\in R\times R$. Suppose that there exist nonnegative functions $a_i(t)$, $b_{ij}(t)$ $(i,j=1,2)$, with $b_{11}(t)>0$, $b_{22}(t)>0$, $b_{ij}(t)$ $(i,j=1,2,\ i\neq j)\le b_{11}(t), b_{22}(t)$ for any $t\in[0,1]$ such that

$$\left|f_i\left(t,t^{\alpha-1}X\right)\right| \le a_i(t)+b_{i1}(t)\left|x_1\right|^{p_i}+b_{i2}(t)\left|x_2\right|^{q_i},\quad i=1,2, \tag{4}$$

where $0\le p_i,q_i\le 1$ $(i=1,2)$, $q_1<p_1$, and $p_2<q_2$. For any real numbers a and b, the functions $f_i(t,t^{\alpha-1}(u,v))$ $(i=1,2)$ satisfy

$$\lim_{v\to+\infty} f_1\left(t,t^{\alpha-1}(u,v)\right) > -\infty,$$
$$\lim_{v\to-\infty} f_1\left(t,t^{\alpha-1}(u,v)\right) < +\infty, \tag{5}$$
$$\text{for any } u\in R,\quad t\in(0,1],$$

$$\lim_{v\to+\infty} f_2\left(t,t^{\alpha-1}(u,v)\right) > -\infty,$$
$$\lim_{v\to-\infty} f_2\left(t,t^{\alpha-1}(u,v)\right) < +\infty, \tag{6}$$
$$\text{for any } v\in R,\quad t\in(0,1].$$

Furthermore, assume that

$$\begin{gathered}\lim_{v\to+\infty} f_1\left(t,t^{\alpha-1}(v,u(v))\right) = +\infty,\\ \text{for any } u(v)\ge-|v|,\quad t\in(0,1],\\ \lim_{v\to-\infty} f_1\left(t,t^{\alpha-1}(v,u(v))\right) = -\infty,\\ \text{for any } u(v)\le|v|,\quad t\in(0,1],\end{gathered} \tag{7}$$

$$\begin{gathered}\lim_{v\to+\infty} f_2\left(t,t^{\alpha-1}(u(v),v)\right) = +\infty,\\ \text{for any } u(v)\ge-|v|,\quad t\in(0,1],\\ \lim_{v\to-\infty} f_2\left(t,t^{\alpha-1}(u(v),v)\right) = -\infty,\\ \text{for any } u(v)\le|v|,\quad t\in(0,1].\end{gathered} \tag{8}$$

We have the following results.

Theorem 1. *Assume that* (H) *holds. If*

$$\max_{1\le i\le 2}\left\{\int_0^1 G^*(s,s)\left(b_{i1}(s)+b_{i2}(s)\right)ds\right\}<1, \tag{9}$$

where

$$G^*(s,s)=\frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \times\begin{cases}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}, & 0\le s\le\eta,\\ (1-s)^{\alpha-1}, & \eta\le s\le 1,\end{cases} \tag{10}$$

then (1) *has at least one solution in* $[0,1]$.

Also, we consider the following special case of (H) as follows.

$(\widetilde{H})$ $f(t,X) \in C([0,1]\times R\times R, R\times R)$, $X=(x_1,x_2)\in R\times R$. Suppose that there exist nonnegative functions $a_i(t)$, $b_{ij}(t)$ $(i,j=1,2)$, with $b_{11}(t)>0$, $b_{22}(t)>0$, $b_{ij}(t)$ $(i,j=1,2,\ i\neq j)\le b_{11}(t), b_{22}(t)$ for any $t\in[0,1]$ such that

$$\left|f_i\left(t,t^{\alpha-1}X\right)\right| \le a_i(t)+b_{i1}(t)\left|x_1\right|^{p_i}+b_{i2}(t)\left|x_2\right|^{q_i},\quad i=1,2, \tag{11}$$

where $0\le p_i,q_i\le 1$ $(i=1,2)$, $q_1<p_1$, and $p_2<q_2$. The functions $f_i(t,t^{\alpha-1}(u,v))$ $(i=1,2)$ satisfy

$$\lim_{v\to\pm\infty}\left|f_1\left(t,t^{\alpha-1}(u,v)\right)\right|<\infty\quad\text{for any } u\in R,\ t\in(0,1], \tag{12}$$

$$\lim_{u\to\pm\infty}\left|f_2\left(t,t^{\alpha-1}(u,v)\right)\right|<\infty\quad\text{for any } v\in R,\ t\in(0,1]. \tag{13}$$

Furthermore, assume that (7) and (8) hold.

From Theorem 1, we have the following corollary.

Corollary 2. *Assume that* $(\widetilde{H})$ *and* (9) *hold; then* (1) *has at least one solution in* $[0,1]$.

2. Some Lemmas

In this section, we first introduce some definitions and preliminary facts and some lemmas which will be used in this paper.

Definition 3 (see [21]). The fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds \tag{14}$$

provided that the right integral converges.

Definition 4 (see [21]). The standard Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:(0,\infty)\to R$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}y(s)\,ds, \tag{15}$$

where $n=[\alpha]+1$, provided that the right integral converges.

Lemma 5 (see [21]). *Assume that* $u\in C(0,1)\cap L(0,1)$ *with a fractional derivative of order* $\alpha>0$ *that belongs to* $C(0,1)\cap L(0,1)$. *Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}y(t)=y(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n}, \tag{16}$$

for some $C_i\in R$, $i=1,2,\ldots,n$, *where* n *is the smallest integer greater than or equal to* α.

The following lemma is a fixed point theorem in a particular Banach space:

$$\Omega = \{(x(t), y(t)) \mid x(t), y(t) \in C([0,1], R)\}, \tag{17}$$

equipped with the norm

$$\|(x(t), y(t))\| = \max\left\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |y(t)|\right\}. \tag{18}$$

It is easy to show that if $X(t) \in \Omega$, then $t^{\alpha-1}X(t) \in \Omega$.

Lemma 6 (see [33]). *Let X be a Banach space with $C \subset X$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is completely continuous. Then either*

(i) *T has a fixed point in $\overline{U}$, or*

(ii) *there exist an $u \in \partial U$ and $\gamma \in (0,1)$ with $u = \gamma T u$.*

To use this lemma to prove our main result, we need transfer (1) into an integral operator.

Lemma 7 (see [34]). *Problem (1) is equivalent to the following integral equation:*

$$X(t) = \int_0^1 G(t,s) f(s, X(s))\, ds + X(1) t^{\alpha-1}, \tag{19}$$

where

$$G(t,s) = \begin{cases} \left(t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}(\eta-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)(t-s)^{\alpha-1}\right) \times \left(\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)\right)^{-1}, & 0 \le s \le \min\{t,\eta\} \le 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le t \le s \le \eta \le 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)(t-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le \eta \le s \le t \le 1; \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le \max\{t,\eta\} \le s \le 1. \end{cases} \tag{20}$$

Lemma 8 (see [34]). *For any $(t,s) \in [0,1]\times[0,1]$, $G(t,s)$ is continuous, and $G(t,s) > 0$ for any $(t,s) \in (0,1)\times(0,1)$.*

Let

$$G(t,s) = t^{\alpha-1} G^*(t,s), \tag{21}$$

where

$$G^*(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)(1-s/t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le s \le \min\{t,\eta\} \le 1; \\ \dfrac{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le t \le s \le \eta \le 1; \\ \dfrac{(1-s)^{\alpha-1} - \left(1-\eta^{\alpha-1}\right)(1-s/t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le \eta \le s \le t \le 1; \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, & 0 \le \max\{t,\eta\} \le s \le 1. \end{cases} \tag{22}$$

Then (1) is equivalent to the following integral equation:

$$X(t) = \int_0^1 t^{\alpha-1} G^*(t,s) f(s, X(s))\, ds + X(1) t^{\alpha-1}. \tag{23}$$

The new Green's function $G^*(t,s)$ has the following properties.

Lemma 9 (see [34]). *$G^*(t,s)$ is continuous for $(t,s) \in (0,1)\times(0,1)$ and*

$$\lim_{t\to 0^+} G^*(t,s) := G^*(0,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \times \left\{(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right\}, & 0 \le s \le \eta; \\ \dfrac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}(1-s)^{\alpha-1}, & \eta \le s \le 1. \end{cases} \tag{24}$$

Furthermore, $G^*(t,s) > 0$ for $(t,s) \in (0,1)\times(0,1)$.

Lemma 10 (see [34]). *$G^*(t,s)$ is nonincreasing with respect to $t \in [0,1]$ for any $s \in (0,1)$. In particular, for any $s \in [0,1]$, $\partial G^*(t,s)/\partial t \le 0$, and $\partial G^*(t,s)/\partial t = 0$ for $t \in [0,s]$. That is, $G^*(1,s) \le G^*(t,s) \le G^*(s,s)$, where*

$$G^*(1,s) = \frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \times \begin{cases} \eta^{\alpha-1}(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}, & 0 \le s \le \eta; \\ \eta^{\alpha-1}(1-s)^{\alpha-1}, & \eta \le s \le 1, \end{cases}$$

$$G^*(s,s) = \frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \times \begin{cases} (1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}, & 0 \le s \le \eta; \\ (1-s)^{\alpha-1}, & \eta \le s \le 1. \end{cases} \tag{25}$$

Let

$$X(t) = t^{\alpha-1} Y(t). \tag{26}$$

Then $X(1) = Y(1)$, *and* (23) *gives*

$$Y(t) = \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1} Y(s)\right) ds + Y(1). \tag{27}$$

Let

$$W(t) = Y(t) - Y(1). \tag{28}$$

Then $Y(t) = W(t) + Y(1)$, *and* $W(1) = Y(1) - Y(1) = 0$. *From* (27), (28) *can be rewritten as*

$$W(t) = \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}(W(s) + Y(1))\right) ds \tag{29}$$

with $W(1) = 0$. *Now the integral equation* (27) *is equivalent to* (29). *It can be seen from* (29) *that the solution* $W(t)$ *of* (29) *is dependent on the value* $Y(1)$. *Now, instead of* (29), *we replace* $Y(1)$ *with a real vector* $\kappa = (\mu, \nu)$ *and consider*

$$W(t) = \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}(W(s) + \kappa)\right) ds. \tag{30}$$

For any $\kappa = (\mu, \nu)$, *let*

$$K = \{W(t) = (w_1(t), w_2(t)) \in \Omega\}, \tag{31}$$

equipped with the norm $\|W(t)\| = \max\{\max_{t\in[0,1]} w_1(t), \max_{t\in[0,1]} w_2(t)\}$. *Define an operator* T *in* K *as follows:*

$$TW(t) = \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}(W(s) + \kappa)\right) ds. \tag{32}$$

Using a similar method of Lemmas 3.5 and 3.6 in [34], we obtain that T *is completely continuous in* K, *and* (30) *has at least a solution* $W(t)$ *for any given real constant vector* κ; *the solution* $W(t)$ *is dependent the given vector* κ. *We note the solution* $W(t) := W_\kappa(t)$.

3. The Proof of Theorem 1

From Lemma 10, for any real vector κ, the integral equation (30) has at least a solution $W(t)$. Therefore, to show that problem (1) has a solution, it remains to show that there exists a $\kappa = (\mu, \nu)$, such that $W(1) = 0$, or $Y(1) = \kappa = (\mu, \nu)$.

In what follows, we will use the method of topological degree to prove our main result.

Let D be an open subset of the plane R^2 with the boundary ∂D being a simple closed curve; $\widetilde{T}$ is a continuous mapping from $\overline{D} = D \cup \partial D$ to R^2. Let $(c,d) \in R^2$. Denote by A a variable point on the boundary ∂D. As A traverses the boundary, assume that its image $\widetilde{T}(A)$ traces out a closed curve that does not pass through the point (c,d). As in complex analysis, we can define the winding number of this curve with respect to (c,d), by measuring the total change of the argument of the vector joining (c,d) and the variable point $\widetilde{T}(A)$. For two-dimensional space, this number is equivalent to the topological degree of the mapping $\widetilde{T}$ at (c,d).

We introduce a proposition from [8] as follows.

Proposition 11. *If the degree of a continuous mapping* $\widetilde{T}$ *with respect to a point* (c,d) *is nonzero, then the equation* $\widetilde{T}(\mu,\nu) = (c,d)$ *has a solution* $(\mu,\nu) \in D$.

From Section 2, for any parameters μ, ν, $\kappa = (\mu,\nu)$, there exists a solution $W_\kappa(t)$ of (30). At the point $t = 1$, we denote $w_1(1) := \theta$, $w_2(1) := \vartheta$. It is obvious that the parameters θ, ϑ depend on the parameters μ, ν, so we define a map $\widetilde{T}$ as follows:

$$\widetilde{T}(\mu,\nu) = (\theta, \vartheta). \tag{33}$$

Therefore, if we can find a domain D with its boundary as a closed curve L, so that its image $\widetilde{T}(L)$ contains the point $(0,0)$ in it, then it is implied by Proposition 11 that there exists a point $\kappa_0 = (\mu_0, \nu_0)$ in D such that $\widetilde{T}(\mu_0,\nu_0) = (0,0)$. Thus, the function $Y(t) = W_{\kappa_0}(t) + \kappa_0 = W(t) + Y(1)$ is a solution of (29), where

$$W_{\kappa_0}(t) = \int_0^1 G^*(t,s) f\left(s, s^{\alpha-1}\left(W_{\kappa_0}(s) + \kappa_0\right)\right) ds. \tag{34}$$

We now proceed to find such L. For convenience, we take a curve $L = PQRS$, where $P = (-\mu^*, -\nu^*)$, $Q = (-\mu^*, \nu^*)$, $R = (\widetilde{\mu}^*, \nu^*)$, $S = (\widetilde{\mu}^*, -\nu^*)$, and PQ, QR, RS, and SP are a part of line. The image $\widetilde{T}(PQRS) = P'Q'R'S'$. We want to show that the point $(\theta, \vartheta) = (0,0)$ is inside the closed curve $P'Q'R'S'$ as the parameters μ^*, $\widetilde{\mu}^*$, and ν^* are large enough. In fact, we will prove that the line $P'Q'(R'S')$ lies in the left (right) side of the ϑ-axis, and the line $Q'R'(S'P')$ lies above (under) the θ-axis as μ^*, $\widetilde{\mu}^*$, and ν^* are large enough.

Let

$$\begin{gathered}
a_1 = \int_0^1 G^*(s,s) a_1(s)\, ds, \qquad b_{11} = \int_0^1 G^*(s,s) b_{11}(s)\, ds, \\
b_{12} = \int_0^1 G^*(s,s) b_{12}(s)\, ds, \\
a_2 = \int_0^1 G^*(s,s) a_2(s)\, ds, \qquad b_{21} = \int_0^1 G^*(s,s) b_{21}(s)\, ds, \\
b_{22} = \int_0^1 G^*(s,s) b_{22}(s)\, ds.
\end{gathered} \tag{35}$$

From (9) and (H), $a_1, a_2 \ge 0$ and $0 < b_{11}, b_{12}, b_{21}, b_{22} < 1$, and $b_{12} < b_{11}$, $b_{21} < b_{22}$, we may take μ^*, $\widetilde{\mu}^*$, and ν^* large enough satisfying

$$\mu^* = \frac{b_{12}}{b_{11}} \nu^*, \tag{36}$$

$$\widetilde{\mu}^* = \frac{b_{22}}{b_{21}} \nu^*. \tag{37}$$

Then, the points P, Q, R, and S can be expressed as follows:

$$\begin{aligned} P &= \left(-\frac{b_{12}}{b_{11}}\nu^*, -\nu^*\right), \\ Q &= \left(-\frac{b_{12}}{b_{11}}\nu^*, \nu^*\right), \\ R &= \left(\frac{b_{22}}{b_{21}}\nu^*, \nu^*\right), \\ S &= \left(\frac{b_{22}}{b_{21}}\nu^*, -\nu^*\right). \end{aligned} \tag{38}$$

Now the proof of Theorem 1 is reduced as the following lemmas.

Lemma 12. *Suppose that* (H) *and* (9) *hold. Then, for* ν^* *large enough,* P' *lies in the third quadrant.*

Proof. From (30), we have

$$W_\kappa(1) := W(1) = \int_0^1 G^*(1,s)\, f\left(s, s^{\alpha-1}(W(s)+\kappa)\right) ds. \tag{39}$$

By the definition of θ, ϑ in (33), we may rewrite (39) as follows:

$$\begin{aligned} \theta_\kappa &= w_1(1) = \int_0^1 G^*(1,s)\, f_1\left(s, s^{\alpha-1}(W(s)+\kappa)\right) ds, \\ \vartheta_\kappa &= w_2(1) = \int_0^1 G^*(1,s)\, f_2\left(s, s^{\alpha-1}(W(s)+\kappa)\right) ds. \end{aligned} \tag{40}$$

Let $\kappa^* = (-\mu^*, -\nu^*) = (-(b_{12}/b_{11})\nu^*, -\nu^*)$. From (40), we have

$$\begin{aligned} \theta_{\kappa^*} &= w_1(1) = \int_0^1 G^*(1,s)\, f_1\left(s, s^{\alpha-1}\left(W(s)+\kappa^*\right)\right) ds, \\ \vartheta_{\kappa^*} &= w_2(1) = \int_0^1 G^*(1,s)\, f_2\left(s, s^{\alpha-1}\left(W(s)+\kappa^*\right)\right) ds. \end{aligned} \tag{41}$$

Now we will show that $\theta_{\kappa^*}, \vartheta_{\kappa^*} \to -\infty$ as $\nu^* \to \infty$. We only show that $\lim_{\nu^*\to\infty}\theta_{\kappa^*} = -\infty$ and the proof of $\lim_{\nu^*\to\infty}\vartheta_{\kappa^*} = -\infty$ is similar. Assume on the contrary that $\overline{\lim}_{\nu^*\to\infty}\theta_{\kappa^*} = l > -\infty$. Thus, there exists a sequence $\{\kappa_n\} = \{(\mu_n, \nu_n)\}$, $\mu_n = (b_{12}/b_{11})\nu_n < 0$ such that $\lim_{\nu_n\to-\infty}\theta_n = l > -\infty$.

Recall that

$$\begin{aligned} w_{1\kappa_n}(t) &= \int_0^1 G^*(t,s)\, f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds, \\ w_{2\kappa_n}(t) &= \int_0^1 G^*(t,s)\, f_2\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds. \end{aligned} \tag{42}$$

Now we claim that it is impossible to have

$$f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t)+\kappa_n\right)\right) \le 0, \quad \forall t \in [0,1] \tag{43}$$

as $-\nu_n$ is sufficiently large. Indeed, assume that (43) is true. Then, by the first equation of (42), we have

$$w_{1\kappa_n}(t) \le 0 \tag{44}$$

for all $t \in [0,1]$. Therefore, we obtain

$$\lim_{\mu_n\to-\infty}\left(w_{1\kappa_n}(t)+\mu_n\right) = \lim_{\nu_n\to-\infty}\left(w_{1\kappa_n}(t)+\frac{b_{12}}{b_{11}}\nu_n\right) = -\infty \tag{45}$$

for $t \in [0,1]$. We define some sets as follows:

$$\begin{aligned} A_n &= \left\{t \in [0,1] : \left|w_{1\kappa_n}(t)+\frac{b_{12}}{b_{11}}\nu_n\right| \ge w_{2\kappa_n}(t)+\nu_n\right\}, \\ B_n &= \left\{t \in [0,1] : \left|w_{1\kappa_n}(t)+\frac{b_{12}}{b_{11}}\nu_n\right| < w_{2\kappa_n}(t)+\nu_n\right\}, \\ C_n &= \left\{t \in [0,1] : w_{2\kappa_n}(t)+\nu_n \ge 0\right\}. \end{aligned} \tag{46}$$

We have assumed in (7) that

$$\begin{gathered} \lim_{\nu\to-\infty} f_1\left(t, t^{\alpha-1}(\nu, u(\nu))\right) = -\infty \\ \text{for any } u(\nu) \le |\nu|, \quad t \in (0,1]. \end{gathered} \tag{47}$$

It is easy to show from (42), (H), and our assumption that the set B_n is not empty, and $B_n \subset C_n$. We have the following:

$$\begin{aligned} \lim_{\nu_n\to-\infty}\left\|w_{2\kappa_n}(t)\right\|_{B_n} &= \lim_{\nu_n\to-\infty}\left\|w_{2\kappa_n}(t)\right\|_{C_n} \\ &= \lim_{\nu_n\to-\infty}\max_{t\in B_n} w_{2\kappa_n}(t) = +\infty. \end{aligned} \tag{48}$$

Using conditions (6) and (8), we have from (45) that there exists a constant l such that

$$f_2\left(t, t^{\alpha-1}\left(w_{1\kappa_n}(t)+\frac{b_{12}}{b_{11}}\nu_n, w_{2\kappa_n}(t)+\nu_n\right)\right) < l \tag{49}$$

for $t \in [0,1] \setminus C_n$ and any n large enough. From the second formula of (42), (45)–(49), one gets

$$\begin{aligned}
w_{2\kappa_n}(t) \le& \left(\int_{[0,1]\setminus C_n} + \int_{C_n\cap A_n} + \int_{C_n \cap B_n}\right) G^*(s,s) \\
&\times f_2\left(s, s^{\alpha-1}\left(w_{1\kappa_n}(s) + \frac{b_{12}}{b_{11}}\nu_n, w_{2\kappa_n}(s) + \nu_n\right)\right) ds \\
\le& \, l \int_{[0,1]\setminus C_n} G^*(s,s)\, ds \\
&+ \int_{C_n\cap B_n} G^*(s,s) \\
&\times f_2\left(s, s^{\alpha-1}\left(w_{1\kappa_n}(s) + \frac{b_{12}}{b_{11}}\nu_n,\right.\right. \\
&\left.\left. w_{2\kappa_n}(s)(+\nu_n)\right)\right) ds \\
=& \, l \int_{[0,1]\setminus C_n} G^*(s,s)\, ds \\
&+ \int_{B_n} G^*(s,s) \\
&\times f_2\left(s, s^{\alpha-1}\left(w_{1\kappa_n}(s) + \frac{b_{12}}{b_{11}}\nu_n,\right.\right. \\
&\left.\left. w_{2\kappa_n}(s) + \nu_n\right)\right) ds \\
\le& \, l \int_{[0,1]\setminus C_n} G^*(s,s)\, ds + \int_0^1 G^*(s,s)\, a_2(s)\, ds \\
&+ \left\| w_{1\kappa_n}(t) + \frac{b_{21}}{b_{11}}\nu_n \right\|_{B_n}^{p_2} \int_0^1 G^*(s,s)\, b_{21}(s)\, ds \\
&+ \left\| w_{2\kappa_n}(t) + \nu_n \right\|_{B_n}^{q_2} \int_0^1 G^*(s,s)\, b_{22}(s)\, ds \\
\le& \, l \int_{[0,1]\setminus C_n} G^*(s,s)\, ds + a_2 + (b_{21} + b_{22}) \left\| w_{2\kappa_n}(t) \right\|_{C_n}, \\
& t \in C_n,
\end{aligned} \tag{50}$$

which implies that

$$\left\| w_{2\kappa_n}(t) \right\|_{C_n} \le \frac{l \int_{[0,1]\setminus C_n} G^*(s,s)\, ds + a_2}{1 - b_{21} - b_{22}} < \infty. \tag{51}$$

It contradicts (48).

Thus, for any $-\nu_n$ large enough, there exists some $t \in (0,1]$, such that

$$f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) > 0. \tag{52}$$

Now we define

$$\begin{aligned}
I_n &= \left\{t \in [0,1] : f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) > 0\right\}, \\
I_n' &= \left\{t \in [0,1] : f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) > 0\right\}.
\end{aligned} \tag{53}$$

Then, I_n is not empty.

We can further divide the set I_n into two sets $\widetilde{I_n}$ and $\widehat{I_n}$, and divide the set I_n' into two sets $\widetilde{I_n'}$ and $\widehat{I_n'}$ as follows:

$$\begin{aligned}
\widetilde{I_n} &= \left\{t \in I_n \mid w_{1\kappa_n}(t) + \frac{b_{12}}{b_{11}}\nu_n \le 0\right\}, \\
\widehat{I_n} &= \left\{t \in I_n \mid w_{1\kappa_n}(t) + \frac{b_{12}}{b_{11}}\nu_n > 0\right\}, \\
\widetilde{I_n'} &= \left\{t \in I_n' \mid w_{2\kappa_n}(t) + \nu_n \le 0\right\}, \\
\widehat{I_n'} &= \left\{t \in I_n' \mid w_{2\kappa_n}(t) + \nu_n > 0\right\}.
\end{aligned} \tag{54}$$

It is easy to know that $\widetilde{I_n} \cap \widehat{I_n} = \phi$, $\widetilde{I_n'} \cap \widehat{I_n'} = \phi$ and $I_n = \widetilde{I_n} \cup \widehat{I_n}$, $I_n' = \widetilde{I_n'} \cup \widehat{I_n'}$.

We claim that the set $\widehat{I_n}$ is not empty for $-\nu_n$ large enough. Otherwise, the function $f_1(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n))$ is bounded from above. In fact, assume that $f_1(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n))$ is unbounded from above for $-\nu_n$ large enough; then we have from (H) that there exist a sequence $\{t_i\}$ and a subsequence $\{\nu_{n_i}\}$ of $\{\nu_n\}$ such that

$$\begin{aligned}
&\lim_{\nu_{n_i} \to -\infty} w_{2\kappa_{n_i}}(t_i) = \infty, \\
&\lim_{\nu_{n_i} \to -\infty} \left| w_{1\kappa_{n_i}}(t_i) + \frac{b_{12}}{b_{11}}\nu_{n_i} \right| \le \lim_{\nu_{n_i} \to -\infty} \left(w_{2\kappa_{n_i}}(t_i) + \nu_{n_i}\right) \\
&\qquad = +\infty.
\end{aligned} \tag{55}$$

Using a similar method of (51), we can derive a contradiction. Therefore, $f_1(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n))$ is bounded from above. From (42), $w_{1\kappa_n}(t)$ is bounded from above, which implies that $w_{1\kappa_n}(t) + (b_{12}/b_{11})\nu_n \to -\infty$ as $\nu_n \to -\infty$. If $B_n = \phi$ (where B_n is defined in (46)), then $\lim_{\nu_n \to -\infty} \theta_n = -\infty$, which contradicts our assumption. Thus, $B_n \neq \phi$. Using a similar method of getting (51) also gives a contradiction. Therefore, $\widehat{I_n}$ is not empty.

Similarly as getting (51) again, we conclude that the function $f_i(t, t^{\alpha-1}X)$ is bounded above by a constant for $t \in [0,1]$ and $x_i \in (-\infty, 0]$ $(i = 1,2)$. From the condition (H), if $w_{1\kappa_n}(t) + \nu_n > 0$ (or $w_{2\kappa_n}(t) + \mu_n > 0$) and $f_2(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n)) < 0$ (or $f_1(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n)) < 0$), then $w_{2\kappa_n}(t) + \nu_n$ (or $w_{1\kappa_n}(t) + \mu_n$) is also bounded from above by a constant for $t \in [0,1]$. Therefore, from the definition of $\widetilde{I_n}$, $\widetilde{I_n'}$, there exists a constant $M > 1$, independent of t and ν_n such that

$$\begin{aligned}
&f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) \le M, \quad \text{for } t \in \widetilde{I_n}, \\
&f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) \le M, \quad \text{for } t \in \widetilde{I_n'}, \\
&w_{2\kappa_n}(t) + \nu_n \le M \\
&\text{for } f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) < 0, \quad t \in \widehat{I_n},
\end{aligned}$$

$$
\begin{aligned}
& w_{1\kappa_n}(t)+\frac{b_{12}}{b_{11}}\nu_n \le M, \\
& \text{for } f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t)+\kappa_n\right)\right)<0, \quad t \in \widehat{I_n'}.
\end{aligned} \tag{56}
$$

Let

$$
\overline{M}_1\left(\kappa_n\right)=\max_{t \in I_n} w_{1\kappa_n}(t). \tag{57}
$$

From the definitions of $\widetilde{I}_n$ and $\widehat{I_n}$, we have

$$
\overline{M}_1\left(\kappa_n\right)=\max_{t \in \widehat{I_n}} w_{1\kappa_n}(t)=\left\|w_{1\kappa_n}(t)\right\|_{I_n}. \tag{58}
$$

Since $\widehat{I_n}$ is not empty, it follows that $\overline{M}_1(\kappa_n) \to \infty$ as $\nu_n \to -\infty$. Recall from (9) and (35) that $b_{ij}<1$ $(i, j=1,2)$. Therefore, we can choose $\nu_{n_1}>0$ large enough so that

$$
\overline{M}_1\left(\kappa_n\right)>\max\left\{1, P_1, P_2\right\} \tag{59}
$$

for $\nu_n<-\nu_{n_1}$, where

$$
P_1=\frac{M\left(\int_0^1 G^*(s, s)\, ds+b_{12}\right)+a_1+b_{11}}{1-b_{11}}, \tag{60}
$$

$$
\begin{aligned}
P_2= & \left(M \int_0^1 G^*(s, s)\, ds\left(1-b_{22}+b_{12}\right)\right. \\
& +b_{12}\left(1-b_{22}+b_{21}\right)+\left(a_2+b_{22}\right) b_{12} \\
& \left.+\left(a_1+b_{11}\right)\left(1-b_{22}\right)\right) \\
& \times\left((1-b_{11})(1-b_{22})-b_{12} b_{21}\right)^{-1}.
\end{aligned} \tag{61}
$$

Now, for later use, for any integral in a domain A

$$
\int_A G^*(s, s)\, b_{ij}(s)\, g(s)\, ds, \quad \text{for } g(s)>0,\ i, j=1,2, \tag{62}
$$

we define a subset $(A)_1$ as

$$
(A)_1=\{t \in A \mid g(t) \ge 1\}. \tag{63}
$$

Thus, the integral in (62) can be rewritten as

$$
\begin{aligned}
\int_A G^*(s, s)\, b_{ij}(s)\, g(s)\, ds= & \int_{(A)_1} G^*(s, s)\, b_{ij}(s)\, g(s)\, ds \\
& +\int_{A \backslash(A)_1} G^*(s, s)\, b_{ij}(s)\, g(s)\, ds.
\end{aligned} \tag{64}
$$

From (H), (42), and the definitions of $\widetilde{I}_n$, $\widehat{I_n}$ and $\widetilde{I_n'}$, $\widehat{I_n'}$, for $\nu_n<-\nu_{n_1}$, we have

$$
\begin{aligned}
w_{1\kappa_n}(t)= & \int_0^1 G^*(t, s)\, f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\
\le & \int_{I_n} G^*(t, s)\, f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\
\le & \int_{\widetilde{I}_n} G^*(s, s)\, f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\
& +\int_{\widehat{I_n}} G^*(s, s)\, a_1(s)\, ds \\
& +\int_{\widehat{I_n}} G^*(s, s)\left(b_{11}(s)\left|w_{1\kappa_n}(s)+\frac{b_{12}}{b_{11}}\nu_n\right|^{p_1}\right. \\
& \left.+b_{12}(s)\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_1}\right) ds \\
\le & \int_{\widetilde{I}_n} G^*(s, s)\, f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\
& +\int_{\widehat{I_n}} G^*(s, s)\, a_1(s)\, ds \\
& +\int_{\widehat{I_n} \backslash(\widehat{I_n})_1} G^*(s, s)\, b_{11}(s)\left|w_{1\kappa_n}(s)+\frac{b_{12}}{b_{11}}\nu_n\right|^{p_1} ds \\
& +\int_{(\widehat{I_n})_1} G^*(s, s)\, b_{11}(s)\left|w_{1\kappa_n}(s)+\frac{b_{12}}{b_{11}}\nu_n\right|^{p_1} ds \\
& +\int_{(\widehat{I_n} \cap I_n')_1} G^*(s, s)\, b_{12}(s)\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_1} ds \\
& +\int_{(\widehat{I_n} \cap([0,1] \backslash I_n')) \cup((\widehat{I_n} \cap I_n') \backslash(\widehat{I_n} \cap I_n')_1)} G^*(s, s)\, b_{12}(s) \\
& \times\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_1} ds,
\end{aligned} \tag{65}
$$

which yields from (56) and the definition in (63) that

$$
\begin{aligned}
w_{1\kappa_n}(t) \le & \int_{\widetilde{I}_n} G^*(s, s)\, f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\
& +\int_{\widehat{I_n}} G^*(s, s)\, a_1(s)\, ds+\int_{\widehat{I_n} \backslash(\widehat{I_n})_1} G^*(s, s)\, b_{11}(s)\, ds \\
& +\int_{(\widehat{I_n})_1} G^*(s, s)\, b_{11}(s)\left|w_{1\kappa_n}(s)+\frac{b_{12}}{b_{11}}\nu_n\right| ds \\
& +\int_{(\widehat{I_n} \cap I_n')_1} G^*(s, s)\, b_{12}(s)\left|w_{2\kappa_n}(s)+\nu_n\right| ds \\
& +M \int_{(\widehat{I_n} \cap([0,1] \backslash I_n')) \cup((\widehat{I_n} \cap I_n') \backslash(\widehat{I_n} \cap I_n')_1)} G^*(s, s)\, b_{12}(s)\, ds.
\end{aligned} \tag{66}
$$

Further, one gets from (56) that

$$\begin{aligned} w_{1\kappa_n}(t) \le & \int_0^1 G^*(s,s) M\,ds + \int_0^1 G^*(s,s) a_1(s)\,ds \\ & + \int_0^1 G^*(s,s) b_{11}(s)\,ds + M\int_0^1 G^*(s,s) b_{12}(s)\,ds \\ & + \int_0^1 G^*(s,s) b_{11}(s)\,ds \left(\overline{M}_1(\kappa_n) - \frac{b_{12}}{b_{11}}\|\nu_n\|\right) \\ & + \int_0^1 G^*(s,s) b_{12}(s)\,ds \left(\|w_{2\kappa_n}(t)\|_{\widehat{I_n}\cap I_n'} + \|\nu_n\|\right) \\ = & M\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11} + b_{11}\overline{M}_1(\kappa_n) \\ & + b_{12}\|w_{2\kappa_n}(t)\|_{\widehat{I_n}\cap I_n'}, \end{aligned} \quad (67)$$

which gives

$$\begin{aligned} \overline{M}_1(\kappa_n) < & M\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11} + b_{11}\overline{M}_1(\kappa_n) \\ & + b_{12}\|w_{2\kappa_n}(t)\|_{\widehat{I_n}\cap I_n'}. \end{aligned} \quad (68)$$

That is,

$$\begin{aligned} & \overline{M}_1(\kappa_n) \\ & < \frac{M\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11} + b_{12}\|w_{2\kappa_n}(t)\|_{\widehat{I_n}\cap I_n'}}{1 - b_{11}}. \end{aligned} \quad (69)$$

If $\widehat{I_n} \cap I_n' = \phi$, then we have from (69) that

$$\overline{M}_1(\kappa_n) < \frac{M\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11}}{1 - b_{11}}, \quad (70)$$

which contradicts (59).

If $\widehat{I_n} \cap I_n' \neq \phi$, using a similar method of (69), we can estimate $w_{2\kappa_n}(t)$ as

$$\begin{aligned} & \|w_{2\kappa_n}(t)\|_{\widehat{I_n}\cap I_n'} \\ & \le \|w_{2\kappa_n}(t)\|_{I_n'} \\ & < \frac{M\left(\int_0^1 G^*(s,s)\,ds + b_{21}\right) + a_2 + b_{22} + b_{21}\overline{M}(\kappa_n)}{1 - b_{22}}. \end{aligned} \quad (71)$$

Substituting this into (69), we obtain

$$\begin{aligned} & \overline{M}(\kappa_n) \\ & < \Bigg(M\left(\left(\int_0^1 G^*(s,s)\,ds\,(1 - b_{22} + b_{12}) + b_{12}(1 - b_{22} + b_{21})\right)\right. \\ & \quad \left. + (a_2 + b_{22}) b_{12} + (a_1 + b_{11})(1 - b_{22})\right) \\ & \quad \times \left((1 - b_{11})(1 - b_{22}) - b_{12}b_{21}\right)^{-1}, \end{aligned} \quad (72)$$

which finally contradicts (59). Therefore, our result is proved.

Similarly, we can show that $\lim_{\nu^*\to\infty}\vartheta_{\kappa^*} = -\infty$. Thus, the point $\widetilde{T}(-\mu^*, -\nu^*)$ lies in the third quadrant. The proof is completed. □

Lemma 13. *Suppose that* (H) *and* (9) *hold. Then, for* $\nu^* > 0$ *large enough,* Q' *lies in the second quadrant.*

Proof. It suffices to show that $\lim_{\nu^*\to\infty}\vartheta^* = \infty$ and $\lim_{\nu^*\to\infty}\theta^* = -\infty$.

First, we claim that $\lim_{\nu^*\to\infty}\vartheta^* = \infty$. On the contrary, we assume that there exists a sequence $\{\kappa_n\} = \{(\mu_n, \nu_n)\} = \{(-(b_{12}/b_{11})\nu_n, \nu_n)\}$ such that $\lim_{\nu_n\to\infty}\vartheta_n = l < \infty$. By a similar method in Lemma 12, we know it is impossible to have

$$f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) \ge 0, \quad \forall t \in [0,1] \quad (73)$$

as ν_n is sufficiently large.

Now, for large ν_n, we define

$$\begin{aligned} J_n &= \left\{t \in [0,1] : f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) < 0\right\}, \\ J_n' &= \left\{t \in [0,1] : f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) < 0\right\}. \end{aligned} \quad (74)$$

Then, J_n' is not empty.

As in Lemma 12, we can further divide the set J_n into two sets $\widetilde{J_n}$ and $\widehat{J_n}$ and divide the set J_n' into two sets $\widetilde{J_n'}$ and $\widehat{J_n'}$ as follows:

$$\begin{aligned} \widetilde{J_n} &= \left\{t \in J_n \mid w_{1\kappa_n}(t) - \frac{b_{12}}{b_{11}}\nu_n \ge 0\right\}, \\ \widehat{J_n} &= \left\{t \in J_n \mid w_{1\kappa_n}(t) - \frac{b_{12}}{b_{11}}\nu_n < 0\right\}, \\ \widetilde{J_n'} &= \left\{t \in J_n' \mid w_{2\kappa_n}(t) + \nu_n \ge 0\right\}, \\ \widehat{J_n'} &= \left\{t \in J_n' \mid w_{2\kappa_n}(t) + \nu_n < 0\right\}. \end{aligned} \quad (75)$$

Then $\widetilde{J_n} \cap \widehat{J_n} = \phi$, $\widetilde{J_n'} \cap \widehat{J_n'} = \phi$ and $J_n = \widetilde{J_n} \cup \widehat{J_n}$, $J_n' = \widetilde{J_n'} \cup \widehat{J_n'}$.

Using a similar method as in the proof of Lemma 12, we can show that the set $\widehat{J_n'}$ is not empty. Furthermore, the function $f_i(t, t^{\alpha-1}X)$ is bounded below by a constant for $t \in [0,1]$ and $x_i \in [0,\infty)$ $(i = 1,2)$. If $w_{2\kappa_n}(t) + \mu_n < 0$ (or $w_{1\kappa_n}(t) + \nu_n < 0$) and $f_1(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n)) > 0$ (or $f_2(t, t^{\alpha-1}(W_{\kappa_n}(t) + \kappa_n)) > 0$), then $w_{1\kappa_n}(t) + \nu_n$ (or $w_{2\kappa_n}(t) + \mu_n$)

is also bounded below by a constant for $t \in [0,1]$. From the definition of $\widetilde{J}_n$, $\widetilde{J}_n'$ and the condition (H), there exists a constant $\widetilde{M} < -1$, independent of t and ν_n such that

$$\begin{aligned} & f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t)+\kappa_n\right)\right) \geq \widetilde{M}, \quad \text{for } t \in \widetilde{J}_n, \\ & f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t)+\kappa_n\right)\right) \geq \widetilde{M}, \quad \text{for } t \in \widetilde{J}_n', \\ & w_{2\kappa_n}(t)+\nu_n \geq \widetilde{M} \\ & \text{for } f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t)+\kappa_n\right)\right)>0, \quad t \in \widehat{J}_n, \\ & w_{1\kappa_n}(t)-\frac{b_{12}}{b_{11}} \nu_n \geq \widetilde{M}, \\ & \text{for } f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t)+\kappa_n\right)\right)>0, \quad t \in \widehat{J}_n'. \end{aligned} \tag{76}$$

Let

$$\overline{m}_2\left(\kappa_n\right)=\min_{t \in J_n'} w_{2\kappa_n}(t). \tag{77}$$

From the definitions of $\widetilde{J}_n'$ and $\widehat{J}_n'$, we have

$$\overline{m}_2\left(\kappa_n\right)=\min_{t \in \widehat{J}_n'} w_{2\kappa_n}(t)=-\left\|w_{2\kappa_n}(t)\right\|_{J_n'}, \tag{78}$$

and it follows that $\overline{m}_2(\kappa_n) \to -\infty$ as $\nu_n \to \infty$. Therefore, we can choose ν_{n_1} large enough so that

$$\overline{m}_2\left(\kappa_n\right)<\min\left\{-1, Q_1, Q_2\right\} \tag{79}$$

for $\nu_n > \nu_{n_1}$, where

$$\begin{aligned} Q_1 = & \frac{\widetilde{M} \int_0^1 G^*(s, s)\, ds\left(1+b_{21}\right)-a_2-b_{22}}{1-b_{22}}, \\ Q_2 = & \frac{\int_0^1 G^*(s, s)\, ds\left(1-b_{11}-b_{21}\right)+b_{21}\left(1-b_{11}-b_{21}\right)}{\left(1-b_{22}\right)\left(1-b_{11}\right)-b_{21} b_{12}} \widetilde{M} \\ & -\frac{\left(a_2+b_{22}\right)\left(1-b_{11}\right)+\left(a_1+b_{11}\right) b_{21}}{\left(1-b_{22}\right)\left(1-b_{11}\right)-b_{21} b_{12}}. \end{aligned} \tag{80}$$

Notice that $b_{12} < b_{11}$, $b_{21} < b_{22}$. From (H), (42), and the definitions of $\widetilde{J}_n$, $\widehat{J}_n$ and $\widetilde{J}_n'$, $\widehat{J}_n'$, for $\nu_n > \nu_{n_1}$, we have

$$\begin{aligned} w_{2\kappa_n}(t) \geq & \int_{J_n'} G^*(s, s) f_2\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\ \geq & \int_{\widetilde{J}_n'} G^*(s, s) f_2\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\ & -\int_{\widehat{J}_n'} G^*(s, s) a_2(s)\, ds \\ & -\int_{\widehat{J}_n'} G^*(s, s)\left(b_{21}(s)\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}} \nu_n\right|^{p_2}\right. \\ & \left.+b_{22}(s)\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_2}\right) ds \\ \geq & \int_{\widetilde{J}_n'} G^*(s, s) f_2\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\ & -\int_{\widehat{J}_n'} G^*(s, s) a_2(s)\, ds \\ & -\int_{\widehat{J}_n' \cap J_n} G^*(s, s) b_{21}(s)\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}} \nu_n\right|^{p_2} ds \\ & -\int_{\widehat{J}_n' \cap([0,1] \backslash J_n)} G^*(s, s) b_{21}(s)\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}} \nu_n\right|^{p_2} ds \\ & -\int_{\widehat{J}_n'} G^*(s, s) b_{22}(s)\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_2} ds. \end{aligned} \tag{81}$$

Thus,

$$\begin{aligned} & w_{2\kappa_n}(t) \\ & \quad \geq \int_{\widetilde{J}_n'} G^*(s, s) f_2\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right) ds \\ & \quad -\int_{\widehat{J}_n'} G^*(s, s) a_2(s)\, ds \\ & \quad -\int_{(\widehat{J}_n' \cap J_n)_1} G^*(s, s) b_{21}(s)\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}} \nu_n\right|^{p_2} ds \\ & \quad -\int_{(\widehat{J}_n' \cap([0,1] \backslash J_n)) \cup((\widehat{J}_n' \cap J_n) \backslash(\widehat{J}_n' \cap J_n)_1)} G^*(s, s) b_{21}(s) \\ & \quad \times\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}} \nu_n\right|^{p_2} ds \\ & \quad -\int_{(\widehat{J}_n')_1} G^*(s, s) b_{22}(s)\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_2} ds \\ & \quad -\int_{\widehat{J}_n' \backslash(\widehat{J}_n')_1} G^*(s, s) b_{22}(s)\left|w_{2\kappa_n}(s)+\nu_n\right|^{q_2} ds, \end{aligned} \tag{82}$$

which follows from (76) and the definition of (63) that

$$\begin{aligned}
w_{2\kappa_n}(t) \geq{} & \widetilde{M}\int_{\widetilde{J}_n'} G^*(s,s)\,ds - \int_{\widehat{J}_n'} G^*(s,s)\,a_2(s)\,ds \\
& + \widetilde{M}\int_{\widehat{J}_n'} G^*(s,s)\,b_{21}(s)\,ds \\
& - \int_{(\widehat{J}_n'\cap J_n)_1} G^*(s,s)\,b_{21}(s)\,ds \\
& \times \left\| w_{1\kappa_n}(t) - \frac{b_{12}}{b_{11}}\nu_n \right\|_{(\widehat{J}_n'\cap J_n)_1} \\
& - \int_{(\widehat{J}_n')_1} G^*(s,s)\,b_{22}(s)\,ds \left\| w_{2\kappa_n}(t) + \nu_n \right\|_{\widehat{J}_n'} \\
& - \int_{\widehat{J}_n'\setminus(\widehat{J}_n')_1} G^*(s,s)\,b_{22}(s)\,ds \\
\geq{} & \widetilde{M}\int_0^1 G^*(s,s)\,ds - \int_0^1 G^*(s,s)\,a_2(s)\,ds \\
& + \widetilde{M}\int_0^1 G^*(s,s)\,b_{21}(s)\,ds \\
& - \int_0^1 G^*(s,s)\,b_{21}(s)\,ds \\
& \times \left(\left\| w_{1\kappa_n}(t) \right\|_{\widehat{J}_n'\cap J_n} + \left\| \frac{b_{12}}{b_{11}}\nu_n \right\| \right) \\
& - \int_0^1 G^*(s,s)\,b_{22}(s)\,ds \\
& - \int_0^1 G^*(s,s)\,b_{22}(s)\,ds \left(-\overline{m}_2(\kappa_n) - \|\nu_n\| \right) \\
& - \int_0^1 G^*(s,s)\,b_{22}(s)\,ds \\
={} & \widetilde{M}\left(\int_0^1 G^*(s,s)\,ds + b_{21} \right) - a_2 - b_{22} \\
& + \frac{\|\nu_n\|\left(b_{11}b_{22} - b_{12}b_{21}\right)}{b_{11}} \\
& - b_{21}\left\| w_{1\kappa_n}(t) \right\|_{\widehat{J}_n'\cap J_n} + b_{22}\overline{m}_2(\kappa_n).
\end{aligned} \tag{83}$$

Thus,

$$\begin{aligned}
\overline{m}_2(\kappa_n) >{} & \widetilde{M}\left(\int_0^1 G^*(s,s)\,ds + b_{21} \right) - a_2 - b_{22} \\
& - b_{21}\left\| w_{1\kappa_n}(t) \right\|_{\widehat{J}_n'\cap J_n} + b_{22}\overline{m}_2(\kappa_n),
\end{aligned} \tag{84}$$

which implies that

$$\begin{aligned}
& \overline{m}_2(\kappa_n) \\
& \quad > \frac{\widetilde{M}\left(\int_0^1 G^*(s,s)\,ds + b_{21} \right) - a_2 - b_{22} - b_{21}\left\| w_{1\kappa_n}(t) \right\|_{\widehat{J}_n'\cap J_n}}{1 - b_{22}}.
\end{aligned} \tag{85}$$

If $\widehat{J}_n' \cap J_n = \phi$, from (85), we have

$$\overline{m}_2(\kappa_n) > \frac{\widetilde{M}\left(\int_0^1 G^*(s,s)\,ds + b_{21} \right) - a_2 - b_{22}}{1 - b_{22}}, \tag{86}$$

which contradicts (79).

If $\widehat{J}_n' \cap J_n \neq \phi$. Using a similar method of (85), we have

$$\begin{aligned}
& - \left\| w_{1\kappa_n}(t) \right\|_{\widehat{J}_n'\cap J_n} \\
& \quad \geq - \left\| w_{1\kappa_n}(t) \right\|_{J_n} \\
& \quad > \frac{\widetilde{M}\left(\int_0^1 G^*(s,s)\,ds + b_{12} \right) - a_1 - b_{11} - b_{12}\overline{m}_2(\kappa_n)}{1 - b_{11}}.
\end{aligned} \tag{87}$$

Substituting it into (85), we obtain

$$\begin{aligned}
& \overline{m}_2(\kappa_n) \\
& \quad > \frac{\int_0^1 G^*(s,s)\,ds\left(1 - b_{11} - b_{21}\right) + b_{21}\left(1 - b_{11} - b_{21}\right)}{\left(1 - b_{22}\right)\left(1 - b_{11}\right) - b_{21}b_{12}}\widetilde{M} \\
& \quad - \frac{\left(a_2 + b_{22}\right)\left(1 - b_{11}\right) + \left(a_1 + b_{11}\right)b_{21}}{\left(1 - b_{22}\right)\left(1 - b_{11}\right) - b_{21}b_{12}},
\end{aligned} \tag{88}$$

which also contradicts (79). Therefore, $\lim_{\nu^*\to\infty}\vartheta = \infty$.

Now, we show that $\lim_{\nu^*\to\infty}\theta = -\infty$.

On the contrary, assume that there exists a vector sequence $\{\kappa_n\} = \{(\mu_n, \nu_n)\}$ such that $\mu_n = -(b_{12}/b_{11})\nu_n$ and

$$\lim_{\nu_n\to\infty}\theta_n = l > -\infty. \tag{89}$$

Similarly as before, it is impossible to have

$$f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) \leq 0, \quad \forall t \in [0,1] \tag{90}$$

as ν_n is sufficiently large.

Now for large ν_n, we define

$$\begin{aligned}
\overline{I}_n &= \left\{ t \in [0,1] : f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) > 0 \right\}, \\
\overline{I}_n' &= \left\{ t \in [0,1] : f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) > 0 \right\}.
\end{aligned} \tag{91}$$

Then, $\overline{I}_n$ is not empty.

We can further divide the set $\bar{I}_n$ into two sets $\tilde{\bar{I}}_n$ and $\hat{\bar{I}}_n$ and divide the set $\bar{I}'_n$ into two sets $\tilde{\bar{I}}'_n$ and $\hat{\bar{I}}'_n$ as follows:

$$\begin{aligned}
\tilde{\bar{I}}_n &= \left\{t \in \bar{I}_n \mid w_{1\kappa_n}(t) - \frac{b_{12}}{b_{11}}\nu_n \le 0\right\}, \\
\hat{\bar{I}}_n &= \left\{t \in I_n \mid w_{1\kappa_n}(t) - \frac{b_{12}}{b_{11}}\nu_n > 0\right\}, \\
\tilde{\bar{I}}'_n &= \left\{t \in \bar{I}'_n \mid w_{2\kappa_n}(t) + \nu_n \le 0\right\}, \\
\hat{\bar{I}}'_n &= \left\{t \in \bar{I}'_n \mid w_{2\kappa_n}(t) + \nu_n > 0\right\}.
\end{aligned} \tag{92}$$

It is easy to know that $\tilde{\bar{I}}_n \cap \hat{\bar{I}}_n = \phi$, $\tilde{\bar{I}}'_n \cap \hat{\bar{I}}'_n = \phi$ and $\bar{I}_n = \tilde{\bar{I}}_n \cup \hat{\bar{I}}_n$, $\bar{I}'_n = \tilde{\bar{I}}'_n \cup \hat{\bar{I}}'_n$.

Using a similar method of the proof of Lemma 12, we obtain that the set $\hat{\bar{I}}_n$ is not empty. Furthermore, there exists a constant $\widehat{M} > 1$, independent of t and ν_n such that

$$\begin{aligned}
& f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) \le \widehat{M}, \quad \text{for } t \in \tilde{\bar{I}}_n, \\
& f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) \le \widehat{M}, \quad \text{for } t \in \tilde{\bar{I}}'_n, \\
& w_{2\kappa_n}(t) + \nu_n \le \widehat{M}, \\
& \text{for } f_2\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) < 0, \quad t \in \hat{\bar{I}}_n, \\
& w_{1\kappa_n}(t) - \frac{b_{12}}{b_{11}}\nu_n \le \widehat{M}, \\
& \text{for } f_1\left(t, t^{\alpha-1}\left(W_{\kappa_n}(t) + \kappa_n\right)\right) < 0, \quad t \in \hat{\bar{I}}'_n.
\end{aligned} \tag{93}$$

Let

$$\widehat{M}_1(\kappa_n) = \max_{t \in \bar{I}_n} w_{1\kappa_n}(t). \tag{94}$$

From the definitions of $\tilde{\bar{I}}_n$ and $\hat{\bar{I}}_n$, we have

$$\widehat{M}_1(\kappa_n) = \max_{t \in \hat{\bar{I}}_n} w_{1\kappa_n}(t) = \left\|w_{1\kappa_n}(t)\right\|_{\bar{I}_n}. \tag{95}$$

Since $\hat{\bar{I}}_n$ is not empty, it follows that $\widehat{M}_1(\kappa_n) \to \infty$ as $\nu_n \to \infty$. Therefore, we can choose ν_{n1} large enough so that

$$\widehat{M}_1(\kappa_n) > \max\left\{1, \widehat{P}_1, \widehat{P}_2\right\}, \tag{96}$$

for $\nu_n > \nu_{n_1}$, where

$$\begin{aligned}
\widehat{P}_1 &= \frac{\widehat{M}\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11}}{1 - b_{11}}, \\
\widehat{P}_2 &= \frac{\int_0^1 G^*(s,s)\,ds\left[(1 - b_{22}) + b_{12}\right] + b_{12}\left(1 - b_{22} + b_{21}\right)}{(1 - b_{11})(1 - b_{22}) - b_{12}b_{21}}\widehat{M} \\
&\quad + \frac{(a_2 + b_{22})b_{12} + (a_1 + b_{11})(1 - b_{22})}{(1 - b_{11})(1 - b_{22}) - b_{12}b_{21}}.
\end{aligned} \tag{97}$$

From (H) and (42), we have

$$\begin{aligned}
& w_{1\kappa_n}(t) \\
&\quad = \int_0^1 G^*(t,s) f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s) + \kappa_n\right)\right) ds \\
&\quad \le \int_{\tilde{\bar{I}}_n} G^*(s,s) f_1\left(s, s^{\alpha-1}\left(W_{\kappa_n}(s) + \kappa_n\right)\right) ds \\
&\qquad + \int_{\hat{\bar{I}}_n} G^*(s,s) a_1(s)\,ds \\
&\qquad + \int_{(\hat{\bar{I}}_n)_1} G^*(s,s) b_{11}(s) \left|w_{1\kappa_n}(s) - \frac{b_{12}}{b_{11}}\nu_n\right|^{p_1} ds \\
&\qquad + \int_{\hat{\bar{I}}_n \setminus (\hat{\bar{I}}_n)_1} G^*(s,s) b_{11}(s) \left|w_{1\kappa_n}(s) - \frac{b_{12}}{b_{11}}\nu_n\right|^{p_1} ds \\
&\qquad + \int_{(\hat{\bar{I}}_n \cap \bar{I}'_n)_1} G^*(s,s) b_{12}(s) \left|w_{2\kappa_n}(s) + \nu_n\right|^{q_1} ds \\
&\qquad + \int_{(\hat{\bar{I}}_n \cap ([0,1] \setminus \bar{I}'_n)) \cup ((\hat{\bar{I}}_n \cap \bar{I}'_n) \setminus (\hat{\bar{I}}_n \cap \bar{I}'_n)_1)} G^*(s,s) b_{12}(s) \\
&\qquad\qquad \times \left|w_{2\kappa_n}(s) + \nu_n\right|^{q_1} ds,
\end{aligned} \tag{98}$$

which follows from (93) and the definition in (63) that

$$\begin{aligned}
w_{1\kappa_n}(t) \le{} & \int_0^1 G^*(s,s)\,\widehat{M}\,ds + \int_0^1 G^*(s,s) a_1(s)\,ds \\
& + \widehat{M}\int_0^1 G^*(s,s) b_{12}(s)\,ds \\
& + \int_{\hat{\bar{I}}_n} G^*(s,s) b_{11}(s)\,ds \left(\widehat{m}_1(\kappa_n) - \left\|\frac{b_{12}}{b_{11}}\nu_n\right\|\right) \\
& + \int_{\hat{\bar{I}}_n \setminus (\hat{\bar{I}}_n)_1} G^*(s,s) b_{11}(s)\,ds \\
& + \int_{(\hat{\bar{I}}_n \cap \bar{I}'_n)_1} G^*(s,s) b_{12}(s)\,ds \\
& \times \left(\left\|w_{2\kappa_n}(t)\right\|_{(\hat{\bar{I}}_n \cap \bar{I}'_n)_1} + \left\|\nu_n\right\|\right).
\end{aligned} \tag{99}$$

Thus,

$$\begin{aligned}
w_{1\kappa_n}(t) \le{} & \int_0^1 G^*(s,s)\,\widehat{M}\,ds + \int_0^1 G^*(s,s) a_1(s)\,ds \\
& + \widehat{M}\int_0^1 G^*(s,s) b_{12}(s)\,ds \\
& + \int_0^1 G^*(s,s) b_{11}(s)\,ds \left(\widehat{m}_1(\kappa_n) - \left\|\frac{b_{12}}{b_{11}}\nu_n\right\|\right) \\
& + \int_0^1 G^*(s,s) b_{11}(s)\,ds
\end{aligned}$$

$$+\int_0^1 G^*(s,s)\,b_{12}(s)\,ds$$
$$\times\left(\left\|w_{2\kappa_n}(t)\right\|_{(\hat{\bar{I}}_n\cap\bar{I}'_n)_1}+\left\|\nu_n\right\|\right)$$
$$=\widehat{M}\left(\int_0^1 G^*(s,s)\,ds+b_{12}\right)+a_1+b_{11}$$
$$+b_{11}\widehat{M}_1(\kappa_n)+b_{12}\left\|w_{2\kappa_n}(t)\right\|_{\hat{\bar{I}}_n\cap\bar{I}'_n}, \tag{100}$$

which implies that

$$\widehat{M}_1(\kappa_n)<\widehat{M}\left(\int_0^1 G^*(s,s)\,ds+b_{12}\right)+a_1+b_{11}+b_{11}\widehat{M}_1(\kappa_n)$$
$$+b_{12}\left\|w_{2\kappa_n}(t)\right\|_{\hat{\bar{I}}_n\cap\bar{I}'_n}. \tag{101}$$

Therefore, we have

$$\widehat{M}_1(\kappa_n)$$
$$<\frac{\widehat{M}\left(\int_0^1 G^*(s,s)\,ds+b_{12}\right)+a_1+b_{11}+b_{12}\left\|w_{2\kappa_n}(t)\right\|_{\hat{\bar{I}}_n\cap\bar{I}'_n}}{1-b_{11}}. \tag{102}$$

If $\hat{\bar{I}}_n\cap\bar{I}'_n=\phi$, then we have from (76) that

$$\widehat{M}_1(\kappa_n)<\frac{\widehat{M}\left(\int_0^1 G^*(s,s)\,ds+b_{12}\right)+a_1+b_{11}}{1-b_{11}}, \tag{103}$$

which contradicts (102).

If $\hat{\bar{I}}_n\cap\bar{I}'_n\neq\phi$. Using a similar method to that in Lemma 12, we have

$$\widehat{M}_1(\kappa_n)$$
$$<\frac{\int_0^1 G^*(s,s)\,ds\left[(1-b_{22})+b_{12}\right]+b_{12}(1-b_{22}+b_{21})}{(1-b_{11})(1-b_{22})-b_{12}b_{21}}\widehat{M}$$
$$+\frac{(a_2+b_{22})b_{12}+(a_1+b_{11})(1-b_{22})}{(1-b_{11})(1-b_{22})-b_{12}b_{21}}, \tag{104}$$

which also contradicts (96). Thus, the point $\widetilde{T}(-\mu^*,\nu^*)=(-(b_{12}/b_{11})\nu^*,\nu^*)$ lies in the second quadrant. The proof is completed. □

Lemma 14. *Suppose that* (H) *and* (9) *hold. Then, for* ν^* *large enough, the line* $P'Q'$ *lies in the left of* ϑ*-axis.*

Proof. For any point $A(\theta,\vartheta)$ in $P'Q'$, it suffices to show that $\theta\to-\infty$ as $\nu^*\to\infty$ for any $\nu\in[-\nu^*,\nu^*]$.

On the contrary, we assume that there exists a vector sequence $\{\kappa_n\}=\{(\mu_n,\nu_n)\}$ satisfying $\mu_n=-(b_{12}/b_{11})\nu_n$ and a point $\nu(\nu_n)\in[-\nu_n,\nu_n]$ such that $\theta(-(b_{12}/b_{11})\nu_n,\nu(\nu_n))\to l>-\infty$ as $\nu_n\to\infty$. We define some sets $\bar{I}_n$, $\tilde{\bar{I}}_n$, $\hat{\bar{I}}_n$ and $\bar{I}'_n$, $\tilde{\bar{I}}'_n$, $\hat{\bar{I}}'_n$, and some numbers $\widehat{M}$, $\widehat{M}_1(\kappa_n)$ as in Lemma 13. Using a similar method of the proof of Lemma 13, we have

$$w_{1\kappa_n}(t)$$
$$\leq\int_{\tilde{\bar{I}}_n} G^*(s,s)\,f_1\left(s,s^{\alpha-1}\left(W_{\kappa_n}(s)+\kappa_n\right)\right)ds$$
$$+\int_{\hat{\bar{I}}_n} G^*(s,s)\,a_1(s)\,ds$$
$$+\int_{(\hat{\bar{I}}_n)_1} G^*(s,s)\,b_{11}(s)\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}}\nu_n\right|^{p_1}ds$$
$$+\int_{\hat{\bar{I}}_n\setminus(\hat{\bar{I}}_n)_1} G^*(s,s)\,b_{11}(s)\left|w_{1\kappa_n}(s)-\frac{b_{12}}{b_{11}}\nu_n\right|^{p_1}ds$$
$$+\int_{(\hat{\bar{I}}_n\cap\bar{I}'_n)_1} G^*(s,s)\,b_{12}(s)\left|w_{2\kappa_n}(s)+\nu(\nu_n)\right|^{q_1}ds$$
$$+\int_{(\hat{\bar{I}}_n\cap([0,1]\setminus\bar{I}'_n))\cup((\hat{\bar{I}}_n\cap\bar{I}'_n)\setminus(\hat{\bar{I}}_n\cap\bar{I}'_n)_1)} G^*(s,s)\,b_{12}(s)$$
$$\times\left|w_{2\kappa_n}(s)+\nu(\nu_n)\right|^{q_1}ds. \tag{105}$$

It follows from (93)–(96) that

$$w_{1\kappa_n}(t)\leq\int_0^1 G^*(s,s)\,\widehat{M}\,ds+\int_0^1 G^*(s,s)\,a_1(s)\,ds$$
$$+\widehat{M}\int_0^1 G^*(s,s)\,b_{12}(s)\,ds$$
$$+\int_0^1 G^*(s,s)\,b_{11}(s)\,ds\left(\widehat{M}_1(\kappa_n)-\left\|\frac{b_{12}}{b_{11}}\nu_n\right\|\right)$$
$$+\int_0^1 G^*(s,s)\,b_{11}(s)\,ds$$
$$+\int_0^1 G^*(s,s)\,b_{12}(s)\,ds$$
$$\times\left(\left\|w_{2\kappa_n}(t)\right\|_{\hat{\bar{I}}_n\cap\bar{I}'_n}+\left\|\nu(\nu_n)\right\|\right). \tag{106}$$

Notice that $-\nu_n\leq\nu(\nu_n)\leq\nu_n$; from (35), one gets

$$w_{1\kappa_n}(t)\leq\int_0^1 G^*(s,s)\,\widehat{M}\,ds+a_1+\widehat{M}b_{12}+b_{11}$$
$$+b_{11}\left(\widehat{M}_1(\kappa_n)-\left\|\frac{b_{12}}{b_{11}}\nu_n\right\|\right)$$
$$+b_{12}\left(\left\|w_{2\kappa_n}(t)\right\|_{\hat{\bar{I}}_n\cap\bar{I}'_n}+\left\|\nu_n\right\|\right) \tag{107}$$
$$=\widehat{M}\left(\int_0^1 G^*(s,s)\,ds+b_{12}\right)+a_1+b_{11}$$
$$+b_{11}\widehat{M}_1(\kappa_n)+b_{12}\left\|w_{2\kappa_n}(t)\right\|_{\hat{\bar{I}}_n\cap\bar{I}'_n},$$

which implies that

$$\begin{aligned} &\widehat{M}_1(\kappa_n) \\ &\quad < \frac{\widehat{M}\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11} + b_{12}\left\|w_{2\kappa_n}(t)\right\|_{\widehat{\overline{I}}_n \cap \overline{I}'_n}}{1 - b_{11}}. \end{aligned} \tag{108}$$

It is easy to show that

$$\widehat{M}_1(\kappa_n) < \frac{\widehat{M}\left(\int_0^1 G^*(s,s)\,ds + b_{12}\right) + a_1 + b_{11}}{1 - b_{11}} \tag{109}$$

for $\widehat{\overline{I}}_n \cap \overline{I}'_n = \phi$, which contradicts (102), and

$$\begin{aligned} &\widehat{M}_1(\kappa_n) \\ &\quad < \frac{\int_0^1 G^*(s,s)\,ds\left[(1-b_{22}) + b_{12}\right] + b_{12}(1-b_{22}) + b_{12}b_{21}}{(1-b_{11})(1-b_{22}) - b_{12}b_{21}}\widehat{M} \\ &\qquad + \frac{(a_2 + b_{22})b_{12} + (a_1 + b_{11})(1-b_{22})}{(1-b_{11})(1-b_{22}) - b_{12}b_{21}} \end{aligned} \tag{110}$$

for $\widehat{\overline{I}}_n \cap \overline{I}'_n \neq \phi$, which contradicts (96) also. Thus, the line $P'Q'$ lies in the left of ϑ-axis. The proof is completed. □

Similar to the proof of Lemma 12, we can show that the image point R' of the point R lies in the first quadrant. From (37), we have $\widetilde{\mu}^* = (b_{22}/b_{21})\nu^*$. Using a similar method of Lemma 13, we can show that the image point S' of the point S lies in the fourth quadrant.

Using the conditions (36) and (37), similar to Lemma 14, we can show that the image line $Q'R'$ of the line QR lies above the θ-axis, $R'S'$ lies in the right of the ϑ-axis, and $S'P'$ lies under the θ-axis. Therefore, we have the following lemmas.

Lemma 15. *Suppose that (H) and (9) hold. For ν^* large enough, $Q'R'$ lies above the θ-axis, Q' lies in the second quadrant, and R' lies in the first quadrant.*

Lemma 16. *Suppose that (H) and (9) hold. For ν^* large enough, $R'S'$ lies in the right of the ϑ-axis and S' lies in the fourth quadrant.*

Lemma 17. *Suppose that (H) and (9) hold. For ν^* large enough, $S'P'$ lies below the θ-axis.*

Proof of Theorem 1. From Lemmas 12–17, when μ^*, ν^*, and $\widetilde{\mu}^*$ are large enough and satisfy (36) and (37), then the image $P'Q'R'S'$ of the curve $PQRS$ will contain the zero in it. From Proposition 11, it follows that there exists a vector $\kappa_0 = (\mu_0, \nu_0)$ such that the solution $W_{\kappa_0}(t)$ of (30) satisfies $W_{\kappa_0}(1) = 0$, which implies that the integral equation (27) has a solution $Y(t)$. From (26), it follows that (19) has a solution. Therefore, the problem (1) has at least one solution. The proof is completed. □

4. Examples

Example 1. Consider the following boundary value system:

$$\begin{aligned} &D_{0+}^{3/2}x(t) + \frac{t^{1/3}}{2}x^{1/3}(t) + \frac{t^{1/3}}{4}\frac{y^{1/3}(t)}{1 + \left|y^{1/3}(t)\right|} + t = 0, \\ &\qquad t \in (0,1), \\ &D_{0+}^{3/2}y(t) + \frac{t^{1/3}}{3}\frac{x^{1/3}(t)}{1 + \left|x^{1/3}(t)\right|} + \frac{t^{1/3}}{2}y^{1/3}(t) + \frac{t}{4} = 0, \\ &\qquad t \in (0,1), \\ &x(0) = y(0) = 0, \qquad x(1) = 2^{1/2}x\left(\frac{1}{2}\right), \\ &y(1) = 2^{1/2}y\left(\frac{1}{2}\right), \end{aligned} \tag{111}$$

where

$$\begin{aligned} f_1\left(t, t^{\alpha-1}(x,y)\right) &= f_1\left(t, t^{1/2}(x,y)\right) \\ &= \frac{t^{1/2}}{2}x^{1/3}(t) + \frac{t^{1/2}}{4}\frac{y^{1/3}(t)}{1 + t^{1/6}\left|y^{1/3}(t)\right|} + t, \\ f_2\left(t, t^{\alpha-1}(x,y)\right) &= f_2\left(t, t^{1/2}(x,y)\right) \\ &= \frac{t^{1/2}}{3}\frac{x^{1/3}(t)}{1 + t^{1/6}\left|x^{1/3}(t)\right|} + \frac{t^{1/2}}{2}y^{1/3}(t) + \frac{t}{4}. \end{aligned} \tag{112}$$

It is obvious that

$$\begin{aligned} \left|f_1\left(t, t^{1/2}(x,y)\right)\right| &\le \frac{t^{1/2}}{2}|x(t)|^{1/3} + \frac{t^{1/2}}{4}|y(t)|^{1/3} + t, \\ \left|f_2\left(t, t^{1/2}(x,y)\right)\right| &\le \frac{t^{1/2}}{3}|x(t)|^{1/3} + \frac{t^{1/2}}{2}|y(t)|^{1/3} + \frac{t}{4}, \end{aligned} \tag{113}$$

where

$$\begin{aligned} &b_{11}(t) = \frac{t^{1/2}}{2}, \qquad b_{12}(t) = \frac{t^{1/2}}{4}4, \qquad b_{21}(t) = \frac{t^{1/2}}{3}, \\ &b_{22}(t) = \frac{t^{1/2}}{2}, \\ &a_1(t) = t, \qquad a_2(t) = \frac{t}{4}, \qquad \eta = \frac{1}{2}, \\ &p_1 = p_2 = q_1 = q_2 = \frac{1}{3}. \end{aligned} \tag{114}$$

It is easy to check that the function f satisfies (8)–(12). Notice that

$$G^*(s,s) = \frac{1}{\Gamma(1/2)\left(1-(1/2)^{1/2}\right)} \times \begin{cases} (1-s)^{1/2} - \left(\frac{1}{2}-s\right)^{1/2}, & 0 \le s \le \frac{1}{2}, \\ (1-s)^{1/2}, & \frac{1}{2} \le s \le 1, \end{cases} \quad (115)$$

$$b_{21}(t), b_{12}(t) < b_{11}(t), b_{22}(t) = \max_{i,j=1,2}\left\{b_{ij}(t)\right\} = \frac{t^{1/2}}{2}.$$

Consider

$$\begin{aligned} &\max_{1\le i\le 2}\int_0^1 G^*(s,s)\left(b_{i1}(s)+b_{i2}(s)\right)ds \\ &\le \int_0^1 G^*(s,s)\left(\frac{s^{1/2}}{2}+\frac{s^{1/2}}{2}\right)ds \\ &= \int_0^1 G^*(s,s)\,s^{1/2}ds \\ &= \frac{1}{\Gamma(1/2)\left(1-(1/2)^{1/2}\right)} \\ &\quad\times\left(\int_0^1 \left[(1-s)\,s\right]^{1/2}ds - \int_0^{1/2}\left[\left(\frac{1}{2}-s\right)s\right]^{1/2}ds\right) \\ &= \frac{1}{\sqrt{\pi}\left(1-\sqrt{2}/2\right)}\left(\frac{\pi}{8}-\frac{\pi}{32}\right) \approx \frac{0.2945}{1.2533} < 1 \end{aligned} \quad (116)$$

which satisfies (9). Therefore, all conditions of Theorem 1 hold and thus the problem (111) has at least a solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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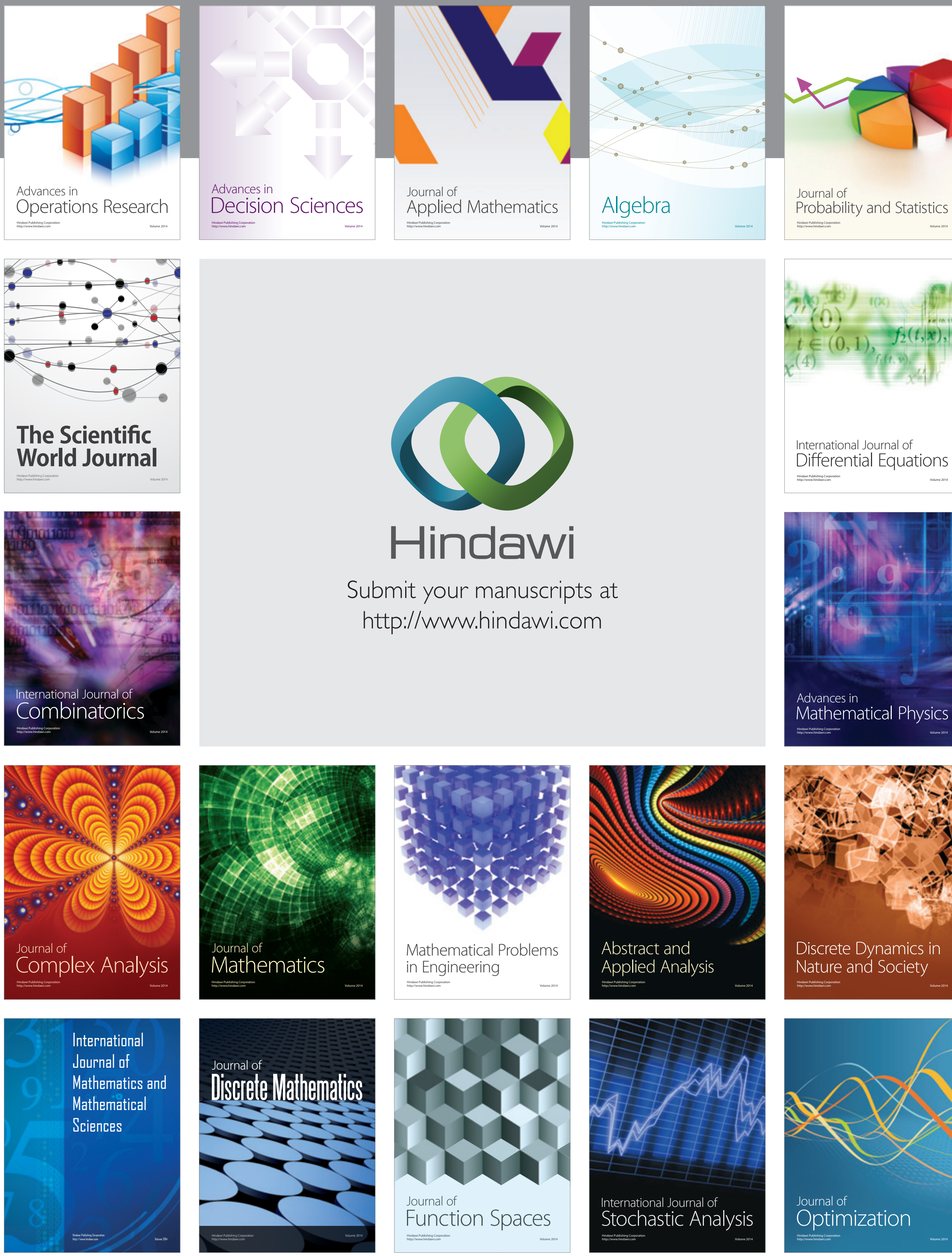
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