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Research Article

Cusped and Smooth Solitons for the Generalized Camassa-Holm Equation on the Nonzero Constant Pedestal

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We investigate the traveling solitary wave solutions of the generalized Camassa-Holm equation $u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}$ on the nonzero constant pedestal $\lim_{\xi\to\pm\infty} u(\xi) = A$. Our procedure shows that the generalized Camassa-Holm equation with nonzero constant boundary has cusped and smooth soliton solutions. Mathematical analysis and numerical simulations are provided for these traveling soliton solutions of the generalized Camassa-Holm equation. Some exact explicit solutions are obtained. We show some graphs to explain our these solutions.

1. Introduction

In 1993, Camassa and Holm [1] derived a nonlinear wave equation (Camassa-Holm equation)

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \tag{1}$$

and obtained the peakon wave solution of the form $u = ce^{-|x-ct|}$. Whereafter, (1) has been researched by many authors [2–7]. Because (1) possesses rich dynamics and complex properties, recently, many authors are interested in its generalized forms. In particular, Liu and Qian [8] suggested a generalized Camassa-Holm equation,

$$u_t + 2ku_x - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}, \tag{2}$$

and obtained the explicit expressions of the peakon solution of (2). Afterwards, Tian and Song [9] gave some physical significance of this equation and obtained some peakon solutions with special wave speeds. Kalisch [10] studied the stability of solitary wave solution of (2). He et al. [11] constructed some exact traveling wave solutions by using the integral bifurcation method. Liu and Liang [12] studied the explicit nonlinear wave solutions and their bifurcations of (2).

When $k = 0$, (2) transforms into the following equation:

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{3}$$

For (3), there are some related works. Shen and Xu [13] discussed the existence of smooth and nonsmooth traveling waves. Khuri [14] obtained a singular wave solution composed of triangle functions. Wazwaz [15, 16] acquired eleven exact traveling wave solutions composed of triangle functions or hyperbolic functions. Liu and Ouyang [17] obtained a peakon solution composed of hyperbolic functions. Liu and Guo [18] investigated the periodic blow-up solutions and their limit forms. Wang and Tang [19] obtained two exact solutions. Yomba [20, 21] gave two methods, the sub-ODE method and the generalized auxiliary equation method, to obtain the exact solution of (3). Liu and Pan [22] studied the coexistence of multifarious solutions.

In this paper, we use the Qiao and Zhang method [23] to investigate the traveling solitary wave solutions of (3) on the nonzero constant pedestal

$$\lim_{\xi\to\pm\infty} u(\xi) = A \neq 0. \tag{4}$$

Since Qiao and Zhang presented this method, many authors applied it to different nonlinear models and obtained a variety of new type soliton solutions. Zhang and Qiao [24] discussed the traveling wave solutions for the Degasperis-Procesi equation

$$m_t + m_xu + 3mu_x = 0, \quad m = u - u_{xx} \tag{5}$$

on the nonzero constant pedestal and found new cusped and peaked soliton solutions. Qiao [25] proposed a new completely integrable wave equation:

$$m_t + m_x\left(u^2 - u_x^2\right) + 2m^2u_x = 0, \quad m = u - u_{xx}, \tag{6}$$

and obtained new cusped, one-peak, W/M-shape-peaks soliton solutions. Later, Chen et al. [26, 27] studied the osmosis $K(2,2)$ equation

$$u_t \pm \left(u^2\right)_x \pm \left(u^2\right)_{xxx} = 0 \tag{7}$$

under the inhomogeneous boundary condition and obtained smooth, peaked, cusped soliton solutions of the osmosis $K(2,2)$ equation by using the phase portrait analytical technique. Wei et al. [28] investigated the generalized KP-MEW(2,2) equation

$$\left(u_t + \left(u^2\right)_x + \left(u^2\right)_{xxt}\right)_x + u_{yy} = 0 \tag{8}$$

on the nonzero constant pedestal and acquired smooth, peaked, cusped, and loop soliton solutions. More works on single peak soliton are reported [29–32].

2. Some Preliminary Results

Substituting $u(x,t) = u(\xi)$ and $\xi = x - ct$ into (3), we have

$$-cu' + cu''' + 3u^2u' = 2u'u'' + uu''', \tag{9}$$

where "$\prime$" is the derivative with respect to ξ. Integrating (9) once, we yield

$$-cu + cu'' + u^3 = \frac{1}{2}\left(u'\right)^2 + uu'' + g_1, \tag{10}$$

where $g_1 \in R$ is an integration constant.

Further, we get

$$\left(u'\right)^2 = \frac{u^4 - 2cu^2 - 4g_1u - 4g_2}{2(u-c)}, \tag{11}$$

where $g_2 \in R$ is an integration constant.

Let us solve (11) with the following boundary condition:

$$\lim_{\xi\to\pm\infty} u(\xi) = A \neq 0, \tag{12}$$

where A is a constant. Equation (11) can be cast into the following ordinary differential equation:

$$\left(u'\right)^2 = \frac{(u-A)^2\left(u^2 + 2Au + 3A^2 - 2c\right)}{2(u-c)}. \tag{13}$$

When $c - A^2 \geq 0$, then (13) reduces to

$$\left(u'\right)^2 = \frac{(u-A)^2(u-B_1)(u-B_2)}{2(u-c)}, \tag{14}$$

where

$$B_1 = -A + \sqrt{2\left(c - A^2\right)}, \qquad B_2 = -A - \sqrt{2\left(c - A^2\right)}. \tag{15}$$

Obviously, $B_1 \geq B_2$.

Remark 1. In the existing research on this method, the cases on $(u-A)^2(u-v_1)/(u-v_2)$ and $(u-A)^2(u-v_3)(u-v_4)/(u-v_5)^2$ have been discussed, but the case on $(u-A)^2(u-v_6)(u-v_7)/(u-v_8)$ (v_i ($i = 1,\ldots,8$) $\doteq$ constant) has not been discussed. So we consider it is very meaningful researching this new case on this method, and we can obtain some new soliton solutions from this case.

Definition 2. A wave function $u(\xi)$ is called smooth soliton solution, if $u(\xi)$ is smooth and $\lim_{\xi\uparrow\xi_0} u'(\xi) = -\lim_{\xi\downarrow\xi_0} u'(\xi) = 0$.

Definition 3. A wave function $u(\xi)$ is called cuspon solution, if $u(\xi)$ is smooth locally on either side of ξ_0 and $\lim_{\xi\uparrow\xi_0} u'(\xi) = -\lim_{\xi\downarrow\xi_0} u'(\xi) = +\infty$ (or $-\infty$).

Without loss of generality, we assume $\xi_0 = 0$.

3. The Parametric Conditions and Phase Portraits of Existence of Soliton Solutions of the Generalized Camassa-Holm Equation (3)

By virtue of the above analysis, we know that soliton solitons for the generalized Camassa-Holm Equation (3) must satisfy the following initial and boundary values problem:

$$\begin{gathered}\left(u'\right)^2 = \frac{(u-A)^2\left(u^2 + 2Au + 3A^2 - 2c\right)}{2(u-c)}, \\ u(0) \in \{c, B_1, B_2\}, \\ \lim_{\xi\to\pm\infty} u(\xi) = A.\end{gathered} \tag{16}$$

Lemma 4. *Suppose that one of the following five conditions holds:*

(i) $c < A^2$, $A \leq c$;

(ii) $c = A^2$, $A \leq c$;

(iii) $A^2 < c < 3A^2$, $A \leq c$;

(iv) $3A^2 = c$, $c < A$;

(v) $3A^2 < c$, $c < A$.

Then (3) *has trivial solution* $u(\xi) \equiv A$.

Proof. (i) If $c < A^2$ and $A \leq c$, then we have $u^2 + 2Au + 3A^2 - 2c > 0$. When $A < c$, (13) leads to $(u')^2 = (u-A)^2(u^2 + 2Au + 3A^2 - 2c)/2(u-c) \leq 0$. For $A = c$, (13) can be cast into $(u')^2 = (1/2)(u-A)(u^2 + 2Au + 3A^2 - 2c) \leq 0$.

(ii) When $c = A^2$ and $A \leq c$, then we have $u^2 + 2Au + 3A^2 - 2c = (u+A)^2 \geq 0$. If $A < c$, (14) changes into $(u')^2 = (u-A)^2(u+A)^2/2(u-c) \leq 0$. If $A = c$, (14) transforms into $(u')^2 = (1/2)(u-A)(u+A)^2 \leq 0$.

(iii) For $A^2 < c < 3A^2$ and $A \leq c$, then we obtain $u^2 + 2Au + 3A^2 - 2c = (u-B_1)(u-B_2) > 0$. If $A < c$, (14) leads to $(u')^2 = (u-A)^2(u-B_1)(u-B_2)/2(u-c) \leq 0$. If $A = c$, (14) changes into $(u')^2 = (1/2)(u-A)(u-B_1)(u-B_2) \leq 0$.

(iv) If $3A^2 = c$ and $c < A$, then we get $u^2 + 2Au + 3A^2 - 2c = (u-A)(u+3A) < 0$ and (14) can be cast into $(u')^2 = (u-A)^3(u+3A)/2(u-3A^2) \leq 0$.

(v) When $3A^2 < c$ and $c < A$, then we have $u^2 + 2Au + 3A^2 - 2c = (u - B_1)(u - B_2) < 0$ and (14) transforms into $(u')^2 = (u - A)^2(u - B_1)(u - B_2)/2(u - c) \leq 0$.

The fact that $(u')^2 \geq 0$ implies $u' = 0$ and $u(\xi) \equiv A$. □

Obviously, we get that the generalized Camassa-Holm Equation (3) with nonzero boundary condition has soliton solutions when A and c do not belong to the above five cases. Then we obtain the generalized Camassa-Holm Equation (3) with nonzero boundary condition having soliton solutions, when $c < A^2, c < A$; $c = A^2, c < A$; $A^2 < c < 3A^2, c < A$; and $3A^2 = c, A \leq c$; $3A^2 < c, A \leq c$.

For the cases on $A^2 < c < 3A^2, c < A, B_1 = c$; $3A^2 = c$, $A = c$, and $3A^2 < c$, $A \leq c$, $B_1 = c$, Liu and Qian [8] and Tian and Song [9] researched that the generalized Camassa-Holm Equation (3) has smooth soliton and peakon solutions as similar as follows:

$$u(\xi) = m\left(\frac{1 - e^{-n|\xi|}}{1 + e^{-n|\xi|}}\right)^2 + p,$$
$$u(\xi) = m\left(\frac{1 - e^{-n(|\xi|+s)}}{1 + e^{-n(|\xi|+s)}}\right)^2 + p, \tag{17}$$

where m, n, and p are constants, and $s > 0$ is an integration constant.

In fact, when $c < A^2, c < A$; $c = A^2, c < A$; $A^2 < c < 3A^2$, $c < A$; and $3A^2 = c$, $A < c$; $3A^2 < c$, $A \leq c$, the generalized Camassa-Holm Equation (3) has also other forms of the smooth soliton and cuspon. Because (11) is equivalent to the two-dimensional system

$$u' = y,$$
$$y' = \frac{u^3 - cu - (1/2)y^2 + g_1}{u - c}. \tag{18}$$

From (18), we can obtain the phase portraits of existence of soliton solutions of the generalized Camassa-Holm Equation (3) under the inhomogeneous boundary condition, when A and c belong to the above five cases (see Figure 1).

The phase portraits of (3) are shown in Figure 1 under different parametric conditions.

(1-1) $c < A^2, c < A$; (1-2) $c = A^2, c < A$; (1-3) $A^2 < c < 3A^2, c < A, c < B_1$; (1-4) $A^2 < c < 3A^2, c < A, B_1 < c$; (1-5) $c = 3A^2, 0 < A < c$; (1-6) $c = 3A^2, A < -1 < c$; (1-7) $c = 3A^2$, $-1 < A < 0 < c$; (1-8) $3A^2 < c, A = c$; (1-9) $3A^2 < c, A < c$, $c < B_1$; (1-10) $3A^2 < c, A < c, B_1 < c$.

4. Cusped and Smooth Solitons for the Generalized Camassa-Holm Equation (3)

In this section, by using the phase portrait analytical technique, which has been developed by Li and Dai [33], we get cusped and smooth soliton solutions of the generalized Camassa-Holm Equation (3) under the inhomogeneous boundary condition.

Case 1 ($c < A^2$, $c < A$). By the standard phase portrait analysis (see Figure 1(1-1)), we have $u(0) = c < A$. From (13), we yield

$$u' = -\frac{(u - A)\sqrt{u^2 + 2Au + 3A^2 - 2c}}{\sqrt{2(u - c)}}\,\mathrm{sign}(\xi). \tag{19}$$

Taking the integration of both sides of (19), we can obtain the implicit cuspon solution $u(\xi)$ defined by

$$-\frac{\sqrt{2W}}{c - W - A}\left[-S_1(u) + \frac{1}{1-\alpha}\left(S_2(u) - \alpha\frac{\mathrm{sn}(u)}{\mathrm{dn}(u)}\right)\right] = |\xi| + K_1, \tag{20}$$

where $K_1 = 0$ is an integration constant,

$$S_1(u) = F\left[\arccos\left(\frac{W + c - u}{W - c + u}\right), k\right],$$
$$S_2(u) = \prod\left[\arccos\left(\frac{W + c - u}{W - c + u}\right), \frac{\alpha^2}{\alpha^2 - 1}, k\right],$$
$$W = \sqrt{(A + c)^2 + 2(A^2 - c)},$$
$$k = \sqrt{\frac{W - A - c}{2W}}, \qquad \alpha = \frac{c - W - A}{c + W - A},$$
$$\mathrm{sn}(u) = \sqrt{1 - \left(\frac{W + c - u}{W - c + u}\right)^2},$$
$$\mathrm{dn}(u) = \sqrt{1 - k^2\mathrm{sn}^2(u)}. \tag{21}$$

Remark 5. $F(\phi, k)$ is the elliptic integral of first kind, and $\Pi(\phi, \tau, k)$ is the elliptic integral of third kind [34].

The profile of cusped soliton solution is shown in Figure 2(2-1).

Case 2 ($c = A^2$, $c < A$). Equation (14) can be cast into

$$(u')^2 = \frac{(u - A)^2(u + A)^2}{2(u - A^2)}. \tag{22}$$

By the standard phase portrait analysis (see Figure 1(1-2)), we have $u(0) = c < A$. From (22), we get

$$u' = -\frac{(u - A)(u + A)}{\sqrt{2(u - A^2)}}\,\mathrm{sign}(\xi). \tag{23}$$

Let $h(u) = -1/(u - A)(u + A)$; then $h(c) = -1/(c - A)(c + A)$, and

$$\int \sqrt{2(u - A^2)}h(u)\,du = \int \mathrm{sign}(\xi)\,d\xi. \tag{24}$$

Inserting $h(u) = h(c) + O(u)$ into (24) and using the initial condition $u(0) = c$, we obtain

$$\frac{1}{3}\left[2\left(u - A^2\right)\right]^{3/2}h(c)(1 + O(1)) = |\xi|. \tag{25}$$

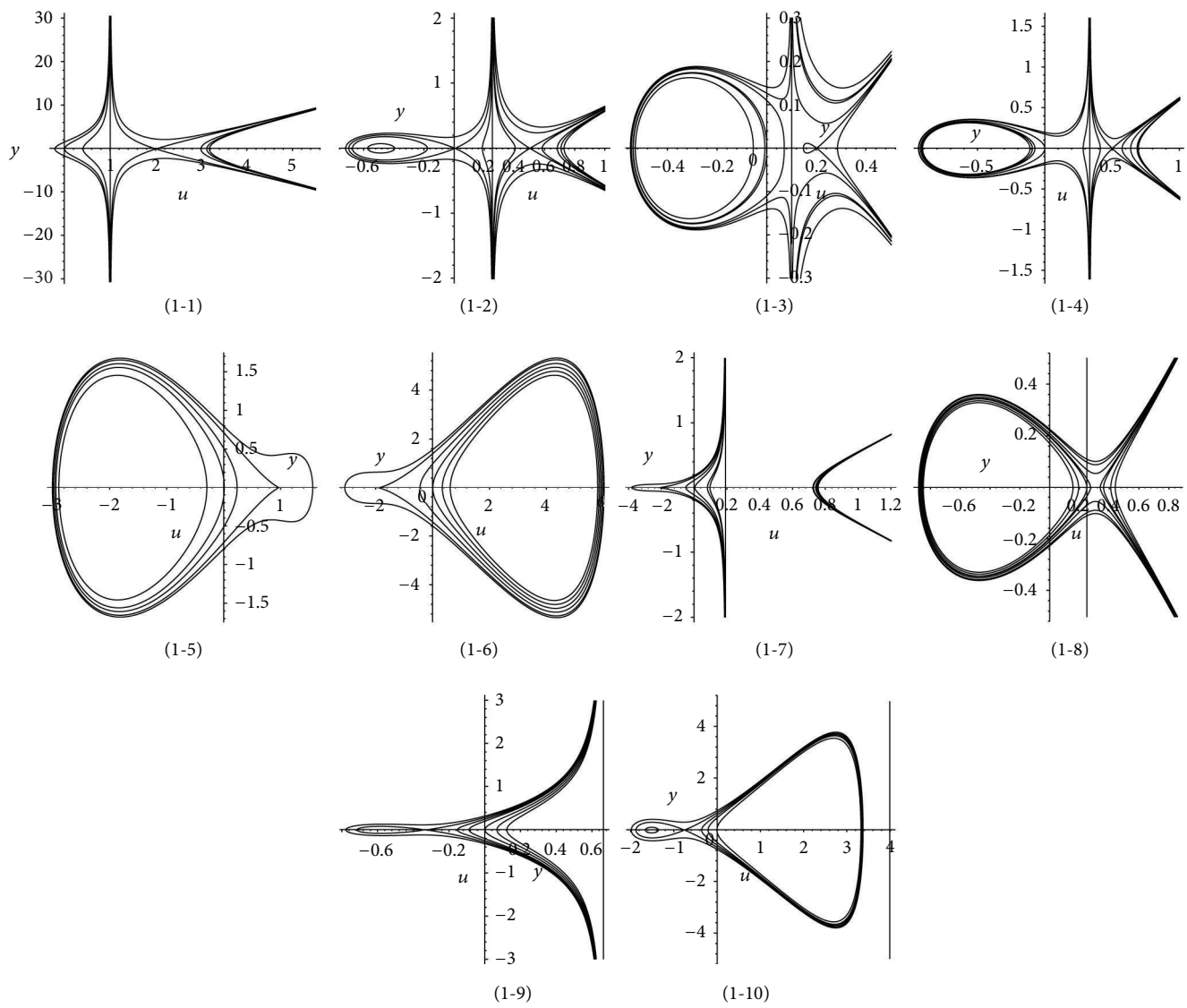

Figure 1: Phase portraits of (3) under the inhomogeneous boundary condition.

Thus,

$$u = \frac{1}{2}|\xi|^{2/3}\left(\frac{3}{h(c)}\right)^{2/3}(1+O(1))^{-2/3} + A^2, \quad \xi \longrightarrow 0, \tag{26}$$

which implies $u = O(|\xi|^{2/3})$. Therefore, we have

$$\begin{aligned} u &= \frac{1}{2}\left(\frac{3}{h(c)}\right)^{2/3}|\xi|^{2/3} + O(|\xi|) + A^2, \quad \xi \longrightarrow 0, \\ u' &= \frac{1}{3}\left(\frac{3}{h(c)}\right)^{2/3}|\xi|^{-1/3} + O(1), \quad \xi \longrightarrow 0. \end{aligned} \tag{27}$$

So we can get the implicit cuspon solution $u(\xi)$ defined by

$$\frac{A-1}{\sqrt{2(A-A^2)}}I_1(u) - \frac{\sqrt{2}(A+1)}{\sqrt{A+A^2}}I_2(u) = |\xi| + K_2, \tag{28}$$

where

$$\begin{aligned} I_1(u) &= \ln\left|\frac{\sqrt{u-A^2}-\sqrt{A-A^2}}{\sqrt{u-A^2}+\sqrt{A-A^2}}\right|, \\ I_2(u) &= \arctan\left(\sqrt{\frac{u-A^2}{A+A^2}}\right). \end{aligned} \tag{29}$$

Remark 6. The proof of other cuspons is similar to the above proof.

Because $u(0) = c$, the constant K_2 is defined by

$$K_{2_c} = \frac{A-1}{\sqrt{2(A-A^2)}}I_1(c) - \frac{\sqrt{2}(A+1)}{\sqrt{A+A^2}}I_2(c) = 0. \tag{30}$$

The profile of cusped soliton solution is shown in Figure 2 (2-2).

Case 3 ($A^2 < c < 3A^2$, $c < A$). In this case, we discuss two conditions: (1) $B_1 > c$; (2) $B_1 < c$.

(1) When $B_1 > c$, by the standard phase portrait analysis (see Figure 1(1-3)), we have $c < u(0) = B_1 < A$. From (14), we have

$$u' = -\frac{(u-A)\sqrt{(u-B_1)(u-B_2)}}{\sqrt{2(u-c)}}\operatorname{sign}(\xi). \tag{31}$$

As same as the above, we can obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$-\frac{2\sqrt{2}(B_1-c)}{(B_1-A)\sqrt{B_1-B_2}}V(u) = |\xi| + K_3, \tag{32}$$

where

$$V(u) = \prod\left(\arcsin\left(\sqrt{\frac{u-B_1}{u-c}}\right), \frac{A-c}{A-B_1}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right). \tag{33}$$

For $u(0) = B_1$, the constant K_3 is defined by $K_{3_{B_1}} = 0$. For this smooth soliton solution, we get an exact explicit form [35]

$$\begin{aligned} u(\xi) &= \left(B_1 - c\cdot\sin^2\left(\Pi^{-1}\left(\frac{(A-B_1)\sqrt{B_1-B_2}|\xi|}{2\sqrt{2}(B_1-c)}, \frac{A-c}{A-B_1}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right)\right)\right) \\ &\quad\times\left(1-\sin^2\left(\Pi^{-1}\left(\frac{(A-B_1)\sqrt{B_1-B_2}|\xi|}{2\sqrt{2}(B_1-c)}, \frac{A-c}{A-B_1}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right)\right)\right)^{-1}. \end{aligned} \tag{34}$$

The profile of smooth soliton solution is shown in Figure 2(2-3).

(2) When $B_1 < c$, by the standard phase portrait analysis (see Figure 1(1-4)), we have $B_1 < u(0) = c < A$. Taking the integration of both sides of (31), we can yield the implicit cuspon solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}(c-B_1)}{(A-B_1)\sqrt{c-B_2}}(O_1(u) - O_2(u)) = |\xi| + K_4, \tag{35}$$

where

$$\begin{aligned} O_1(u) &= \prod\left(\arcsin\left(\sqrt{\frac{u-c}{u-B_1}}\right), \frac{A-B_1}{A-c}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right), \\ O_2(u) &= F\left(\arcsin\left(\sqrt{\frac{u-c}{u-B_1}}\right), \sqrt{\frac{B_1-B_2}{c-B_2}}\right). \end{aligned} \tag{36}$$

By view of $u(0) = c$, the constant K_4 is defined by $K_{4_c} = 0$. The profile of cusped soliton solution is shown in Figure 2(2-4).

Case 4 ($3A^2 = c$, $A < c$). (1) When $0 < A < c$, by the standard phase portrait analysis (see Figure 1(1-5)), we have $u(0) = B_2 < 0 < B_1 = A < c$. Equation (14) transforms into

$$(u')^2 = \frac{(u-A)^2(u-A)(u+3A)}{2(u-3A^2)}. \tag{37}$$

From (37), we have

$$u' = -(u-A)\sqrt{\frac{(u-A)(u+3A)}{2(u-3A^2)}}\operatorname{sign}(\xi). \tag{38}$$

Taking the integration of both sides of (38), we can obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{\sqrt{6}}{2}\sqrt{\frac{A(A+1)}{A}}\left(\frac{3A-1}{3A+3}P_1(u) + \frac{4}{3A+3}P_2(u)\right) = |\xi| + K_5, \tag{39}$$

where

$$\begin{aligned} P_1(u) &= \prod\left(\arcsin\left(\sqrt{\frac{3A+u}{4A}}\right), 1, \frac{2}{\sqrt{3+3A}}\right), \\ P_2(u) &= F\left(\arcsin\left(\sqrt{\frac{3A+u}{4A}}\right), \frac{2}{\sqrt{3+3A}}\right). \end{aligned} \tag{40}$$

For $u(0) = B_2$, we obtain $K_{5_{B_2}} = 0$. The profile of smooth soliton solution is shown in Figure 2(2-5).

(2) When $A < 0$, by virtue of (37), we have

$$u' = (u-A)\sqrt{\frac{(u-A)(u+3A)}{2(u-3A^2)}}\operatorname{sign}(\xi). \tag{41}$$

In this case, we discuss two conditions: (i) $A < -1$; (ii) $-1 < A < 0$.

(i) When $A < -1$, by the standard phase portrait analysis (see Figure 1(1-6)), we have $A = B_2 \le u \le B_1 < c$. Taking the integration of (41) on the interval $[A, B_1]$, thus, we obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{-3\sqrt{2}(1+A)}{2\sqrt{3A^2-A}}H(u) = |\xi| + Q_1, \tag{42}$$

where

$$H(u) = \prod\left(\arcsin\left(\sqrt{\frac{(3A-1)(3A+u)}{4(3A^2-u)}}\right), 1, \sqrt{\frac{-4}{3A-1}}\right). \tag{43}$$

Because $u(0) = B_1$, we obtain $Q_{1_{B_1}} = 0$. For this smooth soliton solution, we get an exact explicit form

$$u(\xi) = \left(12A^2\sin^2\left(\Pi^{-1}\left(\frac{2\sqrt{3A^2-A}\,|\xi|}{-3\sqrt{2}(1+A)}, 1, \sqrt{\frac{-4}{3A-1}}\right)\right) - 3A(3A-1)\right) \times \left((3A-1) + 4\sin^2\left(\Pi^{-1}\left(\frac{2\sqrt{3A^2-A}\,|\xi|}{-3\sqrt{2}(1+A)}, 1, \sqrt{\frac{-4}{3A-1}}\right)\right)\right)^{-1}. \tag{44}$$

The profile of smooth soliton solution is shown in Figure 2(2-6).

(ii) When $-1 < A < 0$, by the standard phase portrait analysis (see Figure 1(1-7)), we have $A = B_2 \le u < c < B_1$. Integrating (41) on the interval $[A, c)$, we obtain the implicit cuspon solution $u(\xi)$ defined by

$$\frac{3\sqrt{2}(A+1)}{4\sqrt{-A}}\left[R_1(u) - R_2(u)\right] = |\xi| + Q_2, \tag{45}$$

where

$$R_1(u) = \prod\left(\arcsin\left(\sqrt{\frac{4(3A^2-u)}{(3A-1)(3A+u)}}\right), 1, \sqrt{\frac{3A-1}{-4}}\right),$$

$$R_2(u) = F\left(\arcsin\left(\sqrt{\frac{4(3A^2-u)}{(3A-1)(3A+u)}}\right), \sqrt{\frac{3A-1}{-4}}\right). \tag{46}$$

From $u(0) = c$, we obtain $Q_{2_c} = 0$. The profile of cusped soliton solution is shown in Figure 2(2-7).

Case 5 ($3A^2 < c$, $A \le c$). (1) When $A = c$, (14) can be cast into

$$(u')^2 = \frac{1}{2}(u-A)(u-B_1)(u-B_2). \tag{47}$$

Because $3A^2 < c$, we have $B_2 < A < B_1$. By the standard phase portrait analysis (see Figure 1(1-8)), we get $B_2 \le u \le A = c < B_1$. By view of (47), we obtain

$$u' = \frac{\sqrt{2}}{2}\sqrt{(u-A)(u-B_1)(u-B_2)}\,\mathrm{sign}(\xi). \tag{48}$$

As same as the above, we can get the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}}{\sqrt{B_1-B_2}}F\left(\arcsin\left(\sqrt{\frac{u-B_2}{A-B_2}}\right), \sqrt{\frac{A-B_2}{B_1-B_2}}\right) = |\xi| + K_6. \tag{49}$$

For $u(0) = B_2$, the constant K_6 is defined by $K_{6_{B_2}} = 0$. For this smooth soliton solution, we can give an exact explicit form

$$u(\xi) = (A-B_2) \times \sin^2\left(F^{-1}\left(\frac{\sqrt{(B_1-B_2)}\,|\xi|}{2\sqrt{2}}, \sqrt{\frac{A-B_2}{B_1-B_2}}\right)\right) + B_2. \tag{50}$$

The profile of smooth soliton solution is shown in Figure 2(2-8).

(2) When $A < c$, we discuss three cases: (i) $B_1 > c$; (ii) $B_1 < c$; (iii) $B_1 = c$.

(i) When $B_1 > c$, by the standard phase portrait analysis (see Figure 1(1-9)), we have $u(0) = B_2 \le u < A$ or $A \le u < u(0) = c$.

For $u(0) = B_2$, from (14), we have

$$u' = (A-u)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\,\mathrm{sign}(\xi). \tag{51}$$

Taking the integration of (51) on the interval $[B_2, A]$, thus, we obtain the implicit smooth soliton solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}}{\sqrt{B_1-B_2}}\left(\frac{c-A}{A-B_2}G_1(u) + G_2(u)\right) = |\xi| + W_1, \tag{52}$$

where

$$G_1(u) = \prod\left(\arcsin\left(\sqrt{\frac{u-B_2}{c-B_2}}\right), \frac{c-B_2}{A-B_2}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right),$$

$$G_2(u) = F\left(\arcsin\left(\sqrt{\frac{u-B_2}{c-B_2}}\right), \sqrt{\frac{c-B_2}{B_1-B_2}}\right). \tag{53}$$

The constant W_1 is defined by $W_{1_{B_2}} = 0$. The profile of smooth soliton solution is shown in Figure 2(2-9).

For $u(0) = c$, by view of (14), we obtain

$$u' = (u-A)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\,\mathrm{sign}(\xi). \tag{54}$$

Taking the integration of both sides of (54), thus, we can yield the implicit solution $u(\xi)$ defined by

$$\frac{2\sqrt{2}(B_1-c)}{(B_1-A)\sqrt{B_1-B_2}}\left[E_1(u) - E_2(u)\right] = |\xi| + W_2, \tag{55}$$

where

$$E_1(u) = \prod\left[\arcsin\left(\sqrt{\frac{(B_1-B_2)(c-u)}{(c-B_2)(B_1-u)}}\right),\right.$$
$$\left.\frac{(c-B_2)(B_1-A)}{(B_1-B_2)(c-A)}, \sqrt{\frac{c-B_2}{B_1-B_2}}\right],$$
$$E_2(u) = \mathcal{F}\left[\arcsin\left(\sqrt{\frac{(B_1-B_2)(c-u)}{(c-B_2)(B_1-u)}}\right), \sqrt{\frac{c-B_2}{B_1-B_2}}\right]. \tag{56}$$

For $u(0) = c$, W_2 is defined by $W_{2_c} = 0$. The profile of cusped soliton solution is shown in Figure 2(2-10).

(ii) When $B_1 < c$, by the standard phase portrait analysis (see Figure 1(1-10)), we have $B_2 < A < B_1 < c$.

For $u(0) = B_2$, from (14), we get

$$u' = (A-u)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\,\mathrm{sign}(\xi); \tag{57}$$

we yield

$$\Theta_1(u) \equiv \int \theta_1(u)\,du = |\xi| + M_1, \tag{58}$$

where

$$\theta_1(u) = -\frac{1}{(u-A)}\sqrt{\frac{2(u-c)}{(u-B_1)(u-B_2)}}, \tag{59}$$

and M_1 is an integration constant. Taking the integration of $\theta_1(u)$ on the interval $[B_2, A]$, thus, we obtain the implicit solution $u(\xi)$ defined by

$$\begin{aligned}\Theta_1(u) &= \frac{2\sqrt{2(c-B_2)}}{B_1-B_2}\\ &\quad\times\left[\left(\frac{B_1-B_2}{A-B_2}-\frac{B_1-B_2}{c-B_2}\right)T_1(u)+\frac{B_1-B_2}{c-B_2}T_2(u)\right]\\ &= |\xi| + M_1,\end{aligned} \tag{60}$$

where

$$T_1(u) = \prod\left(\arcsin\left(\sqrt{\frac{u-B_2}{B_1-B_2}}\right), \frac{B_1-B_2}{A-B_2}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right),$$
$$T_2(u) = \mathcal{F}\left(\arcsin\left(\sqrt{\frac{u-B_2}{B_1-B_2}}\right), \sqrt{\frac{B_1-B_2}{c-B_2}}\right). \tag{61}$$

The constant M_1 is defined by $M_{1_{B_2}} = 0$. Because $\theta_1(u) > 0$, we know that the $\Theta_1(u)$ is strictly increasing on the interval $[B_2, A]$;

$$\Theta_1(u) = \Theta_{1_{[B_2,A]}}(u) \tag{62}$$

has the inverse denoted by $u(\xi) = \Theta_1^{-1}(|\xi|)$. The profile of smooth soliton solution is shown in Figure 2(2-11).

For $u(0) = B_1$, by view of (14), we obtain

$$u' = (u-A)\sqrt{\frac{(u-B_1)(u-B_2)}{2(u-c)}}\,\mathrm{sign}(\xi),$$
$$\Theta_2(u) \equiv \int \theta_2(u)\,du = |\xi| + M_2, \tag{63}$$

where

$$\theta_2(u) = \frac{1}{(u-A)}\sqrt{\frac{2(u-c)}{(u-B_1)(u-B_2)}}, \tag{64}$$

and M_2 is an integration constant. Taking the integration of $\theta_2(u)$ on the interval $[A, B_1]$, thus, we obtain the implicit solution $u(\xi)$ defined by

$$\Theta_2(u) = -\frac{2\sqrt{2}(c-B_1)}{(A-B_1)\sqrt{c-B_2}}N(u) = |\xi| + M_2, \tag{65}$$

where

$$N(u) = \prod\left(\arcsin\left(\sqrt{\frac{(c-B_2)(B_1-u)}{(B_1-B_2)(c-u)}}\right),\right.$$
$$\left.\frac{(B_1-B_2)(A-c)}{(c-B_2)(A-B_1)}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right). \tag{66}$$

When $u(0) = B_1$, the constant M_2 is defined by $M_{2_{B_1}} = 0$. From $\theta_2(u) > 0$, we know that the $\Theta_2(u)$ is strictly increasing on the interval $[A, B_1]$;

$$\Theta_2(u) = \Theta_{2_{[A,B_1]}}(u) \tag{67}$$

has the inverse denoted by $u(\xi) = \Theta_2^{-1}(|\xi|)$.

For this smooth soliton solution, we give an exact explicit form

$$\begin{aligned}u(\xi) = \Bigg(&c(B_1-B_2)\\ &\times\sin^2\left(\Pi^{-1}\left(\frac{(A-B_1)\sqrt{c-B_2}\,|\xi|}{2\sqrt{2}(B_1-c)},\right.\right.\\ &\left.\left.\frac{(B_1-B_2)(A-c)}{(c-B_2)(A-B_1)}, \sqrt{\frac{B_1-B_2}{c-B_2}}\right)\right) - B_1(c-B_2)\Bigg)\end{aligned}$$

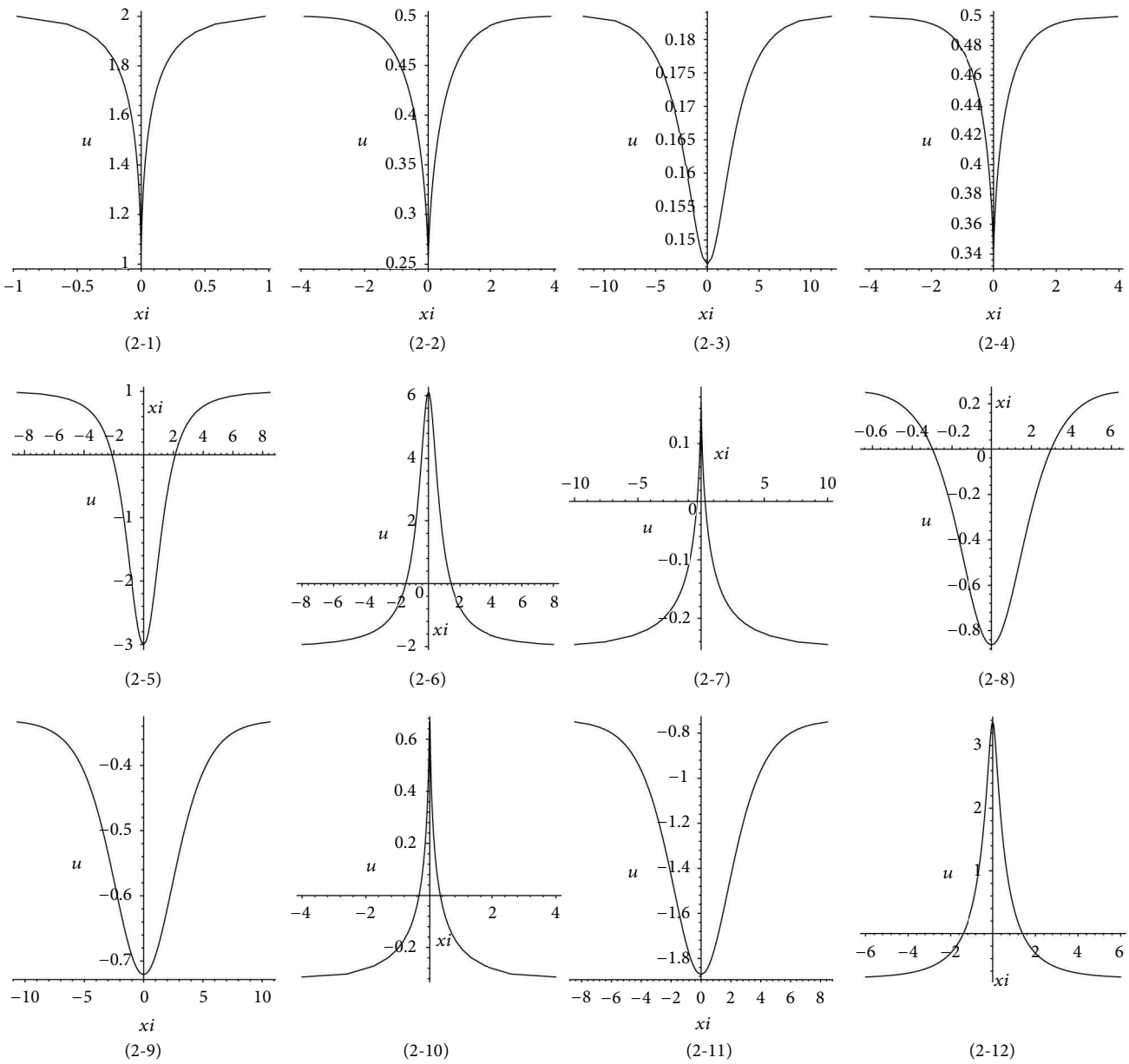

Figure 2: Profiles of soliton solution.

$$\times\left(\left(B_1 - B_2\right)\right.$$
$$\times \sin^2\left(\Pi^{-1}\left(\frac{\left(A - B_1\right)\sqrt{c - B_2}\left|\xi\right|}{2\sqrt{2}\left(B_1 - c\right)},\right.\right.$$
$$\frac{\left(B_1 - B_2\right)\left(A - c\right)}{\left(c - B_2\right)\left(A - B_1\right)},$$
$$\left.\left.\left.\sqrt{\frac{B_1 - B_2}{c - B_2}}\right)\right) - c + B_2\right)^{-1}. \tag{68}$$

The profile of smooth soliton solution is shown in Figure 2 (2-12).

The profile of soliton solution of (3) is shown in Figure 2 under special values of c and A.

(2-1) $c = 1$, $A = 2$; (2-2) $c = 1/4$, $A = 1/2$; (2-3) $c = 1/10$, $A = 1/5$; (2-4) $c = 1/3$, $A = 1/2$; (2-5) $c = 3$, $A = 1$; (2-6) $c = 12$, $A = -2$; (2-7) $c = 3/16$, $A = -1/4$; (2-8) $c = A = 1/4$; (2-9, 10) $c = 2/3$, $A = -1/3$; (2-11, 12) $c = 4$, $A = -3/4$.

5. Conclusion

In this paper, we research the soliton solutions of the generalized Camassa-Holm Equation (3) under inhomogeneous boundary condition. The parametric conditions and phase

portraits of existence of the cuspon and smooth soliton solutions are given. We obtain cuspon and smooth soliton solutions of the generalized Camassa-Holm Equation (3). Some exact explicit solutions are obtained. We show some graphs to explain our these solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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