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Research Article

The Optimal Portfolio Selection Model under g-Expectation

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This paper solves the optimal portfolio selection model under the framework of the prospect theory proposed by Kahneman and Tversky in the 1970s with decision rule replaced by the g-expectation introduced by Peng. This model was established in the general continuous time setting and firstly adopted the g-expectation to replace Choquet expectation adopted in the work of Jin and Zhou, 2008. Using different S-shaped utility functions and g-functions to represent the investors' different uncertainty attitudes towards losses and gains makes the model not only more realistic but also more difficult to deal with. Although the models are mathematically complicated and sophisticated, the optimal solution turns out to be surprisingly simple, the payoff of a portfolio of two binary claims. Also I give the economic meaning of my model and the comparison with that one in the work of Jin and Zhou, 2008.

1. Introduction

In the area of optimal financial portfolio selection, the expected utility maximization theory (EUT), developed by Neumann and Morgenstern [1], has been the most important decision rule for a long time. During the past twenty years, portfolio choice theory has been developed to both discrete time and continuous time models in dynamic setting. Markowitz [2] laid down the basement for modern financial portfolio selection theory by his pioneer work on single-period mean-variance portfolio selection. Li and Ng [3] extended the Markowitz model to the dynamic setting. El-Karoui et al. [4] considered a portfolio-consumption model where the objective is to optimize the recursive utility of consumption and terminal wealth, and they adopted the terminal perturbation method to solve this problem. Bielecki et al. [5] employed the dual approach to deal with a continuous time portfolio selection model without negativity constraint on wealth process. Pliska had earlier introduced this approach for discrete time models in [6]. In [7], Ji and Zhou firstly used the terminal perturbation method and dual approach together, and they reformulated FBSDE controlled system as a backward system by taking the terminal condition of the forward state as a control variable. By applying Ekeland's variational principle, they could deal with additional constraint without convexity assumption on the coefficients of the backward approach to continuous time mean-variance portfolio selection problem in a complete market. Until now, this terminal perturbation method has been widely used in many control problems; see [8–16]. As for these problems, there have been essentially two classical approaches developed to solve in the utility model: one is the stochastic control or the dynamic programming approach, firstly developed by Merton [17, 18], which transforms the problem into solving a partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation. The other one was developed by Harrison and Kreps [19] called the martingale approach.

Although the optimal portfolio selection models under the expected utility theory have been well solved, the risk preference measure or the expected utility theory has some basic tenets which has systematically violated the reality. In other words, the EUT cannot be able to describe the way people make decision in the real world clearly and precisely. Firstly, for example, EUT has an underlying assumption that decision makers are rational and uniformly risk averse when facing uncertainties. But in the real world, people are risk averse on gains and risk taking on losses and appear significantly more sensitive to losses than to gains. Secondly EUT thinks that everyone is able to objectively evaluate probabilities, but the fact is that people usually overweight small probabilities and underweight large probabilities. Thirdly, in EUT, investors evaluate wealth according to final asset position, but evidence shows that people evaluate assets on gains and losses, not on final wealth position. The difference between the theory and the practice leads to many paradoxes and puzzles that the EUT fails to explain, including the famous Allais paradoxes, Ellsberg paradoxes, and the equity premium puzzle. Hence, many alternative preference measures have been put forth. For example, goal reaching, dual theory of choice, and Lope's SP/A model. However, these new models which have successfully solved some paradoxes and puzzles would create new ones.

In 1970s, Kahneman and Tversky proposed the prospect theory (PT) for decision making under uncertainty [20], incorporating human emotions and psychology into their theory. The key elements of this Nobel-prize-winning theory are as follows.

(1) A reference point in wealth which defines gains and losses.

(2) Value function, being similar with utility function, which is concave for gains, convex for losses, and steeper for losses than for gains.

(3) Nonlinear probability distortion, that is, a transformation of the probability scale, enlarging small probabilities and diminishing large probabilities.

The three points above have given PT the power to describe a man's risk attitude and emotions more clearly. So the model under the prospect theory is closer to the reality and the research on it is very interesting and important.

In this framework, owing to the discontinuity and nonglobal convexity of the S-shaped value function, Lagrange method cannot be used. Worse still, the coupling of these two ill-behaved features greatly amplifies the difficulties of the problem. Berkelaar et al. [21] did some research on continuous setting, but they neglected the probability distortion, which is the main difficulty for the problem. Jin et al. [22, 23] studied the continuous model, with both the S-shaped value function and the nonlinear probability distortion. The probability distortions involved the nonlinear Choquet expectation instead of the conventional linear expectation. Using Choquet expectation, the thought in theory of Kahneman and Tversky is described.

This paper extends the thought of prospect theory to another nonlinear case. I replace the Choquet expectation in [22] by another nonlinear expectation, g-expectation. Mathematically g-expectation and Choquet integral are two different ways to describe nonlinear case, where g-expectation is more nonlinear in some sense. It is shown that they coincide only when they become linear expectation; see (Chen and Selum [24]). Although Choquet expectation has many applications in statistics, economics, and finance, it is difficult to define conditional Choquet expectation in terms of Choquet expectation, while it is easy to define conditional expectation via g-expectation. Then there are some important applications for g-expectation in various areas especially in finance. For example, the ambiguity in financial model can be described by the g-expectation; see Chen and Epstein [25]. The g-probabilities and g-expectation have also been found to have intimate connection with the rapidly developed dynamic risk measure theory. Choquet et al. [26] showed that in dynamic setting, risk measures could be formulated via the g-expectation.

This paper firstly adopts g-expectation and g-probability to describe an ambiguous environment. The only difference from prospect theory framework in Jin and Zhou [22] is the decision rule when cost constraint is linear. In prospect theory, there is only a reference or "real" probability in the world, where an agent has a distortion to this probability to describe his attitude. Here, the economic meaning to use g-expectation instead of Choquet integral is that an agent faces an ambiguous world, where there may be a set of priors. So ambiguous attitude replaces the probability distortion. Actually the model I build is a general form, so it can feature many other cases when specific g-function is chosen, which can be searched in the future.

This paper is organized as follows. Section 2 gives out the background of the problem and the optimal portfolio selection model under the prospect theory and g-expectation. In Section 3, the original model is divided into three subproblems owing to the discontinuity, nonconvexity maximization problem. Also this section proves the equivalence between the original problem and the three subproblems. Section 4 solves out the subproblems with the perturbation method and Ekeland's variational principle. Section 5 analyzes the form of the optimal solution under two simple but fundamental examples, gives out the economic meaning under the model, and compares our model with the one in Jin and Zhou [22]. The final part presents some concluding comments.

2. The Model under Prospect Theory and g-Expectation

In this paper T is a fixed terminal time, and $W(\cdot) = (W_1(\cdot), \dots, W_m(\cdot))'$ is a standard m-dimensional Brownian motion defined on a complete filtration probability space $(\Omega, \mathcal{F}, P)$. The information structure is given by a filtration $F = \{\mathcal{F}_t\}_{0\le t\le T}$, which is generated by $W(\cdot)$ and augmented by all the P-null sets. According to Karatzas and Shreve [27], they define a complete capital market in continuous setting: given m kinds of risky assets, for example, m kinds of stocks whose price processes are written as $S_i(t)\, i = 1, 2, \dots, m$ and given a riskless asset, for example, the bank account, whose price process is written as $S_0(t)$. These processes of $m+1$ kinds of assets satisfy the equations below:

$$\begin{gathered} dS_i(t) = S_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{m}\sigma_{ij}(t)\,dW_j(t)\right], \\ S_i(0) = s_i \ge 0, \\ dS_0(t) = S_0(t)\,r(t)\,dt, \\ S_0(0) = s_0 \ge 0; \quad t \in [0, T], \end{gathered} \tag{1}$$

where $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$, $i = 1, 2, \ldots, m$, show the appreciation rate and the disperse rate of risky assets, respectively. They are all $\mathcal{F}_t$-progressively measurable and satisfy

$$\int_0^T \left[\sum_{i=1}^{m} \left| b_i(t) \right| + \sum_{i,j=1}^{m} \left| \sigma_{ij}(t) \right|^2 \right] dt < +\infty, \quad \text{a.s,} \tag{2}$$

and $r(t)$ is the interest rate which is an adapted progressively measurable random process satisfying $\int_0^T |r(t)| dt < \infty$, a.s. Accordingly, in a complete market the total wealth process which is replicated with a portfolio of the $m + 1$ assets can be represented by backward stochastic differential equations (BSDE) introduced in [28]

$$\begin{gathered} -dx(t) = g_0(x(t), z(t), t)\, dt - z'(t)\, dW(t), \\ x(T) = \xi, \end{gathered} \tag{3}$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ is the terminal wealth at T, $g_0 = (-r(t)x(t) - (b(t) - r(t))'\sigma^{-1}(t)z(t))$, and $z(t) = \sigma(t)\pi(t)$. Here, $x(t)$ is the value process replicated by constructing a self-financing portfolio $\pi(t)$ with m kinds of risky assets whose nonsingular volatility matrix is $\sigma(t)$. This wealth process means that there are no transaction costs including price effects in the market. In fact, in a standard complete market, it is possible to construct a portfolio which attains as final wealth the amount ξ, as in (3). Then, the dynamics of the value of the replicating portfolio x are given by a BSDE with linear function g_0, with $z(t)$ (or in fact $\pi(t)$) corresponding to the hedging portfolio. So, the existence of solution restricted to square-integrable ones of (3) should be guaranteed. In [29], Pardoux and Peng got the existence and uniqueness of solutions under some conditions which will be showed later, and because of the uniqueness of solutions, there is only one price as well as hedging portfolio so that valuation of the terminal wealth (e.g., contingent claim) is well possessed without arbitrage.

Furthermore, the g_0 can be expended to the nonlinear form. For example, people can allocate part of the capital to buy the call option, or people can assume the rates between loan and deposit are different. An interesting example of the nonlinear wealth equation is the optimal portfolio choice for large investor that is considered in [30]. A large investor's portfolio choices can affect the securities' price process. The impact of the investor's position on price is specified exogenously and the price may rise because of size or because of other agents in the market believing that the large investor has superior information. In [30], the respective asset price $S_0(\cdot)$ and $S_1, \ldots, S_m(\cdot)$ are described by the following equations:

$$\begin{gathered} dS_0(t) = S_0(t)\left[r(t) + l_0(x(t), \pi(t)) \right] dt, \\ S_0(0) = s_0 \geq 0, \\ dS_i(t) = S_i(t) \left[b_i(t)\, dt + l_i(x(t), \pi(t))\, dt \right. \\ \left. + \sum_{j=1}^{m} \sigma_{ij}(t)\, dW_j(t) \right], \\ S_i(0) = s_i \geq 0; \quad t \in [0, T], \end{gathered} \tag{4}$$

where l_i, $i = 1, \ldots, m$, are functions describing the effect of the wealth and the strategies possessed by the large investor. The corresponding wealth process is governed by

$$\begin{aligned} -dx(t) &= g_0(x(t), z(t), t) - z(t)\, dt \\ &= g_0\left(x(t), \sigma'(t)\pi(t), t\right) dt \\ &\quad - \pi'(t)\sigma(t)\, dW(t), \\ x(0) &= x_0 \geq 0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} g_0\left(x, \sigma'\pi, t\right) &= -r_t x - \left(x - \pi' I\right) l_0(x, \pi) \\ &\quad - \pi' \left[b_t - r_t I + l(x, \pi) \right], \\ \sigma(t) &:= \left(\sigma_{ij}(t)\right)_{m*m} \end{aligned} \tag{6}$$

is a volatility process.

In this wealth equation, the function $g_0(x, z, t)$ is nonlinear about $x(t)$ and $z(t)$, where $z'(t) := \pi'(t)\sigma(t)$ and $z(t)$ is $\mathbb{R}^m$-valued. So clearly I can break through the limit of the linear wealth equation; then I can incorporate more situations which are closer to the real market.

As I see, BSDE can form more generalized kinds of financial models. In fact, BSDE can define generalized stochastic differential utility (see [28]) and model drift uncertainty by using a dynamic nonlinear expectation called g-expectation (see [25]). The key point is that in BSDE the time consistency related to dynamic model is kept because of the property of solutions. Generally, I need several assumptions about BSDE for getting its solution.

Assumption 1. Let $g = g(\omega, x, z, t) : \Omega \times \mathbb{R} \times \mathbb{R}^m \times [0, T] \to \mathbb{R}$ be a function satisfying the following:

- L_1: g is uniformly Lipschitz with (x, z);
- L_2: $g(x, z, t)$ is continuous about t and $E[\int_0^T g^2(0, 0, t) dt] < +\infty$;
- L_3: $g(x, 0, t) = 0, \forall(\omega, x, t) \in \Omega \times R^1 \times [0, T]$;
- L_4: $g(0, 0, t) \equiv 0, \forall(\omega, t) \in \Omega \times [0, T]$.

I can say that for any given $\xi \in L^2(\Omega, \mathcal{F}_T, R)$ and g satisfying L_1, L_2, the solution of the backward stochastic differential equation satisfying $X(T) = \xi$ is formed by a pair of uniquely adapted solutions $(x(\cdot), z(\cdot)) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R}^1) \times L^2_{\mathcal{F}_t}(0, T; \mathbb{R}^{1\times m})$; see [29].

Then I will give the definition of g-expectation and dynamic pricing mechanism correspondence to a BSDE.

Definition 2. Given g satisfying L_1–L_3 and $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the g-expectation of ξ can be defined as $\mathcal{E}_g[\xi]$, where $\mathcal{E}_g[\xi] = x(0)$ in (3) with respect to g. The properties and applications of g-expectation can be read in [28].

Definition 3. Given $r \le t \le T$, and $x(t) \in L^2(\Omega, \mathcal{F}_t, P)$, $g(t, x, z)$ satisfies L_1, L_2, and L_4. Considering the BSDE (3) above with respect to function g and terminal claim $x(t)$, I will define the dynamic pricing mechanism from [31] by Peng as follows:

$$G_{r,t}[x(t)] = x(r) : L^2(\Omega, \mathcal{F}_t, P) \longrightarrow L^2(\Omega, \mathcal{F}_r, P). \quad (7)$$

Peng proves that the dynamic pricing mechanism has the following properties:

(1) $G_{t_1,t}[x(t)] = G_{t_1,t_2}[G_{t_2,t}[x(t)]]$ for all $0 \le t_1 \le t_2 \le t$;

(2) $G_{t,t}[x(t)] = x(t)$;

(3) $x_1(t) \ge x_2(t)$, a.s then one can have $G_{r,t}[x_1(t)] \ge G_{r,t}[x_2(t)]$;

(4) $I_A G_{t,T}[x(T)] = G_{t,T}[I_A x(T)], \forall A \in \mathcal{F}_t$.

Attention. $G_{t,T}(\cdot)$ is not the same with g-expectation. I could say that if a function satisfies the properties (1), (2), and (4), then it is a kind of dynamic pricing mechanism. Through a testation in [31] people could know whether a dynamic pricing mechanism is a g-expectation. Obviously a dynamic pricing mechanism has weaker assumption compared with the g-expectation.

In this paper, I consider an agent with an endowment $x_0 \in \mathbb{R}$. Without loss of generality, I assume the psychological reference point at terminal T which serves as a base point to distinguish gains from losses is zero (for details, see remarks in Jin and Zhou [22]). As a result, $X(T)^+$ means gains while $X(T)^-$ represents losses. For convenience, this paper adopted the backward method [7, 10] and transformed the limit of the initial capital into the control of form of the dynamic pricing mechanism: $G_{0,T}[X] = x_0$. Next, I give two utility functions $u_+(\cdot)$ and $u_-(\cdot)$, both of which map from $R^+ \to R^+$ to measure the gains and losses, respectively. The technical assumptions on these utility functions, which will be imposed throughout this paper, are summarized as follows.

Assumption 4. $u_+(\cdot), u_-(\cdot) : R^+ \to R^+$ are strictly concave, increasing functions and satisfy $u_+(0) = u_-(0) = 0$ and $u_+(\cdot), u_-(\cdot)$ are continuously differential, with their derivatives bounded.

Note that, under Assumption 4, when $X \in L^2(\Omega, \mathcal{F}_T, P)$, then $u_+(X), u_-(X) \in L^2(\Omega, \mathcal{F}_T, P)$.

Now I give out the contingent claim a value reflecting the investor's risk attitudes and psychological emotions different from the prospect theory. It can be written as $\mathcal{V}(X)$, where $X = x(T)$, that is, the terminal contingent claim value, and X^+, X^- are, respectively, the positive part and the negative part of the contingent claim:

$$\mathcal{V}(X) = \mathcal{V}_+(X^+) - \mathcal{V}_-(X^-). \quad (8)$$

Here I replace Choquet integrals in [22] with g-expectations: $\mathcal{V}_+(X^+) = \mathcal{E}_{g_1}[u_+(X^+)]$, $\mathcal{V}_-(X^-) = \mathcal{E}_{g_2}[u_-(X^-)]$, given two backward stochastic differential equations generating functions g_1, g_2 as below satisfying L_1–L_3 and they can be nonlinear form so as to generate the nonlinear expectations:

$$\begin{aligned} -dx_1(t) &= g_1(x_1(t), z_1(t), t)\,dt - z_1'(t)\,dW(t), \\ x_1(T) &= u_+(X^+), \\ -dx_2(t) &= g_2(x_2(t), z_2(t), t)\,dt - z_2'(t)\,dW(t), \\ x_2(T) &= u_-(X^-). \end{aligned} \quad (9)$$

I consider $\mathcal{V}_+(X^+)$ and $\mathcal{V}_-(X^-)$ as a kind of measure or evaluation on the utility of gains and losses of terminal time. This form will greatly enrich the risk measure methods. Then, I give $\mathcal{V}(X) = \mathcal{V}_+(X^+) - \mathcal{V}_-(X^-) = x_1(0) - x_2(0)$. So, the problem under prospect theory with g-expectation's decision rule can be introduced as follows:

$$\begin{aligned} \text{Maximize} \quad & \mathcal{E}_{g_1}[u_+(X^+)] - \mathcal{E}_{g_2}[u_-(X^-)] \\ & = x_1(0) - x_2(0) \\ \text{subject to} \quad & G_{0,T}[X] \le x_0; \quad X = x(T). \end{aligned} \quad (10)$$

Here, $G_{0,T}[X]$ is the solution of BSDE (3) with respect to function g_0 and terminal variable X and actually it is a cost constraint.

Note the following.

(1) I did not assume that wealth function g_0 satisfies L_1–L_3, which will limit the scope of g_0. For example, the large investor case will be omitted by the assumption L_3. So I could not replace $G_{0,T}[X]$ with the form of g-expectation. However, when g_0 satisfies the assumption L_1, L_2, L_4, it will incorporate some nonlinear cases, for example, the wealth equation which allows borrowing money. This makes my model expand to a larger scope that is important both in the theory meaning and in the practical meaning.

(2) This value $\mathcal{V}$ has some good mathematical properties compared with the rule in [22] (see Chen and Selum [24]). (1) $\mathcal{E}_g$ has dynamic consistency which is important in practice, while the Choquet integral does not. (2) The probability distortion $T_+(\cdot)$ and $T_-(\cdot)$ in [22] is

$$\begin{aligned} V(X) = & \int_0^{+\infty} T_+(P(u_+(X^+) > t))\,dt \\ & - \int_0^{+\infty} T_-(P(u_-(X^-) > t))\,dt, \end{aligned} \quad (11)$$

where $T_+(\cdot)$ and $T_-(\cdot)$ are two differentiable and strictly increasing functions, $[0,1] \mapsto [0,1]$ satisfying $T_+(0) = T_-(0) = 0$ and $T_+(1) = T_-(1) = 1$. In my model,

$$\mathscr{V}_+(X^+) = \mathscr{E}_{g_1}[u_+(X^+)], \tag{12}$$

when g has a special form, for example, g does not depend on y and is linear in z (see Lemma 12), and $\mathscr{E}_{g_1}[u_+(X^+)] = E_{P_{g_1}}[u_+(X^+)] = \int_0^{+\infty} P_{g_1}(u_+(X^+) \geq t)dt$, here $P_{g_1}(A) = \mathscr{E}_{g_1}(I_A)$ is the g_1-probability. Obviously this equivalent probability is on the event sets adapted with $\mathscr{F}_T$ and have the similar effect as the distortion of probability $T_+ \circ P(\cdot)$, but it cannot be represented by a $T_+ \circ P(\cdot)$ in [22].

Before I deal with the original Problem (10), I need the following Theorem 5 and Lemma 6 by Peng in [31] to ensure the dividing below.

Theorem 5. *For the original Problem* (10)*, the constraint* $G_{0,T}[X] \leq x_0$ *is equivalent with* $G_{0,T}[X] = x_0$.

Proof. For any given $x_1 \leq x_0$, if I could prove that the constraint $G_{0,T}[X] = x_0$ is superior to the constraint $G_{0,T}[X] = x_1$, then I could prove the equivalence. For any given investment portfolio selection $\pi(t)$, with the given capital x_1, through the wealth equation I could get $X_1(T)$ at terminal time. With the same portfolio $\pi(t)$, the random part has the same form which could be omitted. Since the original capital $x_0 \geq x_1$, and by the comparison theorem of SDE, we could get $X_0(T) \geq X_1(T)$. Then I can get the conclusion. □

Lemma 6. $G_{t,T}(\cdot)$ *satisfies that* $G_{t,T}(0) = 0$, $\forall 0 \leq t \leq T$ *if and only if the generating function* g_0 *satisfies* L_1, L_2, L_4.

So throughout this paper I assume that g_0 satisfies L_1–L_3 and g_1, g_2 satisfy L_1, L_2, L_4.

3. The Analysis of the Model Structure

This section divides the original problem (10) into three subproblems, and I give out the proof that the solution of problem (15) is exactly the solution of problem (10), which means that I can translate the focus on the original problem (10) into the treatment of subproblems (13), (14), and (15). Because the subproblems can be treated beautifully using the method proposed by Ji and Zhou [7] and Ji and Peng [10], the original problems can be solved at the same time. At first, for any given (x_+, A), we have the following.

Subproblem One

$$\begin{aligned} &\text{Maximize} \quad \mathscr{V}_+(X) = \mathscr{E}_{g_1}[u_+(X)] = x_1(0) \\ &\text{subject to} \quad G_{0,T}[X(T)] = x_+; \quad X \geq 0 \text{ a.s}; \ X = 0 \ \text{ on } A^c. \end{aligned} \tag{13}$$

Subproblem Two

$$\begin{aligned} &\text{Minimize} \quad \mathscr{V}_-(X) = \mathscr{E}_{g_2}[u_-(X)] = x_2(0) \\ &\text{subject to} \quad G_{0,T}[X(T)] = x_+ - x_0; \quad X \geq 0 \text{ a.s}; \\ &\qquad\qquad X = 0 \ \text{ on } A. \end{aligned} \tag{14}$$

I write the extreme values of subproblem (13) and subproblem (14) as $\overline{\mathscr{V}}_+(x_+, A)$ and $\overline{\mathscr{V}}_-(x_+, A)$.

Subproblem Three

$$\begin{aligned} &\text{Maximize} \quad \overline{\mathscr{V}}_+(x_+, A) - \overline{\mathscr{V}}_-(x_+, A) \\ &\text{subject to} \quad x_+ \geq x_0; \quad x_+ = 0 \quad \text{when } P(A) = 0; \\ &\qquad\qquad x_+ = x_0 \quad \text{when } P(A) = 1. \end{aligned} \tag{15}$$

In order to get the equivalence between the original problem (10) and its three subproblems, I need g_0 to be linear. Under the linear cost constraint, I give the following theorem.

Theorem 7. *Let* g_0 *be linear with* (x, z)*. Given* X^**, define* $A^* = (\omega : X^* \geq 0)$ *and* $x_+^* = G_{0,T}[(X^*)^+]$*. Then* X^* *is the optimal solution of problem* (10) *if and only if* (x_+^*, A^*) *is the optimal solution of problem* (15)*; also* $(X^*)^+$ *and* $(X^*)^-$ *are, respectively, the optimal solutions of problems* (13) *and* (14) *with respect to* (x_+^*, A^*).

Proof. (1) $\Leftarrow$ Consider

$$\begin{aligned} G_{0,T}[X^*] &= G_{0,T}\left[(X^*)^+ - (X^*)^-\right] \\ &= G_{0,T}\left[(X^*)^+ I_{A^*} - (X^*)^- I_{(A^*)^c}\right] \\ &= G_{0,T}\left[(X^*)^+ I_{A^*}\right] - G_{0,T}\left[(X^*)^- I_{(A^*)^c}\right] \\ &= G_{0,T}\left[(X^*)^+\right] - G_{0,T}\left[(X^*)^-\right] \\ &= x_+^* - (x_+^* - x_0) = x_0. \end{aligned} \tag{16}$$

So, X^* is a feasible solution for the original problem (10). Then, I have $\mathscr{V}(X^*) = \mathscr{V}_+((X^*)^+) - \mathscr{V}_-((X^*)^-) = \overline{\mathscr{V}}_+(x_+^*, A^*) - \overline{\mathscr{V}}_-(x_+^*, A^*)$. So for any feasible solution of problem (10) written as X, I define $A = (\omega : X \geq 0)$ and $x_+ = G_{0,T}[X^+]$. Then I get $\mathscr{V}_+(X^+) \leq \overline{\mathscr{V}}_+(x_+, A)$ and $\mathscr{V}_-(X^-) \geq \overline{\mathscr{V}}_-(x_+, A)$, so $\mathscr{V}(X) = \mathscr{V}_+(X^+) - \mathscr{V}_-(X^-) \leq \overline{\mathscr{V}}_+(x_+, A) - \overline{\mathscr{V}}_-(x_+, A) \leq \overline{\mathscr{V}}_+(x_+^*, A^*) - \overline{\mathscr{V}}_-(x_+^*, A^*) = \mathscr{V}(X^*)$, which means X^* is the optimal solution of the original problem (10).

(2) $\Rightarrow$ Assume X^* is the optimal solution of the original problem (10); then definitely $\mathscr{V}_+((X^*)^+) \leq \overline{\mathscr{V}}_+(x_+^*, A^*)$ and $\mathscr{V}_-((X^*)^-) \geq \overline{\mathscr{V}}_-(x_+^*, A^*)$. If I assume the inequality is strict, then there exists X_1 which is a feasible solution for problem (13) with (A^*, x_+^*), so that $\mathscr{V}_+((X^*)^+) < \mathscr{V}_+(X_1)$. So I can define $\overline{X} = X_1 I_{A^*} - (X^*)^- I_{(A^*)^c}$ which is a feasible solution for problem (10), and $\mathscr{V}(\overline{X}) > \mathscr{V}(X^*)$; this convicts with the optimality of X^*, so $(X^*)^+$ is an optimal solution of subproblem (13). Similarly I can prove that $(X^*)^-$ is also

the optimal solution of subproblem (14). So, $\mathscr{V}_+((X^*)^+) = \overline{\mathscr{V}}_+(x_+^*, A^*)$ and $\mathscr{V}_-((X^*)^-) = \overline{\mathscr{V}}_-(x_+^*, A^*)$. For any feasible (x_+, A) I only need to prove the following:

$$\begin{aligned}\overline{\mathscr{V}}_+(x_+, A) - \overline{\mathscr{V}}_-(x_+, A) \\ \leq \overline{\mathscr{V}}_+(x_+^*, A^*) - \overline{\mathscr{V}}_-(x_+^*, A^*) = \mathscr{V}(X^*).\end{aligned} \tag{17}$$

There are three different cases in the following.

(1) When $P(A) = 0$, $x_+ = 0$, and $x_0 \leq 0$, so

$$\begin{aligned}&\overline{\mathscr{V}}_+(x_+, A) - \overline{\mathscr{V}}_-(x_+, A) \\ &= -\overline{\mathscr{V}}_-(0, A) = -\overline{\mathscr{V}}_-(x_0^+, A) \\ &= \sup_{G_{0,T}[X]=x_0^-;X\geq 0} [-\mathscr{V}_-(X)] \\ &\leq \sup_{G_{0,T}[-X]=-x_0^-;X\leq 0} [\mathscr{V}(-X)] \\ &= \sup_{G_{0,T}[X]=-x_0^-;X\leq 0} [\mathscr{V}(X)] \\ &\leq \sup_{G_{0,T}[X]=-x_0^-=x_0} [\mathscr{V}(X)] \\ &= \mathscr{V}(X^*).\end{aligned} \tag{18}$$

Attention: owing to the $G_{0,T}[X] + G_{0,T}[-X] = G_{0,T}[0] = 0$, so $G_{0,T}[-X] = -G_{0,T}[X] = -x_0^-$, which contributes to the fourth equality above.

(2) When $P(A) = 1$, then $x_+ = x_0$, and since $\overline{\mathscr{V}}_+(x_0, A) = \sup_{G_{0,T}[X]\leq x_0} [\mathscr{V}(X)]$, so I could get $\overline{\mathscr{V}}_+(x_0, A) - 0 \leq \mathscr{V}(X^*)$.

(3) When $0 < P(A) < 1$, for any $x_+ \geq x_0^+$, problems (13), (14) have nonempty feasible solution space for any (x_+, A), so, for any given $\epsilon > 0$, I can find X_1, X_2, respectively, feasible for (13) and (14), and also I can get $\mathscr{V}_+(X_1) \geq \overline{\mathscr{V}}_+(x_+, A) - \epsilon$, $\mathscr{V}_-(X_2) \leq \overline{\mathscr{V}}_-(x_+, A) + \epsilon$. Since $X = X_1 - X_2$ is a feasible solution for problem (10), then $\overline{\mathscr{V}}_+(x_+, A) - \overline{\mathscr{V}}_-(x_+, A) \leq \mathscr{V}_+(X_1) - \mathscr{V}_-(X_2) + 2\epsilon = \mathscr{V}(X) + 2\epsilon \leq \mathscr{V}(X^*) + 2\epsilon$. □

Theorem 7 gives out the equivalence between the original problem and the three subproblems. In the next section, I will address separately the maximum and minimum control problems under g-expectation environment.

4. The Treatment of the Model

In this section, I deal with the optimal portfolio problem. To get the optimal portfolio of problem (10), I need additional assumption.

L_5 : g_0, g_1 and g_2 are both continuously differentiable function with (x, z), and their derivatives are uniformly bounded.

Then, I give the necessary form of solution in problem (13) as the theorem below.

Theorem 8. *Assume that g_1 and g_0 also satisfy L_5, for any given (x_+, A), where $x_+ \geq x_0$, and $u_+(x)$ satisfies Assumption 4; if X^* is the optimal solution of problem* (13) *with these parameters, then X^* has the following form:*

$$X^* = \left(u_+'\right)^{-1}\left[-\frac{h_{11} m(T)}{h_{01} n_1(T)}\right] I_{A'}, \tag{19}$$

where A' is the subset of A. $m(t), n_1(t)$ are respective solutions to the following stochastic differential equations:

$$\begin{aligned}dm(t) &= m(t)\left[g_x^0(x_0^*(t), z_0^*(t), t)\,dt \right.\\ &\quad \left. + g_z^0(x_0^*(t), z_0^*(t), t)' dW(t)\right]; \\ m(0) &= 1; \\ dn_1(t) &= n_1(t)\left[g_x^1(x_1^*(t), z_1^*(t), t)\,dt \right.\\ &\quad \left. + g_z^1(x_1^*(t), z_1^*(t), t)' dW(t)\right]; \\ n_1(0) &= 1,\end{aligned} \tag{20}$$

where $h_{01} < 0$ and $h_{11} > 0$, $|h_{01}|^2 + |h_{11}|^2 = 1$ when $0 < x_+ < +\infty$. If $x_+ = 0$, then let $h_{01} = 0$, $h_{11} < 0$, and when $x_+ = +\infty$, let $h_{11} = 0$, $h_{01} < 0$. Functions g_x^0, g_z^0, g_x^1, and g_z^1 are respective derivatives of g_0, g_1 with respect to x or z, and $(x_0^*(t), z_0^*(t))$ and $(x_1^*(t), z_1^*(t))$ are respective solutions of BSDE (3) *and* (9) *with terminal random variables X^* and $u_+(X^*)$.*

Proof. Here I adopt the terminal variation method and Ekeland's variational principle to get the form that optimal solution of problem (13) needs to satisfy. First I define the state constraint as follows:

$$U = \left\{XX \geq 0, X \in L^2(\Omega, \mathscr{F}_T, P), X \equiv 0 \text{ on } A^c\right\}. \tag{21}$$

For each $0 < \rho \leq 1$ and $X \in U$, I define: $X^\rho = X^* + \rho(X - X^*)$. Let $(x_0^\rho, z_0^\rho, x_1^\rho, z_1^\rho)$ be the solution to (3) and (9) corresponding to $X = X^\rho$, and let $(\widehat{x}_0, \widehat{z}_0, \widehat{x}_1, \widehat{z}_1)$ be the solutions to the following variational equations:

$$\begin{aligned}-d\widehat{x}_0(t) &= \left[g_x^0(t)\,\widehat{x}_0(t) + g_z^0(t)'\widehat{z}_0(t)\right]dt - \widehat{z}_0' dW(t), \\ \widehat{x}_0(T) &= X - X^*, \\ -d\widehat{x}_1(t) &= \left[g_x^1(t)\,\widehat{x}_1(t) + g_z^1(t)'\widehat{z}_1(t)\right]dt - \widehat{z}_1' dW(t), \\ \widehat{x}_1(T) &= u_+'(X^*)(X - X^*)\end{aligned} \tag{22}$$

and define

$$\begin{aligned}\widetilde{x}_1^\rho(t) &= \rho^{-1}\left(x_1^\rho(t) - x_1^*(t)\right) - \widehat{x}_1(t), \\ \widetilde{z}_1^\rho(t) &= \rho^{-1}\left(z_1^\rho(t) - z_1^*(t)\right) - \widehat{z}_1(t), \\ \widetilde{x}_0^\rho(t) &= \rho^{-1}\left(x_0^\rho(t) - x_0^*(t)\right) - \widehat{x}_0(t), \\ \widetilde{z}_0^\rho(t) &= \rho^{-1}\left(z_0^\rho(t) - z_0^*(t)\right) - \widehat{z}_0(t).\end{aligned} \tag{23}$$

□

Lemma 9. (1) $\lim_{\rho\to 0} E[\sup_{0\le t\le T}(\widetilde{X}_i^{\rho}(t))^2] = 0$, *for* $i = 0, 1$;

(2) $\lim_{\rho\to 0} E[\int_0^T |\widetilde{z}_i^{\rho}(t)|^2 dt] = 0$, *for* $i = 0, 1$.

Proof. I only need to prove the lemma at the case of $i = 0$. From the variational equations and (3), (9) I can get the following equations:

$$\begin{aligned}
-d\widetilde{x}_0^{\rho}(t) = \rho^{-1} & \left[g_0\left(x_0^{\rho}(t), z_0^{\rho}(t), t\right) \right. \\
& - g_0\left(x_0^{*}(t), z_0^{*}(t), t\right) \\
& - \rho g_x^0\left(x_0^{*}(t), z_0^{*}(t), t\right)\widehat{x}_0 \\
& \left. -\rho g_z^0\left(x_0^{*}(t), z_0^{*}(t), t\right)'\widehat{z}_0 \right] dt \\
& - \widetilde{z}_0^{\rho}(t)' dW(t), \\
\widetilde{x}_0^{\rho}(T) = 0. &
\end{aligned} \tag{24}$$

Let

$$\begin{aligned}
A^{\rho}(t) &= \int_0^1 g_x^0\left(x_0^{*}(t) + \lambda\rho\left(\widehat{x}_0(t) + \widetilde{x}_0^{\rho}\right),\right. \\
&\qquad \left. z_0^{*}(t) + \lambda\rho\left(\widehat{z}_0(t) + \widetilde{z}_0^{\rho}\right), t\right) d\lambda, \\
B^{\rho}(t) &= \int_0^1 g_z^0\left(x_0^{*}(t) + \lambda\rho\left(\widehat{x}_0(t) + \widetilde{x}_0^{\rho}\right),\right. \\
&\qquad \left. z_0^{*}(t) + \lambda\rho\left(\widehat{z}_0(t) + \widetilde{z}_0^{\rho}\right), t\right) d\lambda, \\
C^{\rho}(t) &= \left[A^{\rho}(t) - g_x^0\left(x_0^{*}, z_0^{*}, t\right)\right]\widehat{x}_0(t) \\
&\quad + \left[B^{\rho}(t) - g_z^0\left(x_0^{*}, z_0^{*}, t\right)\right]'\widehat{z}_0(t).
\end{aligned} \tag{25}$$

Then,

$$\begin{aligned}
-d\widetilde{x}_0^{\rho}(t) &= \left(A^{\rho}(t)\widetilde{x}_0^{\rho}(t) + B^{\rho}(t)'\widetilde{z}_0^{\rho}(t) + C^{\rho}(t)\right) dt \\
&\quad - \widetilde{z}_0^{\rho}(t)' dW(t); \\
\widetilde{x}_0^{\rho}(T) &= 0.
\end{aligned} \tag{26}$$

A standard estimation on the above BSDE yields

$$\begin{aligned}
& E\left[\sup_{0\le t\le T}\left(\widetilde{x}_i^{\rho}(t)\right)^2\right] + E\int_t^T \left|\widetilde{z}_0^{\rho}(s)\right|^2 ds \\
&\quad \le KE\int_t^T \left(\widetilde{x}_0^{\rho}(s)\right)^2 ds + KE\int_t^T \left(\widetilde{C}^{\rho}(s)\right)^2 ds,
\end{aligned} \tag{27}$$

where $K > 0$ is a constant. However the Lebesgue dominated convergence theorem implies that

$$\lim_{\rho\to 0} E\int_0^T \left(C^{\rho}(t)\right)^2 dt = 0. \tag{28}$$

The desired result follows by applying Gronwall's inequality. □

Let $d(\cdot)$ be metric in U naturally introduced by its norm and introduce a mapping $F_{\epsilon}(\cdot) : U \to R$ which has the form as follows:

$$\begin{aligned}
F_{\epsilon}(X) = \Big(& \left(G_{0,T}[X] - x_{+}\right)^2 \\
& + \left\{\max\left(0, \mathscr{E}_{g_1}\left(u_{+}(X^{*})\right)\right.\right. \\
& \left.\left. - \mathscr{E}_{g_1}\left(u_{+}(X)\right) + \epsilon\right)\right\}^2 \Big)^{1/2}.
\end{aligned} \tag{29}$$

Lemma 10. *There must exist* $h_{11} \in R^1$, $h_{01} \in R^1$ *with* $h_{01} \le 0$, $h_{11} \ge 0$, *and* $|h_{01}|^2 + |h_{11}|^2 = 1$ *such that the following variational inequality holds:*

$$h_{11}\widehat{x}_0(0) + h_{01}\widehat{x}_1(0) \ge 0. \tag{30}$$

Proof. It can be easily checked that $F_{\epsilon}(X^{*}) = \epsilon$ and $F_{\epsilon}(X) > 0$, $\forall X \in U$. This leads to $F_{\epsilon}(X^{*}) \le \inf_{x\in U} F_{\epsilon}(X) + \epsilon$. Thus by Ekeland's variational principle, there exists $X^{\epsilon} \in U$ such that

$$\begin{gathered}
F_{\epsilon}\left(X^{\epsilon}\right) \le F_{\epsilon}\left(X^{*}\right), \\
d\left(X^{\epsilon}, X^{*}\right) \le \sqrt{\epsilon}, \\
F_{\epsilon}(X) + \sqrt{\epsilon} d\left(X, X^{\epsilon}\right) \ge F_{\epsilon}\left(X^{\epsilon}\right), \quad \forall X \in U.
\end{gathered} \tag{31}$$

Because X^{*} is the optimal solution of problem (13), it is easy to see that $G_{0,T}[X^{\epsilon}] - x_{+} \ge 0$. For any $X \in U$ and $0 < \rho \le 1$, I introduce the following notation:

$$\widehat{X} = X - X^{*}, \qquad \widehat{X}^{\epsilon} = X - X^{\epsilon}, \qquad X_{\rho}^{\epsilon} = X^{\epsilon} + \rho\widehat{X}^{\epsilon}. \tag{32}$$

Then, $F_{\epsilon}(X_{\rho}^{\epsilon}) + \sqrt{\epsilon} d(X_{\rho}^{\epsilon}, X^{\epsilon}) - F_{\epsilon}(X^{\epsilon}) \ge 0$ and $d(X_{\rho}^{\epsilon}, X^{\epsilon}) = \{E|\rho\widehat{X}^{\epsilon}|^2\}^{1/2}$.

Consider the following variational equation:

$$\begin{aligned}
-d\widehat{x}_0^{\epsilon}(t) &= \left\{g_x^0\left(x_0^{\epsilon}, z_0^{\epsilon}, t\right)\widehat{x}_0^{\epsilon}(t) \right. \\
&\qquad \left. + g_z^0\left(x_0^{\epsilon}, z_0^{\epsilon}, t\right)'\widehat{z}_0^{\epsilon}(t)\right\} dt \\
&\qquad - \left(\widehat{z}_0^{\epsilon}\right)' dW(t), \\
\widehat{x}_0^{\epsilon}(T) &= X - X^{\epsilon} = \widehat{X}^{\epsilon}, \\
-d\widehat{x}_1^{\epsilon}(t) &= \left\{g_x^1\left(x_1^{\epsilon}, z_1^{\epsilon}, t\right)\widehat{x}_1^{\epsilon}(t) \right. \\
&\qquad \left. + g_z^1\left(x_1^{\epsilon}, z_1^{\epsilon}, t\right)'\widehat{z}_1^{\epsilon}(t)\right\} dt \\
&\qquad - \left(\widehat{z}_1^{\epsilon}\right)' dW(t), \\
\widehat{x}_1^{\epsilon}(T) &= u_{+}'\left(X^{\epsilon}\right)\widehat{X}^{\epsilon},
\end{aligned} \tag{33}$$

where $(x_0^\epsilon, z_0^\epsilon, x_1^\epsilon, z_1^\epsilon)$ corresponds to X^ϵ. As for Lemma 9 we can get the following:

$$\lim_{\rho\to 0}\rho^{-1}\left\{G_{0,T}\left[X_\rho^\epsilon\right]-G_{0,T}\left[X^\epsilon\right]\right\}-\widehat{x}_0^\epsilon(0)=0,$$

$$\text{namely, } G_{0,T}\left[X_\rho^\epsilon\right]-G_{0,T}\left[X^\epsilon\right]=\rho\widehat{x}_0^\epsilon(0)+o(\rho),$$

$$\lim_{\rho\to 0}\rho^{-1}\left\{\mathscr{E}_{g_1}\left[u_+\left(X_\rho^\epsilon\right)\right]-\mathscr{E}_{g_1}\left[u_+\left(X^\epsilon\right)\right]\right\}-\widehat{x}_1^\epsilon(0)=0,$$

$$\text{namely, } \mathscr{E}_{g_1}\left[u_+\left(X_\rho^\epsilon\right)\right]-\mathscr{E}_{g_1}\left[u_+\left(X^\epsilon\right)\right]=\rho\widehat{x}_1^\epsilon(0)+o(\rho). \tag{34}$$

This leads to the following expansions:

$$\begin{aligned}&\left|G_{0,T}\left[X_\rho^\epsilon\right]-x_+\right|^2-\left|G_{0,T}\left[X^\epsilon\right]-x_+\right|^2\\&\quad=2\rho\left(G_{0,T}\left[X^\epsilon\right]-x_+\right)\widehat{x}_0^\epsilon(0)+o(\rho),\\&\left|\mathscr{E}_{g_1}\left(u_+\left(X^*\right)\right)-\mathscr{E}_{g_1}\left(u_+\left(X_\rho^\epsilon\right)\right)+\epsilon\right|^2\\&\qquad-\left|\mathscr{E}_{g_1}\left(u_+\left(X^*\right)\right)-\mathscr{E}_{g_1}\left(u_+\left(X^\epsilon\right)\right)+\epsilon\right|^2\\&\quad=-2\rho\widehat{x}_1^\epsilon\left(\mathscr{E}_{g_1}\left(u_+\left(X^*\right)\right)\right.\\&\qquad\left.-\mathscr{E}_{g_1}\left(u_+\left(X^\epsilon\right)\right)+\epsilon\right)+o(\rho).\end{aligned}\tag{35}$$

Now I consider two cases.

Case 1. $\mathscr{E}_{g_1}(u_+(X^*))-\mathscr{E}_{g_1}(u_+(X^\epsilon))+\epsilon>0$, so there exists $\rho_n\downarrow 0$ such that $\mathscr{E}_{g_1}(u_+(X^*))-\mathscr{E}_{g_1}(u_+(X_{\rho_n}^\epsilon))+\epsilon>0$. Consider

$$\begin{aligned}&\lim_{n\to+\infty}\frac{F_\epsilon\left(X_{\rho_n}^\epsilon\right)-F_\epsilon\left(X^\epsilon\right)}{\rho_n}\\&\quad=\lim_{n\to+\infty}\frac{F_\epsilon^2\left(X_{\rho_n}^\epsilon\right)-F_\epsilon^2\left(X^\epsilon\right)}{\rho_n}\frac{1}{F_\epsilon\left(X_{\rho_n}^\epsilon\right)+F_\epsilon\left(X^\epsilon\right)}\\&\quad=\frac{G_{0,T}\left[X^\epsilon\right]-x_+}{F_\epsilon\left(X^\epsilon\right)}\widehat{x}_0^\epsilon(0)\\&\quad-\frac{\mathscr{E}_{g_1}\left(u_+\left(X^*\right)\right)-\mathscr{E}_{g_1}\left(u_+\left(X^\epsilon\right)\right)+\epsilon}{F_\epsilon\left(X^\epsilon\right)}\widehat{x}_1^\epsilon(0).\end{aligned}\tag{36}$$

I define

$$\begin{aligned}h_{11}^\epsilon&=\frac{G_{0,T}\left[X^\epsilon\right]-x_+}{F_\epsilon\left(X^\epsilon\right)},\\h_{01}^\epsilon&=-\frac{\mathscr{E}_{g_1}\left(u_+\left(X^*\right)\right)-\mathscr{E}_{g_1}\left(u_+\left(X^\epsilon\right)\right)+\epsilon}{F_\epsilon\left(X^\epsilon\right)}.\end{aligned}\tag{37}$$

So, $h_{11}^\epsilon\widehat{x}_0^\epsilon(0)+h_{01}^\epsilon\widehat{x}_1^\epsilon(0)\geq-\sqrt{\epsilon}E|\widehat{X}^\epsilon|^{1/2}$, where $h_{01}^\epsilon<0$.

Case 2. $\mathscr{E}_{g_1}(u_+(X^*))-\mathscr{E}_{g_1}(u_+(X^\epsilon))+\epsilon\leq 0$. There exists $\rho_n\downarrow 0$ such that $\mathscr{E}_{g_1}(u_+(X^*))-\mathscr{E}_{g_1}(u_+(X_{\rho_n}^\epsilon))+\epsilon\leq 0$, so

$$\lim_{n\to+\infty}\frac{F_\epsilon\left(X_{\rho_n}^\epsilon\right)-F_\epsilon\left(X^\epsilon\right)}{\rho_n}=\frac{G_{0,T}\left[X^\epsilon\right]-x_+}{F_\epsilon\left(X^\epsilon\right)}\widehat{x}_0^\epsilon(0).\tag{38}$$

Also I define that $h_{11}^\epsilon=(G_{0,T}(X^\epsilon)-x_+)/F_\epsilon(X^\epsilon)$, $h_{01}^\epsilon=0$. I can get the following:

$$h_{11}^\epsilon\widehat{x}_0^\epsilon(0)+h_{01}^\epsilon\widehat{x}_1^\epsilon(0)\geq-\sqrt{\epsilon}E\left|\widehat{X}^\epsilon\right|^{1/2}.\tag{39}$$

At last, $h_{01}^\epsilon\leq 0$, $h_{11}^\epsilon\geq 0$, and $|h_{01}^\epsilon|^2+|h_{11}^\epsilon|^2=1$ for both cases. There exists a converging subsequence of $(h_{01}^\epsilon,h_{11}^\epsilon)$ with the limit (h_{01},h_{11}). Since $h_{01}^\epsilon\leq 0$, we have $h_{01}\leq 0$, so $h_{11}\geq 0$. On the other hand, it is easy to check that $\widehat{x}_0^\epsilon(0)\to\widehat{x}_0(0)$, $\widehat{x}_1^\epsilon(0)\to\widehat{x}_1(0)$.

Now, I apply Ito's lemma to $h_{01}\widehat{x}_1(t)n_1(t)+h_{11}\widehat{x}_0(t)m(t)$ and then have $E[h_{01}\widehat{x}_1(T)n_1(T)+h_{11}\widehat{x}_0(T)m(T)-h_{01}\widehat{x}_1(0)n_1(0)-h_{11}\widehat{x}_0(0)m(0)]=0$.

So by Lemma 10

$$\begin{aligned}&E\left[h_{01}\widehat{x}_1(T)n_1(T)+h_{11}\widehat{x}_0(T)m(T)\right]\\&\quad=E\left[h_{01}u_+'\left(X^*\right)n_1(T)+h_{11}m(T)\right]\cdot\left[X-X^*\right]\\&\quad=h_{11}\widehat{x}_0(0)+h_{01}\widehat{x}_1(0)\geq 0.\end{aligned}\tag{40}$$

Since the above is true for any $X\in U=\{X\mid X\geq 0, X\in L^2(\Omega,\mathscr{F}_T,P), X=0 \text{ on } A^c\}$, so I can have $[h_{01}u_+'(X^*)n_1(T)+h_{11}m(T)]\cdot[u-X^*]\geq 0$, P-a.s, $\forall u\in R^+$.

Owing to $X^*, X\in U$, so $X^*=X^*I_A$, $X=XI_A$ and $X-X^*=(X-X^*)I_A$

$$\begin{aligned}&h_{01}u_+'\left(X^*\right)n_1(T)+h_{11}m(T)\geq 0\quad\text{when } X^*=0 \text{ on } A,\\&h_{01}u_+'\left(X^*\right)n_1(T)+h_{11}m(T)=0\quad\text{when } X^*\neq 0 \text{ on } A.\end{aligned}\tag{41}$$

Then the proof is completed. □

For problem (14) I can solve it in the similar way.

Theorem 11. *Assume that g_2 and g_0 also satisfy L_5, for any given (x_+,A), where $x_+\geq x_0$, and $u_-(x)$ satisfies Assumption 4; if X^* is the optimal solution of problem* (14) *with these parameters, then X^* has the following form:*

$$X^*=\left(u_-'\right)^{-1}\left[-\frac{h_{12}m(T)}{h_{02}n_2(T)}\right]I_{\overline{A}^c},\tag{42}$$

where $h_{12}<0$, $h_{02}>0$, and $|h_{02}|^2+|h_{12}|^2=1$. $\overline{A}^c$ is a subset of A^c. $(x_0^*(t),z_0^*(t))$ and $(x_2^*(t),z_2^*(t))$ are respective solutions of BSDE (3) *and* (9) *with terminal random variables X^* and $u_-(X^*)$. $m(t),n_2(t)$ are, respectively, the solutions of the following stochastic differential equations:*

$$\begin{aligned}dm(t)&=m(t)\left[g_x^0\left(x_0^*(t),z_0^*(t),t\right)dt\right.\\&\qquad\left.+g_z^0\left(x_0^*(t),z_0^*(t),t\right)'dW(t)\right];\\&\qquad m(0)=1,\\dn_2(t)&=n_2(t)\left[g_x^2\left(x_2^*(t),z_2^*(t),t\right)dt\right.\\&\qquad\left.+g_z^2\left(x_2^*(t),z_2^*(t),t\right)'dW(t)\right];\\&\qquad n_2(0)=1.\end{aligned}\tag{43}$$

Combined with Theorems 7, 8, and 11, I can get the necessary form of optimal solution for the original problem (10). Let g_0 be of linear form; if X^* is the optimal solution,

$$X^* = \left(u_+'\right)^{-1}\left[-\frac{h_{11}^* m_1^*(T)}{h_{01}^* n_1^*(T)}\right] I_{A_1} - \left(u_-'\right)^{-1}\left[-\frac{h_{12}^* m_2^*(T)}{h_{02}^* n_2^*(T)}\right] I_{A_2}, \tag{44}$$

where $A_1 \cap A_2 = 0$.

5. The Economic Explanation and Model Comparison

In this section, I try to illustrate the difference between my model and that in [22] and the economic explanation. Because g_0 is a linear form, the cost constraint is the same form; then the only difference is the contingent claim's value: I use the g-expectation to replace the Choquet integral. The relation between two nonlinear expectations is searched in [24].

Lemma 12. *Suppose g satisfies L_1–L_3. Define $P_g(A) := \mathcal{E}_g(1_A)$ for a given event $A \in \mathcal{F}_T$. Then, $\mathcal{E}_g[\xi]$ can be represented as a Choquet integral for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$; that is,*

$$\mathcal{E}_g[\xi] = \int_{-\infty}^{0}\left[P_g(\xi \geq t) - 1\right] dt + \int_0^{+\infty} P_g(\xi \geq t)\, dt \tag{45}$$

if and only if there exists a function $b(t)$ such that g is of the following form:

$$g(x, z, t) = b(t) z. \tag{46}$$

This lemma implies the g-expectation coincides to the Choquet integral only when it becomes a classic linear expectation. Note that this kind of Choquet integral represented by g-probability is different from the form in [22]. Here, when g is a linear form, the original probability P is transformed to an equivalent probability, P_g. Actually, g-expectation and Choquet integral are two parallel ways to search nonlinear expectation and nonadditive probability. Besides, it has been proved that the g-expectation derived in the BSDE framework is more nonlinear in some sense.

The economic motive in model [22] is from prospect theory founded by Kahneman and Tversky. The key point is that an agent considers the world differently when facing gains and losses. In the world, there is a reference probability or I could say it is a "real" probability. Because of the emotion and psychology of an agent, the assumption of rational behavior is slack; then this leads to a probability distortion. The distortion makes the probability of events in agent's thought nonadditive. So, a Choquet integral is used to express the expectation. Moreover, the agent treats the model differently when facing gains and losses, so there are two utilities and probability distortions.

In my model, I accept the idea that an agent treats the world differently when facing gains and losses; that is, there is a change in agents' thought (even I could rationally consider that the agent is absolutely different between facing gains and losses). The reason I use g-expectation to replace the Choquet integral is that I try to describe another economic background. Here, I consider an ambiguous setting, a complicated world. In an ambiguous world, there is a set of priors instead of one reference probability. So the ambiguous environment replaces the probability distortion environment in [22]. Also, I use g_1 and g_2 to describe two sets of priors in the agent's thought, which reflects two uncertainty attitude to gains and losses. The uncertainty under g-expectation framework is considered as drift uncertainty. Recently, Epstein and Ji consider the volatility uncertainty case; see [32, 33].

Then, I will give two fundamental examples to describe some economic explanations.

Example 13. For simplicity I will define that $g_0 = -r(t)x(t) - \theta(t)' z(t)$, $g_1 = g_2 = 0$, which means that investors adopted the same risk attitude as the market. Now the optimal model can be transformed to

$$\begin{aligned} \text{Max:}\quad & \mathcal{V}(X) = E_P\left[u_+\left(X^+\right)\right] - E_P\left[U_-\left(X^-\right)\right] \\ \text{St}\quad & G_{0,,T}[X(T)] = \mathcal{E}_{g_0}(X) = E_{P_0}[X] \leq x_0. \end{aligned} \tag{47}$$

For given (x_+, A), it could be divided into three parts as the method above.

Subproblem One

$$\begin{aligned} \text{Max:}\quad & \mathcal{V}_+(X) = E_P\left[u_+(X)\right] = \int_\Omega u_+(X(\omega))\, dp(\omega) \\ & \qquad = \int_0^{+\infty} P\left(u_+ > y\right) dy \\ \text{St}\quad & G_{0,T}[X] = E_{P_0}[X] = x_+, \quad X \geq 0,\ X = 0 \ \text{ on } A^c. \end{aligned} \tag{48}$$

Subproblem Two

$$\begin{aligned} \text{Min:}\quad & \mathcal{V}_-(X) = E_P\left[u_-(X)\right] = \int_\Omega u_-(X(\omega))\, dp(\omega) \\ & \qquad = \int_0^{+\infty} P\left(u_- > y\right) dy \\ \text{St}\quad & G_{0,T}[X] = E_{P_0}[X] = x_+ - x_0, \quad X \geq 0, \\ & \qquad X = 0 \ \text{ on } \ A. \end{aligned} \tag{49}$$

Subproblem Three

$$\begin{aligned} \text{Max:}\quad & \overline{\mathcal{V}}_+\left(x_+, A\right) - \overline{\mathcal{V}}_-\left(x_+, A\right) \\ \text{St}\quad & x_+ \geq x_0; \quad x_+ = 0 \quad \text{when } P(A) = 0; \\ & x_+ = x_0 \quad \text{when } P(A) = 1. \end{aligned} \tag{50}$$

This is also the problem posed by Jin and Zhou, when $T_+ = T_- = x$. For subproblem one by method in [22], we could solve this problem as follows:

$$X^* = (u'_+)^{-1}\left(\frac{\lambda^*(c^*, x^*_+)\rho}{T'_+(F(\rho))}\right) I_{\rho\le c^*} \\ = (u'_+)^{-1}(\lambda^*\rho) I_{\rho\le c^*}; \quad (51)$$

however, by my method above I could know that they have the similar forms with my conclusion in Theorem 8 when using λ to replace $-h_{11}/h_{01}$, $\rho(T) = m(T)$, and $n_1(T) = n_2(T) = 1$. This leads to a hint that the two methods have the similar effect on solving the conventional cases. This means that the special forms of g-expectation could also be solved by the method in [22].

Example 14. Fixing $g_1 = -k_1(t)'z(t)$, $g_2 = -k_2(t)'z(t)$, g_0 can be any form which could demonstrate all the wealth movement forms. So the optimal model can be the following forms:

$$\text{Max:} \quad \mathscr{V}(X) = E_{P_1}[u_+(X^+)] - E_{P_2}[U_-(X^-)] \\ \text{St} \quad G_{0,T}[X] \le x_0. \quad (52)$$

Here P_1, P_2 show the risk attitudes facing earning and losses by defining

$$\frac{dP_1}{dP} = \exp\left\{-\int_0^T k_1(s)'dW(s) - \frac{1}{2}\int_0^T |k_1(s)|^2 ds\right\}, \\ \frac{dP_2}{dP} = \exp\left\{-\int_0^T k_2(s)'dW(s) - \frac{1}{2}\int_0^T |k_2(s)|^2 ds\right\}. \quad (53)$$

By the lemma above people can know that these two kinds of g-expectations can be replaced by the special form of Choquet expectations. In this case, the probability distortion functions T_1, T_2 have different forms owing to the difference between P_1 and P_2. The probability distortion reflects the psychology of the investors, so P_1 and P_2 reflect the different psychologies when facing gains and losses.

In view of the form (53), it can be seen that the optimal strategy should deliver a wealth in good states A^* and a shortfall in bad states $(A^*)^c$. To realize this goal, the investor should buy a contingent claim with the payoff $(u'_+)^{-1}[-h^*_{11}m^*(T)/h^*_{01}n^*_1(T)]$ at the cost x^*_+. Since $x^*_+ \ge x_0$, he needs to sell a contingent claim $(u'_-)^{-1}[-h^*_{12}m^*(T)/h^*_{02}n^*_2(T)]$ at the price of $x^*_+ - x_0$ to finance the shortfall. Given the investors' risk attitudes and their special S-shaped utility, the investor should try his best to get optimized $\mathscr{V}(X)$ with the initial x_0, so he has to consider how to allocate the limited money on the call option and the put option. This also means that the investor not only invests in stocks but also takes leverage to gamble on the good state of the market. As a result, the ratio of x^*_+/x_0 reflects the risk attitudes of the investors in some extent. Also in the optimal solution, $m(T)/n_i(T)$ reflects the risk probability ratio between the investors and the market.

Finally, I consider the nonlinear case of g, which means ambiguous environment. Consider a g-expectation with a g function; when g is sublinear in (x, z), $\mathscr{E}_g(X)$ is a sublinear expectation and it could be proved that there is a nonempty convex closed set of probabilities absolutely continuous to original probability, Q, such that $\mathscr{E}_g(X) = \max_{p\in Q} E_P[X]$. Also, if g is superlinear, there is a set of probabilities, $\mathscr{P}$, such that $\mathscr{E}_g(X) = \min_{p\in\mathscr{P}} E_P[X]$. Choosing specific g_1, g_2 with corresponding conditions, the contingent claim's value can be expressed as $\mathscr{V}(X) = \min_{p\in\mathscr{P}} E_P[u_+(X^+)] - \max_{p\in Q} E_P[u_-(X^-)]$. This form features an agent who is extremely pessimistic or cautious about his judgment for gains and losses. Note that it is reasonable that he has two sets of priors, Q and $\mathscr{P}$, for two kinds of environment as well as two kinds of mood. But if $g_1(x) = -g_2(-x)$, the sets Q and $\mathscr{P}$ coincide.

From the above linear and nonlinear cases, I give the detailed introduction about the similar and different points between the probability distortion used in [22] and g-expectation used in my model. The key point is that only when there is no probability distortion considered in [22] and there is only one probability P (no ambiguous belief) in my model like Example 13, both of the problems convert to the conventional optimal problem, and, of course, they are the same problem. Otherwise, the model in [22] and my model consider two different economic situations as mentioned above; that is, one is about probability distortion and the other considers the ambiguous case. From the mathematical viewpoint, Choquet integral and g-expectations are two different and parallel ways to search nonlinear expectation. The last point I want to say is that my model is a general form, because g_1, g_2 can be any nonlinear case satisfying the conditions of Theorems 8 or 11. So my model can be applied in other economic cases by choosing other g-functions in the future that cannot be interpreted now.

6. Conclusions

The optimal portfolio selection problem under the conventional expected utility theory has been well researched. There are also some classical methods to be adopted. Owing to the bad performance in realistic market, some new decision rules have been proposed by many economists. The famous prospect theory provided by Kahneman and Tversky has been considered as the most acceptable rule for its three points. Under the framework of this theory, the optimal model becomes difficult for the S-shaped utility function and the nonlinear probability distortion. The previous research usually neglected one or two points of the PT theory, which will greatly simplify the problem. My paper replaces the nonlinear Choquet expectation in [22] by the nonlinear g-expectation, so the nonlinear probability distortion $T_+(\cdot)$ could be replaced by the g function. Also I adopt the S-shaped utility function and different g-functions to show the different uncertainty attitudes towards gains and losses. If the wealth movement equations can be expanded to the nonlinear cases, it will incorporate many famous cases, for example, the larger investor case, the borrowing rate different from the lending rate case, and the different risk premium for short and long position case. In [22], they only research the classical linear case. Unfortunately, I can only simplify the problem in linear wealth movement equations. So, how to divide the original problem (10) under nonlinear wealth function case (nonlinear cost constraint) needs further research. Attention that for the decision rule, I use the g-expectation, but the wealth equation adopted the $G_{t,T}(\cdot)$ raised by Peng. This dynamic pricing mechanism has some similar properties with the g-expectation, but they are different. $G_{t,T}(\cdot)$ could incorporate more cases than the g-expectation.

Using some techniques, for example, dividing the original problem into three subproblems, Ekeland's variational principle, and the terminal perturbation method, I could get the necessary form of the optimal solution in our model. At last, I provide the economic meaning for my model.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] V. J. Neumann and O. Morgenstern, *Theory of Games and Economic Theory*, Princeton University Press, Princeton, NJ, USA, 1944.

[2] H. Markowitz, "Portfolio selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77–91, 1952.

[3] D. Li and W. L. Ng, "Optimal dynamic portfolio selection: multiperiod mean-variance formulation," *Mathematical Finance*, vol. 10, no. 3, pp. 387–406, 2000.

[4] N. El Karoui, S. Peng, and M. C. Quenez, "A dynamic maximum principle for the optimization of recursive utilities under constraints," *The Annals of Applied Probability*, vol. 11, no. 3, pp. 664–693, 2001.

[5] T. R. Bielecki, H. Q. Jin, S. R. Pliska, and X. Y. Zhou, "Continuous-time mean-variance portfolio selection with bankruptcy prohibition," *Mathematical Finance*, vol. 15, no. 2, pp. 213–244, 2005.

[6] S. R. Pliska, "A discrete time stochastic decision model," in *Advances in Filtering and Optimal Stochastic Control*, W. H. Fleming and L. G. Gorostiza, Eds., vol. 42 of *Lecture Notes in Control and Information Sciences*, pp. 290–304, Springer, New York, NY, USA, 1982.

[7] S. L. Ji and X. Y. Zhou, "A maximum principle for stochastic optimal control with terminal state constraints, and its applications," *Communications in Information and Systems*, vol. 6, no. 4, pp. 321–337, 2006.

[8] C. Bernard, S. L. Ji, and W. Tian, "An optimal insurance design problem under Knightian uncertainty," *Decisions in Economics and Finance*, vol. 36, no. 2, pp. 99–124, 2013.

[9] S. L. Ji, "Dual method for continuous-time Markowitz's problems with nonlinear wealth equations," *Journal of Mathematical Analysis and Applications*, vol. 366, no. 1, pp. 90–100, 2010.

[10] S. L. Ji and S. Peng, "Terminal perturbation method for the backward approach to continuous time mean-variance portfolio selection," *Stochastic Processes and their Applications*, vol. 118, no. 6, pp. 952–967, 2008.

[11] S. L. Ji and Q. M. Wei, "An overview on the principal-agent problems in continuous time," in *Real Options, Ambiguity, Risk and Insurance*, A. Bensoussan, S. Peng, and J. Sung, Eds., vol. 5 of *Studies in Probability, Optimization and Statistics*, pp. 126–143, IOS Press, 2013.

[12] S. L. Ji and Q. M. Wei, "A maximum principle for fully coupled forward-backward stochastic control systems with terminal state constraints," *Journal of Mathematical Analysis and Applications*, vol. 407, no. 2, pp. 200–210, 2013.

[13] S. L. Ji, Q. M. Wei, and X. M. Zhang, "A maximum principle for controlled time-symmetric forward-backward doubly stochastic differential equation with initial-terminal sate constraints," *Abstract and Applied Analysis*, vol. 2012, Article ID 537376, 29 pages, 2012.

[14] S. L. Ji and Z. Wu, "The maximum principle for one kind of stochastic optimization problem and application in dynamic measure of risk," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2189–2204, 2007.

[15] S. L. Ji and X. Y. Zhou, "The Neyman-Pearson lemma under g-probability," *Comptes Rendus Mathématique*, vol. 346, no. 3-4, pp. 209–212, 2008.

[16] S. L. Ji and X. Y. Zhou, "A generalized Neyman-Pearson lemma for g-probabilities," *Probability Theory and Related Fields*, vol. 148, no. 3-4, pp. 645–669, 2010.

[17] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuous time case," *Review of Economics and Statistics*, vol. 51, no. 3, pp. 239–265, 1969.

[18] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, vol. 3, no. 4, pp. 373–413, 1971, Erratum in *Journal of Economic Theory*, vol. 6, no. 2, pp. 213-214, 1973.

[19] J. M. Harrison and D. M. Kreps, "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, vol. 20, no. 3, pp. 381–408, 1979.

[20] D. Kahneman and A. Tversky, "Prospect theory: an analysis of decision under risk," *Econometrica*, vol. 47, no. 2, pp. 263–291, 1979.

[21] A. B. Berkelaar, R. Kouwenberg, and T. Post, "Optimal portfolio choice under loss aversion," *Review of Economics and Statistics*, vol. 86, no. 4, pp. 973–987, 2004.

[22] H. Q. Jin and X. Y. Zhou, "Behavioral portfolio selection in continuous time," *Mathematical Finance*, vol. 18, no. 3, pp. 385–426, 2008.

[23] H. Q. Jin, Z. Quan Xu, and X. Yu Zhou, "A convex stochastic optimization problem arising from portfolio selection," *Mathematical Finance*, vol. 18, no. 1, pp. 171–183, 2008.

[24] Z. Chen and A. Selum, *A Integral Representation Theorem of g-Expectation*, RR-4284, INRIA, 2001.

[25] Z. Chen and L. Epstein, "Ambiguity, risk, and asset returns in continuous time," *Econometrica*, vol. 70, no. 4, pp. 1403–1443, 2002.

[26] F. Choquet, Y. Hu, J. Mémin, and S. Peng, "Filtration-consistent nonlinear expectations and related g-expectations," *Probability Theory and Related Fields*, vol. 123, no. 1, pp. 1–27, 2002.

[27] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer, New York, NY, USA, 1998.

[28] N. El Karoui, S. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, no. 1, pp. 1–71, 1997.

[29] É. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[30] D. Cuoco and J. Cvitanić, "Optimal consumption choices for a "large" investor," *Journal of Economic Dynamics & Control*, vol. 22, no. 3, pp. 401–436, 1998.

[31] S. G. Peng, "Modeling derivatives pricing mechanism with their generating functions," 2006.

[32] L. Epstein and S. L. Ji, "Ambiguous volatility and asset pricing in continuous time," *The Review of Financial Studies*, vol. 26, no. 7, pp. 1740–1786, 2013.

[33] L. G. Epstein and S. L. Ji, "Ambiguous volatility, possibility and utility in continuous time," *Journal of Mathematical Economics*, vol. 50, pp. 269–282, 2014.