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Research Article

Karhunen-Loève Expansion for the Second Order Detrended Brownian Motion

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Based on the norm in the Hilbert Space $L^2[0,1]$, the second order detrended Brownian motion is defined as the orthogonal component of projection of the standard Brownian motion into the space spanned by nonlinear function subspace. Karhunen-Loève expansion for this process is obtained together with the relationship of that of a generalized Brownian bridge. As applications, Laplace transform, large deviation, and small deviation are given.

1. Introduction

Let $X = \{X(t), 0 \le t \le 1\}$ be a centered and continuous Gaussian process on $[0,1]$ with covariance function

$$K_X(t,s) = EX(t)X(s). \tag{1}$$

The Karhunen-Loève expansion of X is given by the (convergent in mean squares) series

$$X(t) = \sum_{k=1}^{\infty} \eta_k \sqrt{\lambda_k} f_k(t), \tag{2}$$

where $\{\eta_k, k \ge 1\}$ is a sequence of i.i.d. $N(0,1)$ random variables and $\{\lambda_k, k \ge 1\}$ is at most the countable set of eigenvalues of Fredholm integral operator

$$T_X f(t) = \int_0^1 K_X(t,s) f(s)\, ds \tag{3}$$

$\{f_k(t), k \ge 1\}$ and forms an orthogonal sequence in $L^2[0,1]$ and $\int_0^1 K_X(t,t)dt < \infty$.

Deheuvels et al. in [1–4] provided the Karhunen-Loève expansions for the processes that are related with Brownian motion. The Karhunen-Loève expansion for detrended Brownian motion has been studied by Ai et al. [5]. Note that the detrended Brownian motion in [5] can be viewed as projection to a constant function subspace in $L^2[0,1]$. That is,

$$\int_0^1 \widehat{W}_1(t)^2 dt = \min_{c_1,c_2} \int_0^1 (W(t) - c_1 - c_2 t)^2 dt. \tag{4}$$

To generalize the projection idea into nonlinear detrended process, now we consider

$$\min_{c_1,c_2,c_3} \int_0^1 \left(W(t) - c_1 - c_2 t - c_3 t^2\right)^2 dt \tag{5}$$

and the optimal constant c_j satisfy

$$\frac{\partial}{\partial c_j} \int_0^1 \left(W(t) - c_1 - c_2 t - c_3 t^2\right)^2 dt = 0, \quad j = 1,2,3. \tag{6}$$

It is easy to obtain

$$c_1 = 9\int_0^1 W(s)\, ds - 36\int_0^1 W(s)\, s\, ds + 30\int_0^1 W(s)\, s^2 ds,$$

$$c_2 = -36\int_0^1 W(s)\, ds + 192\int_0^1 W(s)\, s\, ds$$

$$- 180 \int_0^1 W(s) s^2 ds,$$

$$c_3 = 30 \int_0^1 W(s) ds - 180 \int_0^1 W(s) s\, ds + 180 \int_0^1 W(s) s^2 ds. \tag{7}$$

Let

$$A = \left(a_{ij}\right)_{3\times 3} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}; \tag{8}$$

we have

$$c_j = \sum_{i=1}^{3} a_{ij} \int_0^1 s^{i-1} W(s) ds, \quad j = 1, 2, 3. \tag{9}$$

Now we can define the second order detrended process

$$\begin{aligned} \widehat{W}_2(t) &= W(t) - \sum_{j=1}^{3} c_j t^{j-1} \\ &= W(t) + \left(-9 + 36t - 30t^2\right) \int_0^1 W(s) ds \\ &\quad + \left(36 - 192t + 180t^2\right) \int_0^1 W(s) s\, ds \\ &\quad + \left(-30 + 180t - 180t^2\right) \int_0^1 W(s) s^2 ds. \end{aligned} \tag{10}$$

2. Main Results

We give the following lemma that provides the explicit covariance function.

Lemma 1. *For convenience, we add $K_X(s,t)$ into formula* (11), *that is*

$$\begin{aligned} K_X(s,t) &= E\left(\widehat{W}_2(t) \widehat{W}_2(s)\right) \\ &= t \wedge s - \sum_{p,q=1}^{3} a_{pq} \left(\frac{t}{p} - \frac{t^{p+1}}{p(p+1)}\right) s^{q-1} \\ &\quad - \sum_{i,j=1}^{3} a_{ij} \left(\frac{s}{i} - \frac{s^{i+1}}{i(i+1)}\right) t^{j-1} \\ &\quad + \sum_{p,q=1}^{3} \sum_{i,j=1}^{3} a_{ij} a_{pq} \frac{p+i+2}{(p+1)(p+i+1)(i+1)} t^{j-1} s^{q-1}, \end{aligned} \tag{11}$$

where $a_{ij}, a_{pq}, i, j, p, q = 1, 2, 3$ is given in (8).

Proof. Consider

$$\widehat{W}_2(t) = W(t) - \sum_{j=1}^{3} c_j t^{j-1}, \quad 0 \le t \le 1 \tag{12}$$

and $\widehat{W}_2(t)$ is a mean zero Gaussian process; we obtain

$$\begin{aligned} &E\left(\widehat{W}_2(t) \widehat{W}_2(s)\right) \\ &\quad = E\widehat{W}_2(t) \widehat{W}_2(s) \\ &\quad = E\left(W(t) - \sum_{j=1}^{3} c_j t^{j-1}\right)\left(W(s) - \sum_{q=1}^{3} c_q s^{q-1}\right) \\ &\quad = E\left(W(t) - \sum_{i,j=1}^{3} a_{ij} \left(\int_0^1 u^{i-1} W(u) du\right) t^{j-1}\right) \\ &\qquad \cdot E\left(W(s) - \sum_{p,q=1}^{3} a_{pq} \left(\int_0^1 v^{p-1} W(v) dv\right) s^{q-1}\right). \end{aligned} \tag{13}$$

We notice that

$$E(W(t) W(s)) = t \wedge s, \tag{14}$$

$$\begin{aligned} &E\left(W(t) \int_0^1 v^{p-1} W(v) dv\right) \\ &\quad = E\left(\int_0^1 W(t) W(v) v^{p-1} dv\right) \\ &\quad = \int_0^1 (t \wedge v) v^{p-1} dv \\ &\quad = \int_0^t v^p dv + \int_t^1 t v^{p-1} dv \\ &\quad = \frac{t}{p} - \frac{t^{p+1}}{p(p+1)}, \end{aligned} \tag{15}$$

$$\begin{aligned} &E\left(\int_0^1 u^{i-1} W(u) du\right)\left(\int_0^1 v^{p-1} W(v) dv\right) \\ &\quad = \int_0^1 u^{i-1} E\left(W(u) \int_0^1 v^{p-1} W(v) dv\right) du \\ &\quad = \int_0^1 u^{i-1} \left(\frac{u}{p} - \frac{u^{p+1}}{p(p+1)}\right) du \\ &\quad = \frac{p+i+2}{(p+1)(p+i+1)(i+1)}. \end{aligned} \tag{16}$$

Substituting (16), (17), and (19) into (15), we derive

$$\begin{aligned} &E\left(\widehat{W}_2(t) \widehat{W}_2(s)\right) \\ &\quad = t \wedge s - \sum_{p,q=1}^{3} a_{pq} \left(\frac{t}{p} - \frac{t^{p+1}}{p(p+1)}\right) s^{q-1} \end{aligned}$$

$$- \sum_{i,j=1}^{3} a_{ij} \left(\frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) t^{j-1}$$
$$+ \sum_{p,q=1}^{3} \sum_{i,j=1}^{3} a_{ij} a_{pq} \frac{p+i+2}{(p+1)(p+i+1)(i+1)} t^{j-1} s^{q-1}. \tag{17}$$

□

Lemma 2 (see [3]). *If $t \in [0,1]$, $\varsigma_j(t) = \sum_{k=1}^{\infty} \omega_k \sqrt{\lambda_{k,j}} e_{k,j}(t)$, $j = 1, 2, \ldots$, then the condition*

$$\int_{[0,1]} \varsigma_1^2(t)\, dt \overset{law}{=} \int_{[0,1]} \varsigma_2^2(t)\, dt \tag{18}$$

is equivalent to the identity

$$\lambda_{k,1} = \lambda_{k,2} \quad \forall k \geq 1. \tag{19}$$

In the following, we will give some preliminaries, notions, and facts that are needed in Theorem 3. For $\nu > -1$, $J_\nu(\cdot)$ is Bessel function [6] with index ν and the positive zeros of $J_\nu(\cdot)$ are infinite sequence $0 < z_{\nu,1} < z_{\nu,2} < \cdots$. When $\nu = 3/2$, $\nu = 5/2$, the positive zeros of $J_{3/2,k}$, $J_{5/2,k}$ are $z_{3/2,k}$, $z_{5/2,k}$, $k = 1, 2, \ldots$, and they are in such a way that

$$0 < z_{3/2,1} < z_{5/2,1} < z_{3/2,2} < \cdots. \tag{20}$$

Now we can state one of the main results of this paper.

Theorem 3. *For the second order detrended Brownian motion $\widehat{W}_2(t)$ and a generalized Brownian bridge $B_2(t)$ with $n = 2$ in [7],*

$$B_2(t) = B(t) - \frac{1}{36} t \left(60t^2 + 18t - 67\right) B(1)$$
$$- t\left(60t^2 - 96t + 11\right) \int_0^1 B(s)\, ds \tag{21}$$
$$+ 10t\left(12t^2 - 18t + 1\right) \int_0^1 B(s)\, s\, ds.$$

One has the distribution identities

$$\int_0^1 \widehat{W}_2(t)^2 dt \overset{law}{=} \int_0^1 B_2(t)^2 dt$$
$$\overset{law}{=} \sum_{k\geq 1} \frac{\eta_k^2}{4z_{3/2,k}^2} + \sum_{k\geq 1} \frac{\eta_k^{*2}}{4z_{5/2,k}^2}, \tag{22}$$

where $\{\eta_k, k \geq 1\}$ and $\{\eta_k^*, k \geq 1\}$ denote two independent sequences of independently and identically distributed $N(0,1)$ random variables.

Proof. By straightforward induction based on the equation and splitting the integration range from t, we get

$$\lambda f(t) = \int_0^t s f(s)\, ds + t \int_t^1 f(s)\, ds$$
$$- \sum_{p,q=1}^{3} a_{pq} \left(\frac{t}{p} - \frac{t^{p+1}}{p(p+1)} \right) \int_0^1 s^{q-1} f(s)\, ds$$
$$- \sum_{i,j=1}^{3} a_{ij} t^{j-1} \int_0^1 \left(\frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s)\, ds$$
$$+ \sum_{p,q=1}^{3} \sum_{i,j=1}^{3} a_{ij} a_{pq} \frac{p+i+2}{(p+1)(p+i+1)(i+1)} t^{j-1}$$
$$\times \int_0^1 s^{q-1} f(s)\, ds. \tag{23}$$

By differentiation of both sides of (23) with respect to t, we have

$$\lambda f'(t) = \int_t^1 f(s)\, ds - \sum_{p,q=1}^{3} a_{pq} \frac{1-t^p}{p} \int_0^1 s^{q-1} f(s)\, ds$$
$$- \sum_{i=1,j=2}^{3} (j-1) a_{ij} t^{j-2} \int_0^1 \left(\frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s)\, ds$$
$$+ \sum_{p,q=1}^{3} \sum_{i=1,j=2}^{3} a_{ij} a_{pq} \frac{(p+i+2)(j-1)}{(p+1)(p+i+1)(i+1)} t^{j-2}$$
$$\times \int_0^1 s^{q-1} f(s)\, ds. \tag{24}$$

By differentiation of both sides of (24) with respect to t, we have

$$\lambda f''(t) + f(t)$$
$$= \sum_{i,p,q=1}^{3} a_{i3} a_{pq} \frac{2(p+i+2)}{(p+1)(p+i+1)(i+1)} \int_0^1 s^{q-1} f(s)\, ds$$
$$- 2 \sum_{i=1}^{3} a_{i3} \int_0^1 \left(\frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s)\, ds$$
$$+ \sum_{q=1}^{3} a_{1q} \int_0^1 s^{q-1} f(s)\, ds$$
$$+ \left(\sum_{q=1}^{3} a_{2q} \int_0^1 s^{q-1} f(s)\, ds \right) t$$
$$+ \left(\sum_{q=1}^{3} a_{3q} \int_0^1 s^{q-1} f(s)\, ds \right) t^2. \tag{25}$$

We can simplify this equation to

$$\lambda f''(t) + f(t) + b_1 + b_2 t + b_3 t^2 = 0, \tag{26}$$

where

$$\begin{aligned} b_1 = & -\sum_{i,p,q=1}^{3} a_{i3} a_{pq} \frac{2(p+i+2)}{(p+1)(p+i+1)(i+1)} \int_0^1 s^{q-1} f(s)\, ds \\ & + 2\sum_{i=1}^{3} a_{i3} \int_0^1 \left(\frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s)\, ds \\ & - \sum_{q=1}^{3} a_{1q} \int_0^1 s^{q-1} f(s)\, ds, \end{aligned} \tag{27}$$

$$b_2 = -\sum_{q=1}^{3} a_{2q} \int_0^1 s^{q-1} f(s)\, ds, \tag{28}$$

$$b_3 = -\sum_{q=1}^{3} a_{3q} \int_0^1 s^{q-1} f(s)\, ds. \tag{29}$$

We solve the inhomogeneous second differential equation to obtain

$$f(t) = c_1 \cos \frac{t}{\sqrt{\lambda}} + c_2 \sin \frac{t}{\sqrt{\lambda}} + 2\lambda b_3 - b_1 - b_2 t - b_3 t^2. \tag{30}$$

We substitute $f(t)$ into (28) and (29) to obtain

$$\begin{aligned} & \left(\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6\lambda \cos \frac{1}{\sqrt{\lambda}} - 12\lambda\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6\lambda \right) c_1 \\ & \quad + \left(-\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 6\lambda \sin \frac{1}{\sqrt{\lambda}} \right. \\ & \qquad \left. + 12\lambda\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 12\lambda\sqrt{\lambda} + \sqrt{\lambda} \right) c_2 \\ & = 0, \\ & \left(-2\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 14\lambda \cos \frac{1}{\sqrt{\lambda}} + 30\lambda\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 16\lambda \right) c_1 \\ & \quad + \left(2\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 14\lambda \sin \frac{1}{\sqrt{\lambda}} \right. \\ & \qquad \left. - 30\lambda\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 30\lambda\sqrt{\lambda} - 3\sqrt{\lambda} \right) c_2 \\ & = 0. \end{aligned} \tag{31}$$

In order that there are nonzero choices for c_1, c_2, the determinant of the above two equations has to be zero, which can be written as

$$D_{11} D_{22} - D_{12} D_{21} = 0, \tag{32}$$

where

$$\begin{aligned} D_{11} = & \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6\lambda \cos \frac{1}{\sqrt{\lambda}} - 12\lambda\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6\lambda, \\ D_{12} = & -\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 6\lambda \sin \frac{1}{\sqrt{\lambda}} + 12\lambda\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} \\ & - 12\lambda\sqrt{\lambda} + \sqrt{\lambda}, \\ D_{21} = & -2\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 14\lambda \cos \frac{1}{\sqrt{\lambda}} \\ & + 30\lambda\sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 16\lambda, \\ D_{22} = & 2\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 14\lambda \sin \frac{1}{\sqrt{\lambda}} - 30\lambda\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} \\ & + 30\lambda\sqrt{\lambda} - 3\sqrt{\lambda}. \end{aligned} \tag{33}$$

We obtain, after some simplification,

$$\begin{aligned} & 24\lambda^2\sqrt{\lambda} + 4\lambda\sqrt{\lambda} \\ & \quad = \left(24\lambda^2 - \lambda\right) \sin \frac{1}{\sqrt{\lambda}} + \left(24\lambda^2\sqrt{\lambda} - 8\lambda\sqrt{\lambda}\right) \cos \frac{1}{\sqrt{\lambda}}. \end{aligned} \tag{34}$$

Then $\lambda \neq 0$ is an eigenvalue if and only if (34) holds. We therefore obtain

$$\begin{aligned} D(\lambda) = & -720 \left(\left(24\lambda^{-7/2} - \lambda^{-5/2}\right) \sin \lambda^{1/2} \right. \\ & \left. + \left(24\lambda^{-4} - 8\lambda^{-3}\right) \cos \lambda^{1/2} - 24\lambda^{-4} - 4\lambda^{-3} \right), \end{aligned} \tag{35}$$

with $D(0) = 1$.

According to the trigonometric function formula

$$\begin{aligned} \sin \frac{1}{\sqrt{\lambda}} &= 2 \sin \frac{1}{2\sqrt{\lambda}} \cos \frac{1}{2\sqrt{\lambda}}, \\ \cos \frac{1}{\sqrt{\lambda}} &= 2\cos^2 \frac{1}{2\sqrt{\lambda}} - 1 = 1 - 2\sin^2 \frac{1}{2\sqrt{\lambda}}, \end{aligned} \tag{36}$$

we can observe that

$$D_{11} D_{22} - D_{12} D_{21} = -12\pi\sqrt{\lambda} J_{3/2}\left(\frac{1}{2\sqrt{\lambda}}\right) J_{5/2}\left(\frac{1}{2\sqrt{\lambda}}\right) = 0, \tag{37}$$

where $J_{3/2}(z)$, $J_{5/2}(z)$ are Bessel functions as follows:

$$\begin{aligned} J_{3/2}(z) &= \frac{\sqrt{2\pi \cdot z}}{\pi} \left(\frac{\sin z}{z^2} - \frac{\cos z}{z} \right), \\ J_{5/2}(z) &= \frac{\sqrt{2\pi \cdot z}}{\pi} \left(\left(-\frac{1}{z} + \frac{3}{z^3} \right) \sin z - \frac{3}{z^2} \cos z \right), \end{aligned} \tag{38}$$

which gives two sequences of eigenvalues of (37), namely, $(2z_{3/2,k})^{-2}$ and $(2z_{5/2,k})^{-2}$.

Similarly, we can obtain the two eigenvalues $(2z_{3/2,k})^{-2}$, $(2z_{5/2,k})^{-2}$ corresponding to those of integral operator of a generalized Brownian bridge $B_2(t)$. Note that the integral operator is

$$\int_0^1 K_2(s,t) f(s)\, ds. \tag{39}$$

Actually, in Lemma 2, we have the distribution identities

$$\begin{aligned}\int_0^1 \widehat{W}_2(t)^2 dt &\overset{\text{law}}{=} \int_0^1 B_2(t)^2 dt \\ &\overset{\text{law}}{=} \sum_{k\geq 1} \frac{\eta_k^2}{4z_{3/2,k}^2} + \sum_{k\geq 1} \frac{\eta_k^{*2}}{4z_{5/2,k}^2}.\end{aligned} \tag{40}$$

□

Remark 4. From (11) and (22), we derive that

$$\begin{aligned}\int_0^1 K_X(t,t)\, dt &= \int_0^1 E\left(\widehat{W}_2(t)^2\right) dt = E\int_0^1 \widehat{W}_2(t)^2 dt \\ &= \sum_{k\geq 1} \frac{1}{4z_{3/2,k}^2} + \sum_{k\geq 1} \frac{1}{4z_{5/2,k}^2} \\ &= \frac{1}{40} + \frac{1}{56} = \frac{3}{140}\end{aligned} \tag{41}$$

by using the Rayleigh's formula, for $\nu = 3/2$ and $\nu = 5/2$ (see, e.g., [3, (1.91), page 77] and [6, page 502]).

To check (41), from (11), we infer that

$$\begin{aligned}&\int_0^1 K_X(t,t)\, dt \\ &= \int_0^1 \left[t - \sum_{p,q=1}^{3} a_{pq}\left(\frac{t^q}{p} - \frac{t^{p+q}}{p(p+1)}\right) \right. \\ &\qquad - \sum_{i,j=1}^{3} a_{ij}\left(\frac{t^j}{i} - \frac{t^{i+j}}{i(i+1)}\right) \\ &\qquad + \sum_{p,q=1}^{3}\sum_{i,j=1}^{3} a_{ij}a_{pq} \\ &\qquad \left. \times \frac{p+i+2}{(p+1)(p+i+1)(i+1)} t^{j+q} \right] dt \\ &= \frac{3}{140}\end{aligned} \tag{42}$$

which is in agreement with (41).

3. Applications

In this section, the relevant applications of Karhunen-Loève expansion are given.

Proposition 5. *For each $\theta \in R$, one has*

$$\begin{aligned}&E\exp\left(-\frac{\theta^2}{2}\int_0^1 \widehat{W}_2(t)^2 dt\right) \\ &= \left\{ -720\left(\left(24\theta^{-7} - \theta^{-5}\right)\sin\theta + \left(-24\theta^{-8} + 8\theta^{-6}\right)\cos\theta \right.\right. \\ &\qquad \left.\left. + 24\theta^{-8} + 4\theta^{-6}\right)\right\}^{-1/2}.\end{aligned} \tag{43}$$

Proof.

$$\begin{aligned}&E\exp\left(-\frac{\theta^2}{2}\int_0^1 \widehat{W}_2(t)^2 dt\right) \\ &= E\exp\left(-\frac{\theta^2}{2}\sum_{k=1}^{\infty}\lambda_k\xi_k^2\right) \\ &= \prod_{k=1}^{\infty}\left(1+\lambda_k\theta^2\right)^{-1/2} = \left(D\left(-\theta^2\right)\right)^{-1/2} \\ &= \left\{-720\left(\left(24\theta^{-7} - \theta^{-5}\right)\sin\theta + \left(-24\theta^{-8} + 8\theta^{-6}\right)\cos\theta \right.\right. \\ &\qquad \left.\left. + 24\theta^{-8} + 4\theta^{-6}\right)\right\}^{-1/2},\end{aligned} \tag{44}$$

where $\lambda_1 > \lambda_2 > \cdots > 0$ and $\sum_{k=1}^{\infty}\lambda_k < \infty$. □

Proposition 6. *If $x > 0$, then*

$$\begin{aligned}&P\left(\int_0^1 \widehat{W}_2^2(t)\, dt > x\right) \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1} \\ &\quad \times \int_{\gamma_{2k-1}}^{\gamma_{2k}} \left(e^{-ux/2} \right. \\ &\qquad \times \left(u\left(\left|-720\left(\left(24u^{-7/2} - u^{-5/2}\right)\sin u^{1/2}\right.\right.\right.\right. \\ &\qquad\qquad + \left(24u^{-4} - 8u^{-3}\right)\cos u^{1/2} \\ &\qquad\qquad \left.\left.\left.\left. - 24u^{-4} - 4u^{-3}\right)\right|\right)^{1/2}\right)^{-1}\right) du,\end{aligned} \tag{45}$$

where $\gamma_k = \lambda_k^{-1}$, $k = 1, 2, \ldots$.

Proof. It can be proved by the Smirnov formula [8, 9], formula (23), and the definition of the Fredholm determinant. Similar proof method can be found from Proposition 3.3 in [10]. □

Next, we give the large deviation and small deviation probabilities of the second order detrended Brownian motion with respect to the norm in the Hilbert Space $L^2[0,1]$.

Proposition 7. *Consider $x \to \infty$,*

$$P\left(\int_0^1 \widehat{W}_2(t)^2 dt > x\right) = (1+o(1))\left(\frac{2}{\pi}\right)^{1/2}(2z_{3/2,1})^{-2}x^{-1/2}\exp\left(-2z_{3/2,1}^2 x\right) \cdot \left\{720\left(\left(-\frac{3}{2^4}z_{3/2,1}^{-9}+\frac{13}{2^8}z_{3/2,1}^{-7}\right)\sin 2z_{3/2,1} + \left(-\frac{3}{2^5}z_{3/2,1}^{-10}+\frac{9}{2^6}z_{3/2,1}^{-8}-\frac{1}{2^7}z_{3/2,1}^{-6}\right) \times \cos 2z_{3/2,1} + \frac{3}{2^5}z_{3/2,1}^{-10}+\frac{3}{2^6}z_{3/2,1}^{-8}\right)\right\}^{-1/2}. \tag{46}$$

Proof. By Deheuvels [2] and Martynov [8], we have for all $x > 0$

$$P\left(\int_0^1 \widehat{W}_2(t)^2 dt > x\right) = (1+o(1))\left(\frac{2}{\pi}\right)^{1/2}\gamma_1^{-1}\left(-D'(\gamma_1)\right)^{-1/2} \times x^{-1/2}\exp\left(-\frac{\gamma_1 x}{2}\right); \tag{47}$$

we take $D(\lambda)$ and $\gamma_1 = (2z_{3/2,1})^2$ into (47), and then the proof is completed. □

Proposition 8. *There exists a constant $c > 0$ such that*

$$P\left(\int_0^1 \widehat{W}_2(t)^2 dt \le \varepsilon\right) = (c+o(1))\,\varepsilon^{-2}\exp\left(-\frac{1}{8\varepsilon}\right), \quad \text{as } \varepsilon \longrightarrow 0. \tag{48}$$

Proof. We start with proving (48) by recalling Li, 1992 [11, 12].

Given two sequences $a_k > 0$ and $b_k > 0$ with

$$\sum_{k\ge1} a_k < \infty, \qquad \sum_{k\ge1} b_k < \infty, \qquad \sum_{k\ge1}\left|1-\frac{a_k}{b_k}\right| < \infty, \tag{49}$$

we have, as $\varepsilon \to 0$,

$$P\left(\sum_{k\ge1} a_k\xi_k^2 \le \varepsilon\right) = (1+o(1))\left(\prod_{k\ge1}\frac{b_k}{a_k}\right)^{1/2} P\left(\sum_{k\ge1} b_k\xi_k^2 \le \varepsilon\right). \tag{50}$$

By the asymptotic formula for zeros of Bessel function

$$z_{3/2,k} = \left(k+\frac{1}{2}\right)\pi + O\left(k^{-1}\right), \quad k \longrightarrow \infty, \qquad z_{5/2,k} = (k+1)\pi + O\left(k^{-1}\right), \quad k \longrightarrow \infty, \tag{51}$$

then $a_k = \lambda_k$, $b_{2k-1} = ((2k+1)\pi)^{-2}$, and $b_{2k} = ((2k+2)\pi)^{-2}$, $k \in N$, which satisfy (49) and by the distribution identity $\int_0^1 \widehat{W}_2^2(t)dt = \sum_{k\ge1}\lambda_{2k-1}\eta_k^2 + \sum_{k\ge1}\lambda_{2k}\eta_{2k}^{*2}$ and (50), there exists a constant c_1, such that

$$\begin{aligned} P\left(\int_0^1 \widehat{W}_2(t)^2 dt \le \varepsilon\right) &= P\left(\sum_{k\ge1}\lambda_{2k-1}\eta_k^2 + \sum_{k\ge1}\lambda_{2k}\eta_k^{*2} \le \varepsilon\right) \\ &= (1+o(1))\prod_{k\ge1}\left(\frac{b_k}{a_k}\right)^{1/2} P\left(\sum_{k\ge1} b_k\xi_k^2 \le \varepsilon\right) \\ &= (1+o(1))\,c_1 \\ &\quad \times P\left(\sum_{k\ge1}\frac{\xi_{2k-1}^2}{((2k+1)\pi)^2} + \sum_{k\ge1}\frac{\xi_{2k}^2}{((2k+2)\pi)^2} \le \varepsilon\right) \\ &= (1+o(1))\,c_1 P\left(\sum_{k\ge1}(k+2)^{-2}\xi_k^2 \le \varepsilon\pi^2\right), \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \tag{52}$$

Also, for all $d > -1$, there exists a constant $c_2 > 0$, such that, as $\varepsilon \to 0$,

$$P\left(\sum_{k\ge1}(k+d)^{-2}\xi_k^2 \le \varepsilon\pi^2\right) = (1+o(1))\,c_2\varepsilon^{-d}\exp\left(-\frac{1}{8\varepsilon}\right). \tag{53}$$

Connecting (52) with (53), we can obtain the proposition. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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