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Research Article

On Extremal Ranks and Least Squares Solutions Subject to a Rank Restriction

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We discuss the feasible interval of the parameter k and a general expression of matrix X which satisfies the rank equation $r(A - BXC) = k$. With these results, we study two problems under the rank constraint $r(A - BXC) = k$. The first one is to determine the maximal and minimal ranks under the rank constraint $r(A - BXC) = k$. The second one is to derive the least squares solutions of $\|BXC - A\|_F = \min$ under the rank constraint $r(A - BXC) = k$.

1. Introduction

We adopt the following notation in this paper. The set of $m \times n$ matrices with complex entries is denoted by $\mathbb{C}^{m\times n}$. The *conjugate transpose* of a matrix A is denoted by A^*. The symbols I_k and $r(A)$ are the $k\times k$ identity matrix and the rank of $A \in \mathbb{C}^{m\times n}$, respectively. $\|\cdot\|$ stands for the matrix *Frobenius norm*. The *Moore-Penrose inverse* of $A \in \mathbb{C}^{m\times n}$ is defined as the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$\begin{aligned} &(1)\ AXA = A, \qquad (2)\ XAX = X, \\ &(3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA, \end{aligned} \tag{1}$$

and is denoted by $X = A^\dagger$ (see [1]). Furthermore, we denote $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$.

In the literature, ranks of solutions of linear matrix equations have been studied widely. Uhlig [2] derived the extremal ranks of solutions of the consistent matrix equation of $AX = B$. Tian [3] derived the extremal ranks of solutions of $BXC = A$. Li and Liu [4] studied the extremal ranks of Hermitian solutions of $AX = B$. Li et al. [5] studied the extremal ranks of solutions with special structure of $AX = B$. Liu [6] derived the extremal ranks of solutions of $AX + YB = C$. Wang and Li [7] established the maximal and minimal ranks of the solution to consistent system $A_1X_1 = C_1, A_2X_2 = C_2$, and $A_3X_1B_1 + A_4X_2B_2 = C_3$. Wang and He [8] derived the extremal ranks of the general solution of the mixed Sylvester matrix equations

$$\begin{aligned} A_1X - YB_1 &= C_1, \\ A_2Z - YB_2 &= C_2. \end{aligned} \tag{2}$$

Liu [9] derived the extremal ranks of least square solutions to $BXC = A$. Sou and Rantzer [10] studied the minimum rank matrix approximation problem in the spectral norm

$$\min_X \ \operatorname{rank}(X) \quad \text{subject to } \|A - BXC\|_2 < 1. \tag{3}$$

Wei and Shen [11] studied a more general problem

$$\min_X \ \operatorname{rank}(X) \quad \text{subject to } \|A - BXC\|_2 < \xi, \tag{4}$$

where $\xi \geq \theta$ and $\theta = \min_Y \|A - BYC\|_2$. More results and applications about ranks of matrix expressions and solutions of matrix equations can be seen in ([2, 3, 8, 11–13], etc.).

Motivated by the work of [2, 3, 7–9, 14, 15], we consider a general problem. Assume that k is a prescribed nonnegative integer and $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, and $C \in \mathbb{C}^{q\times n}$ are given matrices. We now investigate the problem to determine the maximal and minimal ranks of solutions to the rank equation $r(A - BXC) = k$. This problem can be stated as follows.

Problem 1. Given matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, and $C \in \mathbb{C}^{q\times n}$ and nonnegative integer k_1, characterize the set

$$\mathbb{S}_{k_1} = \{X \mid X \in \mathbb{C}^{p\times q}, r(A - BXC) = k_1\}, \tag{5}$$

and determine the maximal and minimal ranks of solutions of the rank equation $r(A-BXC)=k_1$.

In [16–18], Wang, Wei, and Zha studied least squares solutions of line matrix equations under rank constraints, respectively. In [19], Wei and Wang derived a rank-k Hermitian nonnegative definite least squares solution to the equation $BXB^* = A$. In Problem 2, we discuss the least squares solutions X of $\|BXC-A\|_F = \min$ subject to $r(A-BXC)=k$. This problem can be stated as follows.

Problem 2. Given matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, and $C \in \mathbb{C}^{q\times n}$ and nonnegative integer k, determine the range of k, such that there exists a least squares solution X of $\|BXC-A\|_F = \min$ subject to $r(A-BXC)=k$; that is, characterize the set

$$\widehat{\mathbb{S}} = \{X \mid X \in \mathbb{C}^{p\times q}, \quad \|A-BXC\| = \min \text{ subject to } r(A-BXC)=k\}. \tag{6}$$

The paper is organized as follows. In Section 2, we provide some preliminary results; in Sections 3 and 4, we study Problems 1 and 2, respectively; and finally in Section 5, we conclude the paper with some remarks.

2. Preliminaries

In this section we present some preliminary results which will be used in the following sections to study Problems 1 and 2.

Lemma 3 (see [20]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$ be given. Then*

$$r\begin{bmatrix} A & B\end{bmatrix} = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{7}$$

$$r\begin{bmatrix} A \\ C\end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{8}$$

$$r\begin{bmatrix} A & B \\ C & 0\end{bmatrix} = r(B) + r(C) + r(E_BAF_C), \tag{9}$$

where $B_1 = E_AB$ and $C_1 = CF_A$.

Lemma 4. *Let $A \in \mathbb{C}^{m\times n}$ be given. Then*

$$\min_{r(X)=k} r(A-X) = \max\{k-r(A), r(A)-k\}, \tag{10}$$

$$\max_{r(X)=k} r(A-X) = \min\{m, n, k+r(A)\}. \tag{11}$$

Lemma 5 (see [21]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, $C \in \mathbb{C}^{q\times n}$, and $D \in \mathbb{C}^{q\times p}$ be given. Then*

$$r\left(D - CA^{\dagger}B\right) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D\end{bmatrix} - r(A). \tag{12}$$

Lemma 6 (see [22, 23] (the Eckart-Young-Mirsky theorem)). *Let $A \in \mathbb{C}_r^{m\times n}$, k be a given nonnegative integer in which $k \le r$ and the singular value decomposition [24] of A be*

$$A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0\end{bmatrix} V^*, \tag{13}$$

where $\Sigma = \operatorname{diag}\{\lambda_1, \ldots, \lambda_r\}$, $\lambda_1 \ge \cdots \ge \lambda_r > 0$, and U and V are unitary matrices of appropriate sizes. Then

$$\min_{r(X)=k} \|A-X\|^2 = \sum_{i=k+1}^{r} \lambda_i^2. \tag{14}$$

Furthermore, when $\lambda_k > \lambda_{k+1}$,

$$X = U \operatorname{diag}\{\lambda_1, \ldots, \lambda_k, 0, \ldots, 0\} V^*; \tag{15}$$

when $p < k < q \le r$ and $\lambda_p > \lambda_{p+1} = \cdots = \lambda_q > \lambda_{q+1}$,

$$X = U \operatorname{diag}\left\{\lambda_1, \ldots, \lambda_p, \lambda_k QQ^*, 0, \ldots, 0\right\} V^*, \tag{16}$$

where Q is an arbitrary matrix satisfying $Q \in \mathbb{C}^{(q-p)\times(k-p)}$ and $Q^*Q = I_{k-p}$.

3. Solutions to Problem 1

In this section, we study Problem 1 proposed in Section 1.

Suppose that the matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, and $C \in \mathbb{C}^{q\times n}$ are given. Let

$$l = r\begin{bmatrix} A \\ C\end{bmatrix} + r\begin{bmatrix} A & B\end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0\end{bmatrix}. \tag{17}$$

Then from [25] there exists X such that $r(A-BXC)=k_1$, if and only if

$$\min\left\{r\begin{bmatrix} A & B\end{bmatrix}, r\begin{bmatrix} A \\ C\end{bmatrix}\right\} \ge k_1 \ge l. \tag{18}$$

Furthermore, let

$$B = U_1\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0\end{bmatrix} V_1^*, \qquad C = U_2\begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0\end{bmatrix} V_2^* \tag{19}$$

be singular value decompositions of B and C with unitary matrices $U_1 \in \mathbb{C}^{m\times m}$, $U_2 \in \mathbb{C}^{q\times q}$, $V_1 \in \mathbb{C}^{p\times p}$, and $V_2 \in \mathbb{C}^{n\times n}$. Write $U_1^*AV_2$ in partitioned form as

$$U_1^*AV_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \tag{20}$$

where $A_{11} \in \mathbb{C}^{r(B)\times r(C)}$, $A_{12} \in \mathbb{C}^{r(B)\times(n-r(C))}$, $A_{21} \in \mathbb{C}^{(m-r(B))\times r(C)}$, and $A_{22} \in \mathbb{C}^{(m-r(B))\times(n-r(C))}$. Also assume that the singular value decomposition of A_{22} and the corresponding decompositions are given by

$$\begin{gathered} A_{22} = U_3\begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0\end{bmatrix} V_3^*, \qquad U_3^*A_{21} = \begin{bmatrix} J \\ K\end{bmatrix}, \\ A_{12}V_3 = \begin{bmatrix} E & F\end{bmatrix}, \qquad F = U_4\begin{bmatrix} \Sigma_4 & 0 \\ 0 & 0\end{bmatrix} V_4^*, \\ K = U_5\begin{bmatrix} \Sigma_5 & 0 \\ 0 & 0\end{bmatrix} V_5^*, \\ U_4^*\left(A_{11} - E\Sigma_3^{-1}J\right)V_5 = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22}\end{bmatrix}, \end{gathered} \tag{21}$$

where U_i and V_i are unitary matrices of appropriate sizes in which $i = 3, 4, 5$, $J \in \mathbb{C}^{r(A_{22})\times r(C)}$, $K \in \mathbb{C}^{(m-r(B)-r(A_{22}))\times r(C)}$, $E \in \mathbb{C}^{r(B)\times r(A_{22})}$, $F \in \mathbb{C}^{r(B)\times(n-r(C)-r(A_{22}))}$, $\widehat{A}_{11} \in \mathbb{C}^{r(F)\times r(K)}$, $\widehat{A}_{12} \in \mathbb{C}^{r(F)\times(r(C)-r(K))}$, $\widehat{A}_{21} \in \mathbb{C}^{(r(B)-r(F))\times r(K)}$, and $\widehat{A}_{22} \in \mathbb{C}^{(r(B)-r(F))\times(r(C)-r(K))}$.

We have the following result.

Theorem 7. *Suppose that the singular value decompositions of matrices B, C, A_{22}, F, and K are given in* (19)–(21). *$U_1^*AV_2$, $U_3^*A_{21}$, $U_3^*A_{21}$, $A_{12}V_3$, and $U_4^*(A_{11}-E\Sigma_3^{-1}J)V_5$ have the forms in* (20) *and* (21). *If k satisfies* (18), *then any solution X to the rank equation $r(A-BXC) = k_1$ has the form*

$$X = V_1 \begin{bmatrix} \Sigma_1^{-1}U_4 & 0 \\ 0 & I_{p-r(B)} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & \widehat{A}_{22}+T \end{bmatrix} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \times \begin{bmatrix} V_5^*\Sigma_2^{-1} & 0 \\ 0 & I_{q-r(C)} \end{bmatrix} U_2^*, \quad (22)$$

where $X_{12} \in \mathbb{C}^{r(B)\times(q-r(C))}$, $X_{21} \in \mathbb{C}^{(p-r(B))\times r(C)}$, $X_{22} \in \mathbb{C}^{(p-r(B))\times(q-r(C))}$, $Y_{11} \in \mathbb{C}^{r(F)\times r(K)}$, $Y_{12} \in \mathbb{C}^{r(F)\times(r(C)-r(K))}$, and $Y_{21} \in \mathbb{C}^{(r(B)-r(F))\times r(K)}$ are arbitrary, $T \in \mathbb{C}^{(r(B)-r(F))\times(r(C)-r(K))}$, and $r(T) = k_1 - l$.

Proof. From the singular value decompositions of matrices of B, C, F, and K, we observe that

$$\begin{gathered} U_1 \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} V_2^* = BB^\dagger AC^\dagger C, \\ U_1 \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} V_2^* = BB^\dagger AF_C, \\ U_1 \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} V_2^* = E_B AC^\dagger C, \\ U_1 \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix} V_2^* = E_B AF_C, \\ U_1 \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix} V_2^* = AF_C, \\ U_1 \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} V_2^* = E_B A, \\ \begin{bmatrix} 0 & F \end{bmatrix} V_3^* = A_{12}F_{A_{22}}, \qquad U_3 \begin{bmatrix} 0 \\ K \end{bmatrix} = E_{A_{22}}A_{21}, \\ U_4 \begin{bmatrix} 0 & 0 \\ 0 & \widehat{A}_{22} \end{bmatrix} V_5^* = E_F\left(A_{11} - E\Sigma_3^{-1}J\right)F_K. \end{gathered} \quad (23)$$

Then by repeated application of Lemma 3, we have

$$r(A_{22}) = r(E_B AF_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C), \quad (24)$$

$$\begin{aligned} r(F) &= r\left(\begin{bmatrix} 0 & F \end{bmatrix} V_3^*\right) = r\left(A_{12}F_{A_{22}}\right) = r\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} - r(A_{22}) \\ &= r(AF_C) - r(E_B AF_C) \\ &= r(B) + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \end{aligned} \quad (25)$$

$$\begin{aligned} r(K) &= r\left(U_3\begin{bmatrix} 0 \\ K \end{bmatrix}\right) = r\left(E_{A_{22}}A_{21}\right) \\ &= r\begin{bmatrix} A_{21} & A_{22} \end{bmatrix} - r(A_{22}) \\ &= r(E_B A) - r(E_B AF_C) \\ &= r(C) + r\begin{bmatrix} A & B \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \end{aligned} \quad (26)$$

Furthermore, write

$$V_1^*XU_2 = \begin{bmatrix} \Sigma_1^{-1}X_{11}\Sigma_2^{-1} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \qquad U_4^*X_{11}V_5 = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \quad (27)$$

in which $X_{11} \in \mathbb{C}^{r(B)\times r(C)}$, $X_{12} \in \mathbb{C}^{r(B)\times(q-r(C))}$, $X_{21} \in \mathbb{C}^{(p-r(B))\times r(C)}$, $X_{22} \in \mathbb{C}^{(p-r(B))\times(q-r(C))}$, $Y_{11} \in \mathbb{C}^{r(F)\times r(K)}$, $Y_{12} \in \mathbb{C}^{r(F)\times(r(C)-r(K))}$, $Y_{21} \in \mathbb{C}^{(r(B)-r(F))\times r(K)}$, and $Y_{22} \in \mathbb{C}^{(r(B)-r(F))\times(r(C)-r(K))}$. It follows that

$$\begin{aligned} r&(A-BXC) \\ &= r\left(U_1^*AV_2 - \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^*XU_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix}\right) \end{aligned} \quad (28a)$$

$$\begin{aligned} &= r\begin{bmatrix} A_{11}-X_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ &= r\begin{bmatrix} A_{11}-X_{11} & A_{12}V_3 \\ U_3^*A_{21} & \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\ &= r\begin{bmatrix} A_{11}-X_{11} & \begin{bmatrix} E & F \end{bmatrix} \\ \begin{bmatrix} J \\ K \end{bmatrix} & \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\ &= r\begin{bmatrix} A_{11}-E\Sigma_3^{-1}J-X_{11} & F \\ K & 0 \end{bmatrix} + r(A_{22}) \\ &= r\begin{bmatrix} \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix} - U_4^*X_{11}V_5 & \begin{bmatrix} \Sigma_4 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \Sigma_5 & 0 \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \\ &\quad + r(A_{22}) \end{aligned} \quad (28b)$$

$$= r\begin{bmatrix} \widehat{A}_{11}-Y_{11} & \widehat{A}_{12}-Y_{12} & \Sigma_4 & 0 \\ \widehat{A}_{21}-Y_{21} & \widehat{A}_{22}-Y_{22} & 0 & 0 \\ \Sigma_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + r\left(A_{22}\right)$$
$$= r\left(\widehat{A}_{22}-Y_{22}\right)+r\left(A_{22}\right)+r\left(F\right)+r\left(K\right). \tag{28c}$$

Since $r(A-BXC)=k_1$, from (28c), we obtain

$$r\left(\widehat{A}_{22}-Y_{22}\right)=k_1-r\left(A_{22}\right)-r\left(F\right)-r\left(K\right). \tag{29}$$

The identity $r(\widehat{A}_{22}-Y_{22})=k_1-l$ follows by substituting (24)–(26) into (29). Hence, any solution Y_{22} to the rank equation $r(\widehat{A}_{22}-Y_{22})=k_1-l$ has the form

$$Y_{22}=\widehat{A}_{22}+T, \tag{30}$$

where $r(T)=k_1-l$.

Substituting (30) into the second partitioned matrix in (27), we obtain

$$X_{11}=U_4\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & \widehat{A}_{22}+T \end{bmatrix}V_5^{*}, \tag{31}$$

where $r(T)=k_1-l$. The expression of X in (22) follows by substituting (31) into the first partitioned matrix in (27). □

Let $X=0$. From (28b), we have

$$r\begin{bmatrix} A_{11}-E\Sigma_3^{-1}J & F \\ K & 0 \end{bmatrix}=r\left(A\right)-r\left(A_{22}\right). \tag{32}$$

Substituting (24) into the above identity, we have

$$r\begin{bmatrix} A_{11}-E\Sigma_3^{-1}J & F \\ K & 0 \end{bmatrix}=r\left(A\right)+r\left(B\right)+r\left(C\right)-r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{33}$$

By applying Lemma 3 (9) to the final identity in (23), it follows that

$$\begin{aligned} r\left(\widehat{A}_{22}\right) &= r\left(E_F\left(A_{11}-E\Sigma_3^{-1}J\right)F_K\right) \\ &= r\begin{bmatrix} A_{11}-E\Sigma_3^{-1}J & F \\ K & 0 \end{bmatrix}-r\left(F\right)-r\left(K\right) \\ &= r\left(A\right)+r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}-r\begin{bmatrix} A \\ C \end{bmatrix}-r\begin{bmatrix} A & B \end{bmatrix} \\ &= r\left(A\right)-l. \end{aligned} \tag{34}$$

We have the following result.

Theorem 8. *Let A, B, and C be as in Problem 1 and let k_1 satisfy* (18). *Then*

$$\min_{r(A-BXC)=k_1} r\left(X\right)=\max\left\{r\left(A\right)-k_1,k_1-r\left(A\right)\right\}, \tag{35}$$

$$\max_{r(A-BXC)=k_1} r\left(X\right) = \min\left\{p,q,k_1+p+q+r\left(A\right) -2r\left(B\right)-2r\left(C\right)+r\begin{bmatrix} A \\ C \end{bmatrix}+r\begin{bmatrix} A & B \end{bmatrix}\right\}. \tag{36}$$

Proof. From a general expression of X for the rank equation $r(A-BXC)=k_1$ given in (22), (10), and (34), we obtain

$$\begin{aligned} \min_{r(A-BXC)=k_1} r\left(X\right) &= \min_{r(T)=k_1-l} r\left(\widehat{A}_{22}+T\right) \\ &= \max\left\{k_1-l-r\left(\widehat{A}_{22}\right),r\left(\widehat{A}_{22}\right)-k_1+l\right\} \\ &= \max\left\{r\left(A\right)-k_1,k_1-r\left(A\right)\right\}. \end{aligned} \tag{37}$$

From (11), (22), and (34), we obtain

$$\max_{r(A-BXC)=k_1} r\left(X\right)$$
$$= \max_{r(T)=k_1-l,\,Y_{11},Y_{12},Y_{21},X_{12},X_{21},X_{22}} r\begin{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & \widehat{A}_{22}+T \end{bmatrix} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$
$$= \max_{r(T)=k-l,\,Y\in\mathbb{C}^{p\times q},Z\in\mathbb{C}^{p\times q}} r\left(\begin{bmatrix} \widehat{A}_{22}+T & 0 \\ 0 & 0 \end{bmatrix}-Y\begin{bmatrix} 0 & 0 \\ 0 & I_{q-r(C)+r(K)} \end{bmatrix} -\begin{bmatrix} 0 & 0 \\ 0 & I_{p-r(B)+r(F)} \end{bmatrix}Z\right)$$
$$= \min\left\{p,q,p+q, \max_{r(T)=k-l} r\begin{bmatrix} \widehat{A}_{22}+T & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{p-r(B)+r(F)} \\ 0 & 0 & 0 & 0 \\ 0 & I_{q-r(C)+r(K)} & 0 & 0 \end{bmatrix}\right\}$$

$$= \min\left\{p, q, \max_{r(T)=k_1-l} r\left(\widehat{A}_{22}+T\right)+p+q\right.$$
$$\left.+r(F)+r(K)-r(B)-r(C)\right\}$$
$$= \min\left\{p, q, p+q+r(K)-r(C),\right.$$
$$p+q+r(F)-r(B),$$
$$k_1+p+q+r(A)-2r(B)-2r(C)$$
$$\left.+r\begin{bmatrix}A\\C\end{bmatrix}+r\begin{bmatrix}A & B\end{bmatrix}\right\}. \tag{38}$$

Since $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$, we see that $p+q+r(K)-r(C) \geq p$ and $p+q+r(F)-r(B) \geq q$. To simplify expression (38) by the two inequalities, we obtain expression (36) for the maximal rank of solutions to the rank equation $r(A-BXC) = k_1$. □

Remark 9 (see [3]). Let A, B, and C be as in Theorem 7. The matrix equation $BXC = A$ is consistent, if and only if there exists X such that $r(A - BXC) = 0$. Therefore, applying Theorem 8, we have the extremal ranks of solutions to the matrix equation $BXC = A$:

$$\min_{BXC=A} r(X) = r(A),$$
$$\max_{BXC=A} r(X) = \min\{p, q, p+q+r(A)-r(B)-r(C)\}. \tag{39}$$

Remark 10 (see [9]). Let A, B, and C be as in Theorem 7 and let $\mathbb{S} = \{X \in \mathbb{C}^{p\times q} \mid \|BXC - A\| = \min\}$. Since $X \in \mathbb{S}$, if and only if $B^*AC^* = B^*BXCC^*$, and the matrix equation $B^*AC^* = B^*BXCC^*$ is always consistent, we can use B^*AC^*, B^*B, and CC^* to replace A, B, and C in (39). Then we have the extremal ranks of least squares solutions of the matrix equation $BXC = A$:

$$\min_{\|BXC-A\|=\min} r(X) = r\left(B^*AC^*\right),$$
$$\max_{\|BXC-A\|=\min} r(X) = \min\left\{p, q, p+q\right.$$
$$\left.+r\left(B^*AC^*\right)-r(B)-r(C)\right\}. \tag{40}$$

In [14], Liu and Tian derive the extremal ranks of submatrices in a Hermitian solution to the consistent matrix equation $BXB^* = A$. In the following theorem, we derive the range of k_1 such that there exists a Hermitian solution X to the rank equation $r(A - BXB^*) = k_1$, and the maximal and minimal ranks of X which may be proved in the same way as Theorem 8.

Theorem 11. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and let A be Hermitian. Then from [15] there exists a Hermitian matrix X satisfying $r(A - BXB^*) = k_1$, if and only if*

$$r\begin{bmatrix}A & B\end{bmatrix} \geq k_1 \geq 2r\begin{bmatrix}A & B\end{bmatrix} - r\begin{bmatrix}A & B\\ B^* & 0\end{bmatrix}. \tag{41}$$

If k_1 satisfies the above inequalities, then

$$\min_{r(A-BXB^*)=k_1} r(X) = \max\left\{r(A)-k_1, k_1-r(A)\right\},$$
$$\max_{r(A-BXB^*)=k_1} r(X) = \min\left\{n, k_1+2n+r(A)\right.$$
$$\left.-4r(B)+2r\begin{bmatrix}A & B\end{bmatrix}\right\}. \tag{42}$$

4. Solutions to Problem 2

In this section, we study Problem 2 proposed in Section 1.

Let

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad B = C = I_2. \tag{43}$$

It is obvious that

$$2 \geq r(A - BXC) \geq 0 \tag{44}$$

and there do not exist the least squares solutions of $\|BXC - A\|_F = \min$ subject to $r(A - BXC) = 2$. Therefore, we should study the range of k, such that there exists a least squares solution X of $\|BXC - A\|_F = \min$ subject to $r(A - BXC) = k$.

Theorem 12. *Let A, B, and C be as in Theorem 7. Then there exists a least squares solution of $\|A - BXC\| = \min$ under the rank constraint $r(A - BXC) = k$, if and only if*

$$r\begin{bmatrix}A & AC^* & B\\ B^*A & 0 & 0\\ C & 0 & 0\end{bmatrix} - r(B) - r(C) \geq k$$
$$\geq r\begin{bmatrix}A\\C\end{bmatrix} + r\begin{bmatrix}A & B\end{bmatrix} - r\begin{bmatrix}A & B\\ C & 0\end{bmatrix}. \tag{45}$$

Proof. Let E, F, J, K, Σ_i, U_i, and V_i $(i = 1,2,3,4,5)$ be as in Theorem 7, and let $U_4^*(E\Sigma_3^{-1}J)V_5$ be partitioned in the form

$$U_4^*\left(E\Sigma_3^{-1}J\right)V_5 = \begin{bmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{bmatrix}, \tag{46}$$

where $D_{22} \in \mathbb{C}^{(r(B)-r(F))\times(r(C)-r(K))}$, $D_{21} \in \mathbb{C}^{(r(B)-r(F))\times r(K)}$, $D_{12} \in \mathbb{C}^{r(F)\times(r(C)-r(K))}$, and $D_{11} \in \mathbb{C}^{r(F)\times r(K)}$. Let D_{22} have the singular value decomposition

$$D_{22} = U_7\begin{bmatrix}\Sigma_7 & 0\\ 0 & 0\end{bmatrix}V_7^*, \tag{47}$$

where $\Sigma_7 = \mathrm{diag}\{\delta_1, \ldots, \delta_{r(D_{22})}\}, \delta_1 \geq \cdots \geq \delta_{r(D_{22})} > 0$, and U_7 and V_7 are unitary matrices of appropriate sizes.

From the partitioned form for $U_4^*(A_{11}-E\Sigma_3^{-1}J)V_5$ in (21),

$$U_4^* A_{11} V_5 = U_4^* \left(E\Sigma_3^{-1} J\right) V_5 + \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}. \quad (48)$$

Since the Frobenius norm is invariant, we have the following identities by substituting (22) into $\|A - BXC\|$ and applying (46) and (48):

$$\begin{aligned} &\min_{r(A-BXC)=k} \|A - BXC\|^2 \\ &= \min_{Y_{11},Y_{12},Y_{21},r(T)=k-l} \left\| \begin{bmatrix} A_{11} - U_4 \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & \widehat{A}_{22} + T \end{bmatrix} V_5^* & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\|^2 \\ &\quad + \min_{Y_{11},Y_{12},Y_{21},r(T)=k-l} \left\| U_4^* A_{11} V_5 - \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & \widehat{A}_{22} + T \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\|^2 \\ &\quad + \min_{Y_{11},Y_{12},Y_{21},r(T)=k-l} \left\| U_4^* \left(E\Sigma_3^{-1} J\right) V_5 - \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & T \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\|^2 + \min_{r(T)=k-l} \left\| D_{22} - T \right\|^2. \end{aligned} \quad (49)$$

Therefore, there exists a least squares solution X satisfying $\|A - BXC\|^2 = \min$ subject to $r(A - BXC) = k$ if and only if $r(D_{22}) \geq r(T) \geq 0$, that is, if and only if

$$r\left(D_{22}\right) + l \geq k \geq l. \quad (50)$$

From the partitioned form for $U_4^*(E\Sigma_3^{-1}J)V_5$ in (46) and the decompositions of F and K in (21), we have $r(D_{22}) = r(E_F E\Sigma_3^{-1} J F_K)$. Applying (9) gives

$$r\left(D_{22}\right) = r \begin{bmatrix} E\Sigma_3^{-1} J & F \\ K & 0 \end{bmatrix} - r(F) - r(K). \quad (51)$$

The identity

$$r\left(A - BB^{\dagger} AC^{\dagger} C\right) = r \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (52)$$

follows from applying the decompositions of B and C in (19) and the partitioned form for $U_1^* A V_2$ in (20). Substituting the decomposition of A_{22} in (19) into $\left[\begin{smallmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$, applying the partitioned forms for $U_3^* A_{21}$ and $A_{12} V_3$ in (21), we conclude that

$$r\left(A - BB^{\dagger} AC^{\dagger} C\right) = r \begin{bmatrix} E\Sigma_3^{-1} J & F \\ K & 0 \end{bmatrix} + r\left(A_{22}\right). \quad (53)$$

Hence,

$$r\left(D_{22}\right) = r\left(A - BB^{\dagger} AC^{\dagger} C\right) - r\left(A_{22}\right) - r(F) - r(K). \quad (54)$$

It follows from applying (12) that

$$r\left(A - BB^{\dagger} AC^{\dagger} C\right) = r \begin{bmatrix} A & AC^* & B \\ B^* A & 0 & 0 \\ C & 0 & 0 \end{bmatrix} - r(B) - r(C). \quad (55)$$

Substituting (24)–(26) and (55) into (54), we have

$$\begin{aligned} r\left(D_{22}\right) &= r \begin{bmatrix} A & AC^* & B \\ B^* A & 0 & 0 \\ C & 0 & 0 \end{bmatrix} + r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \\ &\quad - r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \end{bmatrix} - r(B) - r(C). \end{aligned} \quad (56)$$

Therefor, the inequalities in (45) follow from substituting (17) and (56) into (50). □

Theorem 13. *Let A, B, C, E, F, J, K, Σ_i, U_i, and V_i $(i = 1, 2, 3, 4, 5)$ be as in Theorem 7, and let $U_4^*(E\Sigma_3^{-1}J)V_5$ and D_{22} be partitioned as in (46) and (47), respectively. If k satisfies (45), then any least squares solution $X \in \{X \mid r(A-BXC) = k\}$ satisfying $\|A - BXC\| = \min$ has the form*

$$\begin{aligned} X &= V_1 \begin{bmatrix} \Sigma_1^{-1} U_4 & 0 \\ 0 & I_{p-r(B)} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & \widehat{A}_{22} + T \end{bmatrix} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\ &\quad \times \begin{bmatrix} V_5^* \Sigma_2^{-1} & 0 \\ 0 & I_{q-r(C)} \end{bmatrix} U_2^*, \end{aligned} \quad (57)$$

where $X_{12} \in \mathbb{C}^{r(B)\times(q-r(C))}$, $X_{21} \in \mathbb{C}^{(p-r(B))\times r(C)}$, $X_{22} \in \mathbb{C}^{(p-r(B))\times(q-r(C))}$, $Y_{11} \in \mathbb{C}^{r(F)\times r(K)}$, $Y_{12} \in \mathbb{C}^{r(F)\times(r(C)-r(K))}$, $Y_{21} \in \mathbb{C}^{(r(B)-r(F))\times r(K)}$, and $T \in \mathbb{C}^{(r(B)-r(F))\times(r(C)-r(K))}$ are arbitrary matrices, such that $r(T) = k - l$.

(1) *When $\delta_{k-l} > \delta_{k-l+1}$,*

$$T = U_7 \operatorname{diag}\left\{\delta_1, \ldots, \delta_{k-l}, 0, \ldots, 0\right\} V_7^*; \quad (58)$$

(2) *when $\widehat{p} < k - l < \widehat{q} \leq r$ and $\lambda_{\widehat{p}} > \lambda_{\widehat{p}+1} = \cdots = \delta_{\widehat{q}} > \delta_{\widehat{q}+1}$,*

$$T = U_7 \operatorname{diag}\left\{\delta_1, \ldots, \delta_{\widehat{p}}, \delta_{k-l} QQ^*, 0, \ldots, 0\right\} V_7^*, \quad (59)$$

where Q is an arbitrary matrix satisfying $Q \in \mathbb{C}^{(\widehat{q}-\widehat{p})\times(k-l-\widehat{p})}$ and $Q^* Q = I_{k-l-\widehat{p}}$.

Proof. When k satisfies the inequalities in (45), then, by applying Lemma 6 and (48), we obtain the desired form of T in (58) and (59), respectively. □

5. Conclusions

In this paper, we have discussed the solutions to Problem 1

$$\mathbb{S}_{k_1} = \left\{X \mid X \in \mathbb{C}^{p\times q}, r(A - BXC) = k_1\right\} \quad (60)$$

and the solutions to Problem 2

$$\widehat{\mathbb{S}} = \left\{ X \mid X \in \mathbb{S}_k, \|A - BXC\|^2 = \min \right\}. \quad (61)$$

We first derived the expression of solutions to $r(A - BXC) = k_1$ when Problem 1 is solvable. Based on these results, we obtained the extremal ranks of the expression of solutions to Problem 1, the solvability conditions of Problem 2, and the expression of least squares solutions when Problem 2 is solvable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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