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Research Article

Rapidly Converging Series for $\zeta(2n+1)$ from Fourier Series

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Ever since Euler first evaluated $\zeta(2)$ and $\zeta(2m)$, numerous interesting solutions of the problem of evaluating the $\zeta(2m)$ $(m \in \mathbb{N})$ have appeared in the mathematical literature. Until now no simple formula analogous to the evaluation of $\zeta(2m)$ $(m \in \mathbb{N})$ is known for $\zeta(2m+1)$ $(m \in \mathbb{N})$ or even for any special case such as $\zeta(3)$. Instead, various rapidly converging series for $\zeta(2m+1)$ have been developed by many authors. Here, using Fourier series, we aim mainly at presenting a recurrence formula for rapidly converging series for $\zeta(2m+1)$. In addition, using Fourier series and recalling some indefinite integral formulas, we also give recurrence formulas for evaluations of $\beta(2m+1)$ and $\zeta(2m)$ $(m \in \mathbb{N})$, which have been treated in earlier works.

1. Introduction and Preliminaries

The Riemann zeta function $\zeta(s)$ is defined by (see, e.g., [1, p. 164])

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \Re(s) > 1. \tag{1}$$

The Riemann zeta function $\zeta(s)$ in (1) plays a central role in the applications of complex analysis to number theory. The number-theoretic properties of $\zeta(s)$ are exhibited by the following result known as *Euler's formula*, which gives a relationship between the set of primes and the set of positive integers:

$$\zeta(s) = \prod_{p} \left(1 - p^{-s}\right)^{-1} \quad \text{for } \Re(s) > 1, \tag{2}$$

where the product is taken over all primes.

The solution of the so-called *Basler problem* (cf., e.g., [2], [3, p. xxii], [4, p.66], [5, pp. 197-198], and [6, p. 364])

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{3}$$

was first found in 1735 by Euler (1707–1783) [7], although Jakob Bernoulli (1654–1705) and Johann Bernoulli (1667–1748) did their utmost to sum the series in (3). The former of these Bernoulli brothers did not live to see the solution of the problem, and the solution became known to the latter soon after Euler found it (see, for details, Knopp [8, p.238]). Five years later in 1740, Euler (see [9]; see also [10, pp. 137–153]) succeeded in evaluating all of $\zeta(2n)$ $(n \in \mathbb{N} := \{1, 2, 3, \ldots\})$:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2\,(2n)!} B_{2n} \quad \text{for } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{4}$$

where B_n $(n \in \mathbb{N}_0)$ are the nth Bernoulli numbers defined by the following generating function (see, e.g., [1, p. 81]):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad \text{for } |z| < 2\pi. \tag{5}$$

The following recursion formula

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k \quad (n \in \mathbb{N} \setminus \{1\}), \; B_0 = 1, \tag{6}$$

can be used for computing Bernoulli numbers. Ever since Euler first evaluated $\zeta(2)$ and $\zeta(2n)$, numerous interesting solutions of the problem of evaluating the $\zeta(2n)$ $(n \in \mathbb{N})$ have appeared in the mathematical literature. Even though there were certain earlier works which gave a rather long list of papers and books together with some useful comments on the methods of evaluation of $\zeta(2)$ and $\zeta(2n)$ (see, e.g., [5, 11, 12]),

the reader may be referred to the very recent work [13] which contains an extensive literature of as many as more than 70 papers.

We may recall here a known recursion formula for $\zeta(2n)$ (see, e.g., [1, p. 167], [1, Section 4.1], and [14, Theorem I]):

$$\zeta(2n) = \frac{2}{2n+1}\sum_{k=1}^{n-1}\zeta(2k)\,\zeta(2n-2k) \quad \text{for } n \in \mathbb{N}\setminus\{1\}, \tag{7}$$

which can also be used to evaluate $\zeta(2n)$ $(n \in \mathbb{N}\setminus\{1\})$.

The eta function or the alternating zeta function $\eta(s)$ is defined by

$$\eta(s) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \quad \text{for } \Re(s) > 0. \tag{8}$$

Then it is easy to have the following relation between $\zeta(s)$ and $\eta(s)$:

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s) \quad \text{for } \Re(s) > 1. \tag{9}$$

The β-function is defined by

$$\beta(s) := \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^s} \quad \text{for } \Re(s) > 0. \tag{10}$$

Remark 1. The β-function (see [15, p. 329]) has been denoted in several ways, such as $\xi(s)$ (see [14]), $\mathscr{L}(s)$ (see [15, p. 329]; see also [1, p. 404]), $L(s)$ (see [16, p. 125] and [17]), $\psi(s)$ (see [18]), $\mathrm{Ti}_n(1)$ (see [19, p. 190]; see also [15, p. 332]), and $S(s, 1/4)$ (see [20, p. 375]). Williams [14, p. 22, Theorem II] gave an interesting companion of the result (7) in the following form:

$$\sum_{k=1}^{n}\beta(2k-1)\,\beta(2n-2k+1) = \left(n-\frac{1}{2}\right)\left(1-2^{-2n}\right)\zeta(2n), \tag{11}$$

which appears erroneously in Hansen [21, p. 357, Entry (54.7.1)]. Since $\beta(1)$ is the well-known Gregory series for $\pi/4$ (with $\beta(2)$ being the familiar Catalan constant G), by setting $n = 1$ in (11), we immediately obtain

$$\zeta(2) = \frac{8}{3}\{\beta(1)\}^2 = \frac{\pi^2}{6}. \tag{12}$$

Until now no simple formula analogous to (4) is known for $\zeta(2m+1)$ or even for any special case such as $\zeta(3)$. It is not even known whether $\zeta(2m+1)$ is rational or irrational, except that the irrationality of $\zeta(3)$ was proved by Apéry [22]. But it is known that there are infinitely many $\zeta(2n+1)$ which are irrational (see [23, 24]). On the other hand, various rapidly converging series for $\zeta(2m+1)$ $(m \in \mathbb{N})$ have been developed by many authors (see, e.g., [25, 26]; see also [1, Chapter 4] and the references cited in the chapter). Very recently, Choi and Chen [27] gave a double inequality approximating $\zeta(2m+1)$ $(m \in \mathbb{N})$ by a more rapidly convergent series. Here, using Fourier series, we aim mainly at presenting a recurrence formula for rapidly converging series for $\zeta(2m+1)$ $(m \in \mathbb{N})$. In addition, using Fourier series, we also give recurrence formulas for evaluations of $\beta(2m+1)$ and $\zeta(2m)$ $(m \in \mathbb{N})$.

2. Evaluation of $\beta(2m+1)$ from Fourier Series

Euler proved (see, e.g., [18, p. 1071], [16, p. 125], [15, p. 330], [17, p. 372], and [19, p. 196]) that

$$\beta(2m+1) = (-1)^m\frac{E_{2m}}{2^{2m+2}(2m)!}\pi^{2m+1} \quad \text{for } m \in \mathbb{N}_0, \tag{13}$$

where E_{2m} are called Euler numbers (see, e.g., [1, pp. 86–89]) defined by

$$\frac{2e^z}{e^{2z}+1} = \operatorname{sech} z = \sum_{m=0}^{\infty}E_m\frac{z^m}{m!} \quad \text{for } |z| < \frac{\pi}{2}. \tag{14}$$

Here we present a recurrence formula for evaluation of the $\beta(2m+1)$ $(m \in \mathbb{N})$ given in (10) by using Fourier series. To do this, we choose the odd 2π-periodic function g_m given by

$$g_m(x) := \pi^2 x^{2m-1} - x^{2m+1} \quad \text{for } -\pi \le x \le \pi,\ m \in \mathbb{N}, \tag{15}$$

which is seen to be continuous and piecewise differentiable on the set of real numbers $\mathbb{R}$. Now we can get the following Fourier series expansion of $g_m(x)$:

$$x^{2m-1}\left(\pi^2 - x^2\right) = \sum_{n=1}^{\infty}b_n\sin(nx) \quad \text{for } x \in \mathbb{R}, \tag{16}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\pi^2 x^{2m-1} - x^{2m+1}\right)\sin(nx)\,dx \quad \text{for } n \in \mathbb{N}. \tag{17}$$

Evaluation of b_n in (17). We use a known indefinite integral formula (see, e.g., [28, p. 211, Entry 2.633(1)]) for $m \in \mathbb{N}_0$

$$\int x^m\sin(ax)\,dx = -\sum_{k=0}^{m}k!\binom{m}{k}\frac{x^{m-k}}{a^{k+1}}\cos\left(ax+\frac{1}{2}k\pi\right) \tag{18}$$

to get the following two involved integral formulas:

$$\begin{aligned}
&\pi^2\int_0^{\pi}x^{2m-1}\sin(nx)\,dx \\
&\quad = -\sum_{k=0}^{m-1}(-1)^{n+k}(2k)!\binom{2m-1}{2k}\frac{\pi^{2m+1-2k}}{n^{2k+1}} \quad (m \in \mathbb{N}), \\
&\int_0^{\pi}x^{2m+1}\sin(nx)\,dx \\
&\quad = -\sum_{k=0}^{m}(-1)^{n+k}(2k)!\binom{2m+1}{2k}\frac{\pi^{2m+1-2k}}{n^{2k+1}} \quad (m \in \mathbb{N}_0).
\end{aligned} \tag{19}$$

Now setting $x = \pi/2$ in (16) and using the evaluation of b_n with (19) we get

$$\begin{aligned} b_n &= 2(-1)^{n+m}\frac{(2m+1)!}{n^{2m+1}} \\ &\quad + 2\sum_{k=0}^{m-1}(-1)^{n+k}(2k)!\cdot\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\} \\ &\qquad \times\frac{\pi^{2(m-k)}}{n^{2k+1}} \\ &= 2\sum_{k=1}^{m}(-1)^{n+k}(2k)!\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\} \\ &\qquad \times\frac{\pi^{2(m-k)}}{n^{2k+1}} \quad (n,m\in\mathbb{N}), \end{aligned} \tag{20}$$

and after some simplifications we finally obtain the result stated in Theorem 2.

Theorem 2. *The following recurrence formula for evaluation of $\beta(2m+1)$ $(m\in\mathbb{N})$ holds. For $m\in\mathbb{N}$,*

$$\begin{aligned} &(-1)^m(2m+1)!\beta(2m+1) \\ &\quad = -\frac{3}{2^{2m+2}}\pi^{2m+1} \\ &\qquad + \sum_{k=1}^{m-1}(-1)^{k+1}(2k)!\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\} \\ &\qquad\quad \times\pi^{2m-2k}\beta(2k+1), \end{aligned} \tag{21}$$

where the empty sum is (as usual) understood to be nil throughout this paper.

For small values of m, we have

$$\begin{aligned} \beta(3) &= \frac{\pi^3}{32}, \qquad \beta(5) = \frac{5}{1536}\pi^5, \\ \beta(7) &= \frac{61}{184320}\pi^7, \qquad \beta(9) = \frac{277}{8257536}\pi^9, \ldots. \end{aligned} \tag{22}$$

Remark 3. In order to get the evaluation of $\beta(3)$ and a rapidly converging series representation of $\zeta(3)$ from Fourier series, instead of using the periodic version of x^3 which is not continuous, Scheufens [29] made a good choice of the odd 2π-periodic function f given by

$$f(x) = \pi^2 x - x^3 \quad \text{for } -\pi\le x\le\pi, \tag{23}$$

which is now continuous and piecewise differentiable. Here we use the function $g_m(x)$ in (15) which is a natural modification of the Scheufens chosen function (23) to give the results in this and the next sections.

Chen [17] used the even 2π-periodic function $f(x) = x^{2k}$ on $[-\pi,\pi]$ to get a recurrence formula for $\beta(2m+1)$. Yue and Williams [20] used residue calculus to derive a recurrence formula for $\beta(2m+1)$. Butzer and Hauss [15] presented diverse single and multiple integral representations of $\beta(m)$.

3. A Recurrence Formula for a Rapidly Converging Series for $\zeta(2m+1)$

We begin by recalling some elementary known or easily derivable formulas for the binomial coefficients as in the lemma given below.

Lemma 4. *Each of the following formulas holds:*

$$(1-x)^n = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j \quad (n\in\mathbb{N}_0; x\in\mathbb{C}), \tag{24}$$

where $\mathbb{C}$ denotes the set of complex numbers,

$$nx(1-x)^{n-1} = \sum_{j=1}^{n}(-1)^{j+1}j\binom{n}{j}x^j \quad (n\in\mathbb{N}; x\in\mathbb{C}), \tag{25}$$

$$\begin{aligned} &nx(1-x)^{n-1} - n(n-1)x^2(1-x)^{n-2} \\ &\quad = \sum_{j=1}^{n}(-1)^{j+1}j^2\binom{n}{j}x^j \quad (n\in\mathbb{N}; x\in\mathbb{C}), \end{aligned} \tag{26}$$

$$\binom{x}{n} = \frac{x}{n}\binom{x-1}{n-1} \Longleftrightarrow \frac{1}{x}\binom{x}{n} = \frac{1}{n}\binom{x-1}{n-1}, \tag{27}$$

where $\binom{x}{n}$ is defined, for $x\in\mathbb{C}$, by

$$\binom{x}{n} = \begin{cases} \dfrac{x(x-1)\cdots(x-n+1)}{n!} & (n\in\mathbb{N}), \\ 1 & (n=0). \end{cases} \tag{28}$$

Lemma 5. *For each $m\in\mathbb{N}$ and $x\in\mathbb{C}$, one has $\mathscr{L}_m(x) = \mathscr{R}_m(x)$, where, for convenience,*

$$\begin{aligned} \mathscr{L}_m(x) &:= x^{2m+1}\sum_{n=0}^{m}\frac{2}{2n+1}\left(\frac{1}{x}\right)^{2n+1} \\ &\quad - x^{2m-1}\sum_{n=0}^{m-1}\frac{2}{2n+1}\left(\frac{1}{x}\right)^{2n+1}, \\ \mathscr{R}_m(x) &:= x^{2m-1} \\ &\quad \times\sum_{j=1}^{2m-1}\frac{(-1)^j}{j}\binom{2m-1}{j}\left\{(x+1)^j-(x-1)^j\right\} \\ &\quad \times\left(\frac{1}{x}\right)^j - x^{2m+1} \\ &\quad \times\sum_{j=1}^{2m+1}\frac{(-1)^j}{j}\binom{2m+1}{j}\left\{(x+1)^j-(x-1)^j\right\} \\ &\quad \times\left(\frac{1}{x}\right)^j. \end{aligned} \tag{29}$$

Proof. We proceed to prove by induction on $m \in \mathbb{N}$. We can give a direct evaluation to check the first three ones:

$$\begin{aligned}
\mathscr{L}_1(x) &= -\frac{4}{3} + 2x^2 = \mathscr{R}_1(x), \\
\mathscr{L}_2(x) &= -\frac{4}{15} - \frac{4}{3}x^2 + 2x^4 = \mathscr{R}_2(x), \\
\mathscr{L}_3(x) &= -\frac{4}{35} - \frac{4}{15}x^2 - \frac{4}{3}x^4 + 2x^6 = \mathscr{R}_3(x).
\end{aligned} \tag{30}$$

Now assume that $\mathscr{L}_m(x) = \mathscr{R}_m(x)$ for some $m \in \mathbb{N}$. Then we have to show that

$$\mathscr{L}_{m+1}(x) = \mathscr{R}_{m+1}(x). \tag{31}$$

By using the induction hypothesis, we find

$$\mathscr{L}_{m+1}(x) = -\frac{4}{(2m+1)(2m+3)} + x^2 \mathscr{R}_m(x). \tag{32}$$

In view of (31) and (32), it is enough to show that

$$\mathscr{P}_m(x) := x^2 \mathscr{R}_m(x) - \mathscr{R}_{m+1}(x) = \frac{4}{(2m+1)(2m+3)} \quad (m \in \mathbb{N}), \tag{33}$$

where, for convenience, $\mathscr{P}_m(x) = \mathscr{P}_{m,1}(x) + \mathscr{P}_{m,2}(x)$ with

$$\begin{aligned}
\mathscr{P}_{m,1}(x) &:= \sum_{j=1}^{2m+3} \frac{(-1)^j}{j} \binom{2m+3}{j} \left\{(x+1)^j - (x-1)^j\right\} \\
&\quad \times x^{2m+3-j} \\
&\quad - \sum_{j=1}^{2m+1} \frac{(-1)^j}{j} \binom{2m+1}{j} \left\{(x+1)^j - (x-1)^j\right\} \\
&\quad \times x^{2m+3-j}, \\
\mathscr{P}_{m,2}(x) &:= \sum_{j=1}^{2m-1} \frac{(-1)^j}{j} \binom{2m-1}{j} \left\{(x+1)^j - (x-1)^j\right\} \\
&\quad \times x^{2m+1-j} \\
&\quad - \sum_{j=1}^{2m+1} \frac{(-1)^j}{j} \binom{2m+1}{j} \left\{(x+1)^j - (x-1)^j\right\} \\
&\quad \times x^{2m+1-j}.
\end{aligned} \tag{34}$$

We first try to evaluate $\mathscr{P}_{m,1}(x)$. We find

$$\begin{aligned}
&\mathscr{P}_{m,1}(x) \\
&= \sum_{j=1}^{2m+3} \frac{(-1)^j}{j} \left\{\binom{2m+3}{j} - \binom{2m+1}{j}\right\} \\
&\qquad \cdot \left\{(x+1)^j - (x-1)^j\right\} x^{2m+3-j} \\
&= \sum_{j=1}^{2m+3} (-1)^j \frac{(4m+5) - j}{(2m+3-j)(2m+2-j)} \binom{2m+1}{j} \\
&\qquad \cdot \left\{(x+1)^j - (x-1)^j\right\} x^{2m+3-j}.
\end{aligned} \tag{35}$$

Let us consider the following partial fraction:

$$\begin{aligned}
\frac{(4m+5) - j}{(2m+3-j)(2m+2-j)} &= \frac{4m+5}{2m+2-j} - \frac{4m+5}{2m+3-j} \\
&\quad - \frac{j}{2m+2-j} + \frac{j}{2m+3-j}.
\end{aligned} \tag{36}$$

Using (27), we have

$$\begin{aligned}
\frac{1}{2m+2-j}\binom{2m+1}{j} &= \frac{1}{2m+2}\binom{2m+2}{2m+2-j} \\
&= \frac{1}{2m+2}\binom{2m+2}{j}, \\
\frac{1}{2m+3-j}\binom{2m+1}{j} &= \frac{2m+2-j}{(2m+2)(2m+3)}\binom{2m+3}{j}.
\end{aligned} \tag{37}$$

Applying the last two identities to the last expression of $\mathscr{P}_{m,1}(x)$, we can separate $\mathscr{P}_{m,1}(x)$ into six polynomials as follows:

$$\mathscr{P}_{m,1}(x) = \sum_{k=1}^{6} \alpha_{m,k}(x), \tag{38}$$

where

$$\begin{aligned}
\alpha_{m,1}(x) &:= \frac{4m+5}{2m+2} \sum_{j=1}^{2m+2} (-1)^j \binom{2m+2}{j} \\
&\qquad \times \left\{(x+1)^j - (x-1)^j\right\} x^{2m+3-j}, \\
\alpha_{m,2}(x) &:= \frac{4m+5}{2m+3} \sum_{j=1}^{2m+3} (-1)^{j+1} \binom{2m+3}{j} \\
&\qquad \times \left\{(x+1)^j - (x-1)^j\right\} x^{2m+3-j},
\end{aligned}$$

$$
\begin{aligned}
&\alpha_{m,3}(x)\\
&:= -\frac{4m+5}{(2m+2)(2m+3)}\sum_{j=1}^{2m+3}(-1)^{j+1}j\binom{2m+3}{j}\\
&\qquad\cdot\left\{(x+1)^j-(x-1)^j\right\}\\
&\qquad\times x^{2m+3-j},\\
&\alpha_{m,4}(x)\\
&:= \frac{1}{2m+2}\sum_{j=1}^{2m+2}(-1)^{j}j\binom{2m+2}{j}\\
&\qquad\times\left\{(x+1)^j-(x-1)^j\right\}x^{2m+3-j},\\
&\alpha_{m,5}(x)\\
&:= \frac{1}{2m+3}\sum_{j=1}^{2m+3}(-1)^{j}j\binom{2m+3}{j}\\
&\qquad\times\left\{(x+1)^j-(x-1)^j\right\}x^{2m+3-j},\\
&\alpha_{m,6}(x)\\
&:= \frac{1}{(2m+2)(2m+3)}\sum_{j=1}^{2m+3}(-1)^{j+1}j^2\binom{2m+3}{j}\\
&\qquad\cdot\left\{(x+1)^j-(x-1)^j\right\}\\
&\qquad\times x^{2m+3-j}.
\end{aligned}
\tag{39}
$$

Choosing to use some identities in Lemma 4, we can evaluate $\alpha_{m,k}(x)$ as follows:

$$
\begin{aligned}
&\alpha_{m,1}(x)=0, \qquad \alpha_{m,2}(x)=4-\frac{2}{2m+3},\\
&\alpha_{m,3}(x)=-4-\frac{1}{m+1}, \qquad \alpha_{m,4}(x)=-2x^2,\\
&\alpha_{m,5}(x)=-2, \qquad \alpha_{m,6}(x)=2+\frac{1}{m+1}+2x^2.
\end{aligned}
\tag{40}
$$

We thus have

$$
\mathscr{P}_{m,1}(x)=\sum_{k=1}^{6}\alpha_{m,k}(x)=-\frac{2}{2m+3}. \tag{41}
$$

Similarly we find

$$
\mathscr{P}_{m,2}(x)=\sum_{k=1}^{6}\beta_{m,k}(x), \tag{42}
$$

where

$$
\begin{aligned}
&\beta_{m,1}(x)\\
&=\frac{1}{2m}\sum_{j=1}^{2m}(-1)^{j}j\binom{2m}{j}\left\{(x+1)^j-(x-1)^j\right\}\\
&\qquad\times x^{2m+1-j},\\
&\beta_{m,2}(x)\\
&=\frac{1}{2m+1}\sum_{j=1}^{2m+1}(-1)^{j+1}j\binom{2m+1}{j}\\
&\qquad\times\left\{(x+1)^j-(x-1)^j\right\}x^{2m+1-j},\\
&\beta_{m,3}(x)\\
&=\frac{1}{2m(2m+1)}\sum_{j=1}^{2m+1}(-1)^{j}j^2\binom{2m+1}{j}\\
&\qquad\cdot\left\{(x+1)^j-(x-1)^j\right\}x^{2m+1-j},\\
&\beta_{m,4}(x)\\
&=-\frac{4m+1}{2m}\sum_{j=1}^{2m}(-1)^{j}\binom{2m}{j}\\
&\qquad\times\left\{(x+1)^j-(x-1)^j\right\}x^{2m+1-j},\\
&\beta_{m,5}(x)\\
&=\frac{4m+1}{2m+1}\sum_{j=1}^{2m+1}(-1)^{j}\binom{2m+1}{j}\\
&\qquad\times\left\{(x+1)^j-(x-1)^j\right\}x^{2m+1-j},\\
&\beta_{m,6}(x)\\
&=\frac{4m+1}{2m(2m+1)}\sum_{j=1}^{2m+1}(-1)^{j+1}j\binom{2m+1}{j}\\
&\qquad\cdot\left\{(x+1)^j-(x-1)^j\right\}x^{2m+1-j}.
\end{aligned}
\tag{43}
$$

Similarly as in evaluating $\alpha_{m,k}(x)$, we have

$$
\begin{aligned}
&\beta_{m,1}(x)=2x^2, \qquad \beta_{m,2}(x)=2,\\
&\beta_{m,3}(x)=-2-\frac{1}{m}-2x^2, \qquad \beta_{m,4}(x)=0,\\
&\alpha_{m,5}(x)=-4+\frac{2}{2m+1}, \qquad \beta_{m,6}(x)=4+\frac{1}{m}.
\end{aligned}
\tag{44}
$$

We thus obtain

$$
\mathscr{P}_{m,2}(x)=\sum_{k=1}^{6}\beta_{m,k}(x)=\frac{2}{2m+1}. \tag{45}
$$

Now it is easy to see that

$$\mathscr{P}_m(x) = \mathscr{P}_{m,1}(x) + \mathscr{P}_{m,2}(x) = \frac{4}{(2m+1)(2m+3)}. \quad (46)$$

This completes the proof of (33) and so does Lemma 5. □

Theorem 6. *One has a recurrence formula for a rapidly converging series for $\zeta(2m+1)$. For $m \in \mathbb{N}$,*

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{k+1}\pi^{2(m-k)}(2k)!\cdot\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\}&\\ \times\left(1-2^{-2k}\right)\zeta(2k+1)&\\ = \frac{2\pi^{2m}}{4m^2-1} - 4\pi^{2m}&\\ \times\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n+2m+1)(2n+2m-1)4^n}.&\end{aligned} \quad (47)$$

Proof. We find from (15), (16), and (20) that, for $x \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$\begin{aligned}g_m(x)&\\ = 2\sum_{n=1}^{\infty}\left[\sum_{k=1}^{m}(-1)^k\pi^{2(m-k)}(2k)!\right.&\\ \left.\cdot\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\}\frac{(-1)^n}{n^{2k+1}}\right]\sin(nx).&\end{aligned} \quad (48)$$

Here we choose a method where the sine function disappears by using the following well-known result:

$$\int_0^{\infty}\frac{\sin(nx)}{x}dx = \frac{\pi}{2} \quad (n \in \mathbb{N}). \quad (49)$$

The series for $g_m(x)/x$ obtained by dividing (48) by x converges uniformly on $\mathbb{R}$. Indeed, the value of $g_m(x)/x$ at $x = 0$ can be considered as $\lim_{x\to 0}\sin(nx)/x = n\cdot\lim_{x\to 0}\sin(nx)/(nx) = n$ and then one may use the Weierstrass M-test. We therefore apply termwise integration to the resulting series and use (49) to get

$$\begin{aligned}\int_0^{\infty}\frac{g_m(x)}{x}dx&\\ = 2\sum_{n=1}^{\infty}\left[\sum_{k=1}^{m}(-1)^k\pi^{2(m-k)}(2k)!\right.&\\ \left.\cdot\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\}\frac{(-1)^n}{n^{2k+1}}\right]&\\ \times\int_0^{\infty}\frac{\sin(nx)}{x}&\\ = \pi\sum_{k=1}^{m}(-1)^{k+1}\pi^{2(m-k)}(2k)!&\\ \times\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\}\eta(2k+1),&\end{aligned} \quad (50)$$

where η is the eta function given in (8). Now using the relationship (9) with $s = 2k+1$ gives

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{k+1}\pi^{2(m-k)}(2k)!&\\ \times\left\{\binom{2m+1}{2k}-\binom{2m-1}{2k}\right\}\cdot\left(1-2^{-2k}\right)\zeta(2k+1)&\\ = \frac{1}{\pi}\int_0^{\infty}\frac{g_m(x)}{x}dx.&\end{aligned} \quad (51)$$

Since g_m is 2π-periodic, we have

$$\begin{aligned}\int_0^{\infty}\frac{g_m(x)}{x}dx &= \int_0^{\pi}\left(\pi^2x^{2m-2}-x^{2m}\right) + \sum_{k=1}^{\infty}\mathscr{I}_k(m)\\ &= \frac{2\pi^{2m+1}}{4m^2-1} + \sum_{k=1}^{\infty}\mathscr{I}_k(m),\end{aligned} \quad (52)$$

where, for convenience,

$$\begin{aligned}\mathscr{I}_k(m)&\\ := \int_{(2k-1)\pi}^{(2k+1)\pi}\frac{\pi^2(x-2k\pi)^{2m-1}-(x-2k\pi)^{2m+1}}{x}dx.&\end{aligned} \quad (53)$$

By using the binomial formula and integrating the resulting identity, we get

$$\begin{aligned}\mathscr{I}_k(m) = \pi^{2m+1}\left\{(2k)^{2m+1}-(2k)^{2m-1}\right\}\ln\frac{1+1/2k}{1-1/2k}&\\ + \pi^{2m+1}\mathscr{C}_k(m),&\end{aligned} \quad (54)$$

where

$$\begin{aligned}\mathscr{C}_k(m)&\\ := (2k)^{2m+1}\sum_{j=1}^{2m+1}\frac{(-1)^j}{j}\binom{2m+1}{j}&\\ \times\left\{(2k+1)^j-(2k-1)^j\right\}\left(\frac{1}{2k}\right)^j&\\ - (2k)^{2m-1}\sum_{j=1}^{2m-1}\frac{(-1)^j}{j}\binom{2m-1}{j}&\\ \times\left\{(2k+1)^j-(2k-1)^j\right\}\left(\frac{1}{2k}\right)^j.&\end{aligned} \quad (55)$$

Applying the following Maclaurin series

$$\ln\frac{1+x}{1-x}=\sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}\quad(-1<x<1)\tag{56}$$

to $\ln(1+1/2k)/(1-1/2k)$, in view of Lemma 5, we can show that

$$\begin{aligned}\mathscr{C}_k(m)&=(2k)^{2m-1}\sum_{n=0}^{m-1}\frac{2}{2n+1}\left(\frac{1}{2k}\right)^{2n+1}\\&\quad-(2k)^{2m+1}\sum_{n=0}^{m}\frac{2}{2n+1}\left(\frac{1}{2k}\right)^{2n+1}.\end{aligned}\tag{57}$$

We therefore have

$$\begin{aligned}&\mathscr{I}_k(m)\\&=2\pi^{2m+1}\left[(2k)^{2m+1}\sum_{n=m+1}^{\infty}\frac{1}{2n+1}\left(\frac{1}{2k}\right)^{2n+1}\right.\\&\qquad\left.-(2k)^{2m-1}\sum_{n=m}^{\infty}\frac{1}{2n+1}\left(\frac{1}{2k}\right)^{2n+1}\right]\\&=2\pi^{2m+1}\left[\sum_{n=1}^{\infty}\frac{1}{2n+2m+1}\left(\frac{1}{2k}\right)^{2n}\right.\\&\qquad\left.-\sum_{n=1}^{\infty}\frac{1}{2n+2m-1}\left(\frac{1}{2k}\right)^{2n}\right]\\&=-4\pi^{2m+1}\sum_{n=1}^{\infty}\frac{1}{(2n+2m+1)(2n+2m-1)4^n}\frac{1}{k^{2n}}.\end{aligned}\tag{58}$$

Finally, setting the last expression of $\mathscr{I}_k(m)$ in (52) and considering (51) yield our desired identity (47). This completes the proof of Theorem 6. □

The special case of (47) when $m=1$ yields

$$\zeta(3)=\frac{4\pi^2}{27}-\frac{8\pi^2}{9}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)4^n},\tag{59}$$

which, upon using $\zeta(0)=-1/2$ (see, e.g., [1, p. 165, (10)]), can be expressed in a more compact form:

$$\zeta(3)=-\frac{8\pi^2}{9}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+3)4^n}.\tag{60}$$

The formula (59) or (60) has already been presented (cf., e.g., [29, p. 31] and [30, p. 837]).

Remark 7. Since $1<\zeta(2n)\le\zeta(2)=\pi^2/6<2$ $(n\in\mathbb{N})$, using the Nth partial sum of the infinite series in (59) or (60), we can compute $\zeta(3)$ with an error R_N satisfying

$$\begin{aligned}|R_N|&<\frac{8\pi^2}{9}\frac{\zeta(2N+2)}{(2N+3)(2N+5)}\sum_{n=N+1}^{\infty}\frac{1}{4^n}\\&<\frac{16\pi^2}{27}\frac{1}{(2N+3)(2N+5)4^N}.\end{aligned}\tag{61}$$

Using the 45th partial sum in (60) we have an error bound $|R_{45}|<6\cdot10^{-31}$, and approximately the value $\zeta(3)=1.202056903159594285399738161511$. For comparison, using the Nth partial sum of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$, we can compute $\zeta(3)$ with an error R_N^o satisfying

$$R_N^o:=\sum_{n=N+1}^{\infty}\frac{1}{n^3}<\int_N^{\infty}\frac{1}{x^3}dx=\frac{1}{2N^2},\tag{62}$$

from which we get the error bound $R_{45}^o<3\cdot10^{-4}$. Scheufens [29, p. 31] estimated the error bounds $|R_{25}|<9\cdot10^{-19}$ and $R_{25}^o<8\cdot10^{-4}$. So it is easy to see that the original series of $\zeta(3)$ converges very slowly while the series representation (59) or (60) of $\zeta(3)$ converges very rapidly.

Srivastava and Choi [1, Chapter 3] presented a rather extensive collection of closed-form sums of series involving the zeta functions such as (59) or (60), together with an interesting historical introduction. In fact, the formula (47) may be obtained in a totally different way (see, e.g., [1, p. 259, (71)]).

4. Evaluation of $\zeta(2m)$ from Fourier Series

There have been earlier works (see, e.g., [17, 29, 31–34]) in which the authors evaluated $\zeta(2m)$ by using Fourier series. Here, for completeness, we also do the same thing. Yet we may very carefully emphasize that, when the involved coefficient in Fourier series is computed, its computation becomes a little easier by using a known indefinite integral formula.

For $m\in\mathbb{N}$, let f be the even 2π-periodic function given by $f(x)=x^{2m}$, $x\in[-\pi,\pi]$. Since f is continuous and piecewise differentiable, we have

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx)\quad(x\in\mathbb{R}),\tag{63}$$

where

$$a_0=\frac{2}{\pi}\int_0^{\pi}x^{2m}dx=\frac{2\pi^{2m}}{2m+1},\tag{64}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^{2m}\cos(nx)\,dx\quad(n\in\mathbb{N}).\tag{65}$$

Evaluation of a_n in (65). Use a known indefinite integral formula (see, e.g., [28, p. 211, Entry 2.633(2)]) to get the following equation:

$$\begin{aligned}&\int_0^{\pi}x^{2m}\cos(nx)\,dx\\&\quad=\sum_{k=0}^{2m}k!\binom{2m}{k}\frac{x^{2m-k}}{n^{k+1}}\sin\left(nx+\frac{1}{2}k\pi\right)\Bigg|_0^{\pi}.\end{aligned}\tag{66}$$

For convenience, let the right-hand side of (66) be denoted by $\mathscr{I}$. Then we have

$$\mathscr{I}=\sum_{k=0}^{2m}k!\binom{2m}{k}\frac{\pi^{2m-k}}{n^{k+1}}\sin\left(n\pi+\frac{1}{2}k\pi\right).\tag{67}$$

By considering the following relation

$$\sum_{k=0}^{2m}\alpha(k)=\sum_{k=0}^{m}\alpha(2k)+\sum_{k=0}^{m-1}\alpha(2k+1)\qquad(m\in\mathbb{N}),\tag{68}$$

we have

$$\begin{aligned}\mathcal{J}&=\sum_{k=0}^{m}(2k)!\binom{2m}{2k}\frac{\pi^{2m-2k}}{n^{2k+1}}\sin(n\pi+k\pi)\\&\quad+\sum_{k=0}^{m-1}(2k+1)!\binom{2m}{2k+1}\frac{\pi^{2m-2k-1}}{n^{2k+2}}\\&\qquad\times\sin\left(n\pi+k\pi+\frac{\pi}{2}\right)\\&=\sum_{k=0}^{m-1}(-1)^{n+k}(2k+1)!\binom{2m}{2k+1}\frac{\pi^{2m-2k-1}}{n^{2k+2}}.\end{aligned}\tag{69}$$

We finally get

$$\begin{aligned}a_n&=\frac{2(-1)^n}{\pi^2}\sum_{k=0}^{m-1}(-1)^k(2k+1)!\binom{2m}{2k+1}\\&\qquad\times\frac{\pi^{2(m-k)}}{n^{2k+2}}\qquad(n,m\in\mathbb{N}).\end{aligned}\tag{70}$$

Setting $f(x)=x^{2m}$ in (63) and setting $x=\pi$ and applying (64) and (70) to the resulting identity, we finally get the following recurrence formula for evaluation of $\zeta(2m)$:

$$\begin{aligned}\zeta(2m)&=(-1)^{m-1}\frac{m}{(2m+1)!}\pi^{2m}+(-1)^m\\&\quad\times\sum_{k=1}^{m-1}(-1)^{k+1}\frac{\pi^{2(m-k)}}{(2m-2k+1)!}\zeta(2k)\qquad(m\in\mathbb{N}).\end{aligned}\tag{71}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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