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Research Article

On Subscalarity of Some 2×2 M-Hyponormal Operator Matrices

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We provide some conditions for 2×2 operator matrices whose diagonal entries are M-hyponormal operators to be subscalar. As a consequence, we obtain that Weyl type theorem holds for such operator matrices.

1. Introduction and Preliminaries

Let H be a complex separable Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. If $T \in B(H)$, we write $N(T)$, $R(T)$, $\sigma(T)$, and $\sigma_a(T)$ for the null space, the range space, the spectrum, and the approximate point spectrum of T, respectively. An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) := \dim N(T) < \infty$, and $\beta(T) := \dim N(T^*) < \infty$. The index of a Fredholm operator T is given by $i(T) = \alpha(T) - \beta(T)$. An operator T is called Weyl if it is Fredholm of index zero. The Weyl spectrum of T [1] is defined by $w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$.

We consider the sets

$$\Phi_+(H) := \{T \in B(H) : R(T) \text{ is closed}, \alpha(T) < \infty\}; \tag{1}$$

$$\Phi_+^-(H) := \{T \in B(H) : T \in \Phi_+(H), i(T) \leq 0\}$$

and define

$$\sigma_{ea}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(H)\};$$

$$\pi_{00}(T) := \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}; \tag{2}$$

$$\pi_{00}^a(T) := \{\lambda \in \text{iso}\,\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},$$

where $\text{iso}\,\sigma(T)$ denotes the isolated points of $\sigma(T)$.

Following [2], we say that Weyl's theorem holds for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$ and that a-Weyl's theorem holds for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$.

Let $T \in B(H)$. As an easy extension of normal operators, hyponormal operators have been studied by many mathematicians. Though there are many unsolved interesting problems for hyponormal operators (e.g., the invariant subspace problem), one of recent trends in operator theory is studying natural extensions of hyponormal operators. So we introduce some of these nonhyponormal operators. An operator T is said to be M-hyponormal if there exists a real positive number M such that

$$M^2(T - \lambda)^*(T - \lambda) \geq (T - \lambda)(T - \lambda)^* \quad \forall \lambda \in \mathbb{C}. \tag{3}$$

Evidently,

$$T \text{ is hyponormal} \Longrightarrow T \text{ is } M\text{-hyponormal}. \tag{4}$$

There is a vast literature concerning M-hyponormal operators (see [3–5], etc.). We also note that an operator T need not be hyponormal even though T and T^* are both M-hyponormal. To see this, consider the operator

$$T = \begin{pmatrix} U & K \\ 0 & U^* \end{pmatrix} : l_2 \oplus l_2 \longrightarrow l_2 \oplus l_2, \tag{5}$$

where U is the unilateral shift on l_2 and $K : l_2 \to l_2$ is given by $K(x_1, x_2, x_3, \ldots) = (2x_1, 0, 0, \ldots)$. Then a direct calculation shows that

$$\frac{1}{2}\|(T - z)x\| \leq \|(T - z)^* x\| \leq 2\|(T - z)x\| \tag{6}$$

for all $z \in \mathbb{C}$ and for all $x \in l_2 \oplus l_2$, which says that T and T^* are both M-hyponormal. But since

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & I + \frac{3}{2}K \end{pmatrix}, \tag{7}$$

while

$$TT^* = \begin{pmatrix} I + \frac{3}{2}K & 0 \\ 0 & I \end{pmatrix}, \tag{8}$$

T is not hyponormal.

Let z be the coordinate in the complex plane $\mathbb{C}$ and let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of $\mathbb{C}$. We will denote by $L^2(U,H)$ the Hilbert space of measurable functions $f : U \to H$, such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 d\mu(z)\right)^{1/2} < \infty. \tag{9}$$

The Bergman space for U is defined by $A^2(U,H) = L^2(U,H) \cap O(U,H)$, where $O(U,H)$ denotes the Fréchet space of H-valued analytic functions on U with respect to uniform topology. Note that $A^2(U,H)$ is a Hilbert space. Let us define now a special Sobolev type space. Let U be again a bounded open subset of $\mathbb{C}$ and let m be a fixed nonnegative integer. The vector valued Sobolev space $W^m(U,H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U,H)$ whose derivatives $\bar{\partial} f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,H)$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \left\|\bar{\partial}^i f\right\|_{2,U}^2 \tag{10}$$

$W^m(U,H)$ becomes a Hilbert space contained continuously in $L^2(U,H)$. A bounded linear operator S on H is called scalar of order m if it possesses a spectral distribution of order m, that is, if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \longrightarrow B(H) \tag{11}$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \le m \le \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. Let U be a (connected) bounded open subset of $\mathbb{C}$ and let m be a nonnegative integer. The linear operator M_f of multiplication by f on $W^m(U,H)$ is continuous and it has a spectral distribution of order m, defined by the functional calculus

$$\Phi_M : C_0^m(\mathbb{C}) \longrightarrow B(W^m(U,H)), \quad \Phi_M(f) = M_f. \tag{12}$$

Therefore, M_z is a scalar operator of order m.

An operator $T \in B(H)$ is said to have the single-valued extension property (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to H$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G.

An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known that

$$\text{Bishop's property } (\beta) \Longrightarrow \text{SVEP}. \tag{13}$$

In 1984, Putinar showed in [6] that every hyponormal operator is subscalar, and then in 1987, Brown used this result to prove that a hyponormal operator with rich spectrum has a nontrivial invariant subspace (see [7]). There have been a lot of generalizations of such beautiful consequences (see [8–11]). In this paper, we provide some conditions for 2×2 operator matrices whose diagonal entries are M-hyponormal operators to be subscalar. As a consequence, we obtain that Weyl type theorem holds for such operator matrices.

2. Subscalarity

Lemma 1 (see [6, Proposition 2.1]). *For a bounded open disk D in the complex plane $\mathbb{C}$, there is a constant C_D such that for an arbitrary operator $T \in B(H)$ and $f \in W^2(D,H)$ we have*

$$\|(I-P)f\|_{2,D} \le C_D\left(\left\|(T-z)^*\bar{\partial} f\right\|_{2,D} + \left\|(T-z)^*\bar{\partial}^2 f\right\|_{2,D}\right), \tag{14}$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$.

Corollary 2. *Let D be as in Lemma 1. If $T \in B(H)$ is an M-hyponormal operator, then there exists a constant C_D such that for all $z \in \mathbb{C}$ and $f \in W^2(D,H)$*

$$\|(I-P)f\|_{2,D} \le MC_D \times \left(\left\|(T-z)\bar{\partial} f\right\|_{2,D} + \left\|(T-z)\bar{\partial}^2 f\right\|_{2,D}\right), \tag{15}$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$.

Proof. This follows from Lemma 1 and the definition of M-hyponormal operator. ☐

Lemma 3. *Let $T \in B(H)$ be an M-hyponormal operator and let D be a bounded disk in $\mathbb{C}$. If $\{f_n\}$ is a sequence in $W^m(D,H)(m > 2)$ such that*

$$\lim_{n\to\infty}\left\|(z-T)\bar{\partial}^i f_n\right\|_{2,D} = 0 \tag{16}$$

for $i = 1, 2, \ldots, m$, then $\lim_{n\to\infty}\|\bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1, 2, \ldots, m-2$, where D_0 is a disk strictly contained in D.

Proof. Since T is an M-hyponormal operator, it follows from Corollary 2 that there exists a constant C_D such that

$$\left\|(I-P)\overline{\partial}^i f_n\right\|_{2,D} \le MC_D\left(\left\|(T-z)\overline{\partial}^{i+1} f_n\right\|_{2,D} + \left\|(T-z)\overline{\partial}^{i+2} f_n\right\|_{2,D}\right) \tag{17}$$

for $i=0,1,2,\ldots,m-2$. From (17), we have

$$\lim_{n\to\infty}\left\|(I-P)\overline{\partial}^i f_n\right\|_{2,D}=0 \tag{18}$$

for $i=0,1,2,\ldots,m-2$. Hence,

$$\lim_{n\to\infty}\left\|(T-z)P\overline{\partial}^i f_n\right\|_{2,D}=0 \tag{19}$$

for $i=1,2,\ldots,m-2$. Since T has Bishop's property (β) [12], we have

$$\lim_{n\to\infty}\left\|P\overline{\partial}^i f_n\right\|_{2,D_0}=0 \tag{20}$$

for $i=1,2,\ldots,m-2$, where D_0 denotes a disk with $\sigma(T)\subsetneq\overline{D_0}\subsetneq D$. From (18) and (20), we get that

$$\lim_{n\to\infty}\left\|\overline{\partial}^i f_n\right\|_{2,D_0}=0 \tag{21}$$

for $i=1,2,\ldots,m-2$. □

Lemma 4. *Let $T=\begin{pmatrix}T_1 & T_2\\ T_3 & T_4\end{pmatrix}\in B(H\oplus H)$, where T_i are mutually commuting, and both T_1 and T_4 are M-hyponormal operators. For any positive integer m and any bounded disk D in $\mathbb{C}$ containing $\sigma(T)$, define the map $V_m : H\oplus H\to H(D)$ by*

$$V_m h = 1\otimes h + \overline{(T-z)W^m(D,H)\oplus W^m(D,H)}\left(=\widetilde{1\otimes h}\right), \tag{22}$$

where

$$H(D) := \frac{W^m(D,H)\oplus W^m(D,H)}{\overline{(T-z)W^m(D,H)\oplus W^m(D,H)}} \tag{23}$$

and $1\otimes h$ denotes the constant function sending any $z\in D$ to $h\in H\oplus H$. Then the following statements hold.

(i) *If $T_2^rT_3^s=0$ for some nonnegative integer r and s, where $T_2^0=T_3^0=I$, then V_{4N+2} is one-to-one and has closed range, where $N:=\max\{r,s\}$.*

(ii) *If $T_1=T_4$, $T_2=T_3$, and T_2 is algebraic of order k, then V_{4k+2} is one-to-one and has closed range.*

(iii) *If T_1+T_4 is an M-hyponormal operator and $T_1T_4=T_2T_3$, then V_6 is one-to-one and has closed range.*

Proof. Let $h_n=h_n^1\oplus h_n^2\in H\oplus H$ and

$$f_n=f_n^1\oplus f_n^2\in W^m(D,H)\oplus W^m(D,H) \tag{24}$$

be sequences such that

$$\lim_{n\to\infty}\left\|(z-T)f_n+1\otimes h_n\right\|_{W^m\oplus W^m}=0. \tag{25}$$

Then (25) implies that

$$\begin{aligned}&\lim_{n\to\infty}\left\|(z-T_1)f_n^1+T_2f_n^2+1\otimes h_n^1\right\|_{W^m}=0,\\&\lim_{n\to\infty}\left\|T_3f_n^1+(z-T_4)f_n^2+1\otimes h_n^2\right\|_{W^m}=0.\end{aligned} \tag{26}$$

By the definition of the norm of Sobolev space and (26) we get

$$\begin{aligned}&\lim_{n\to\infty}\left\|(z-T_1)\overline{\partial}^i f_n^1+T_2\overline{\partial}^i f_n^2\right\|_{2,D}=0,\\&\lim_{n\to\infty}\left\|T_3\overline{\partial}^i f_n^1+(z-T_4)\overline{\partial}^i f_n^2\right\|_{2,D}=0\end{aligned} \tag{27}$$

for $i=1,2,\ldots,m$.

(i) Set $m=4N+2$, where $N:=\max\{r,s\}$. We may assume that $s\le r$. Then $m=4r+2$.

We prove that for every $j=0,1,2,\ldots,s$, the following equations hold

$$\begin{aligned}&\lim_{n\to\infty}\left\|T_2^{r-j}T_3^{s-j+1}\overline{\partial}^i f_n^1\right\|_{2,D_j}=0,\\&\lim_{n\to\infty}\left\|T_2^{r-j}T_3^{s-j}\overline{\partial}^i f_n^2\right\|_{2,D_j}=0\end{aligned} \tag{28}$$

for $i=1,2,\ldots,4(r-j)+2$, where $\sigma(T)\subsetneq D_s\subsetneq\cdots\subsetneq D_2\subsetneq D_1\subsetneq D$.

To prove (28), we will use the induction on j. Since $T_2^rT_3^s=0$, then (28) holds when $j=0$. Suppose that (28) is true for some $j<s$. From (27) and the inductions hypothesis, we have

$$\begin{aligned}&\lim_{n\to\infty}\left\|(T_1-z)T_2^{r-j-1}T_3^{s-j}\overline{\partial}^i f_n^1\right\|_{2,D_j}=0,\\&\lim_{n\to\infty}\left\|T_2^{r-j-1}T_3^{s-j}\overline{\partial}^i f_n^1+(T_4-z)T_2^{r-j-1}T_3^{s-j-1}\overline{\partial}^i f_n^2\right\|_{2,D_j}=0\end{aligned} \tag{29}$$

for $i=1,2,\ldots,4(r-j)+2$. Since T_1 is an M-hyponormal operator, by Lemma 3 we have

$$\lim_{n\to\infty}\left\|T_2^{r-j-1}T_3^{s-j}\overline{\partial}^i f_n^1\right\|_{2,D_j'}=0 \tag{30}$$

for $i=1,2,\ldots,4(r-j)$, where $\sigma(T)\subsetneq D_j'\subsetneq D_j$. From (30) and the second equation of (27),

$$\lim_{n\to\infty}\left\|(T_4-z)T_2^{r-j-1}T_3^{s-j-1}\overline{\partial}^i f_n^2\right\|_{2,D_j'}=0 \tag{31}$$

for $i=1,2,\ldots,4(r-j)$. Since T_4 is an M-hyponormal operator, by Lemma 3 we derive

$$\lim_{n\to\infty}\left\|T_2^{r-j-1}T_3^{s-j-1}\overline{\partial}^i f_n^2\right\|_{2,D_{j+1}}=0 \tag{32}$$

for $i = 1, 2, \ldots, 4(r - j - 1) + 2$, where $\sigma(T) \subsetneq D_{j+1} \subsetneq D'_j$. Therefore, the proof of (28) is completed. Let $j = s$ in (28); we have

$$\lim_{n\to\infty}\left\|T_2^{r-s}T_3\overline{\partial}^i f_n^1\right\|_{2,D_s} = 0,$$
$$\lim_{n\to\infty}\left\|T_2^{r-s}\overline{\partial}^i f_n^2\right\|_{2,D_s} = 0 \quad (33)$$

for $i = 1, 2, \ldots, 4(r-s)+2$. From (27) and (33), it follows that

$$\lim_{n\to\infty}\left\|(T_1 - z)\,T_2^{r-s-1}\overline{\partial}^i f_n^1\right\|_{2,D_s} = 0 \quad (34)$$

for $i = 1, 2, \ldots, 4(r - s) + 2$. Since T_1 is an M-hyponormal operator, from Lemma 3 we obtain that

$$\lim_{n\to\infty}\left\|T_2^{r-s-1}\overline{\partial}^i f_n^1\right\|_{2,D_{s'}} = 0 \quad (35)$$

for $i = 1, 2, \ldots, 4(r - s)$, where $\sigma(T) \subsetneq D'_s \subsetneq D_s$, and hence

$$\lim_{n\to\infty}\left\|(T_4 - z)\,T_2^{r-s-1}\overline{\partial}^i f_n^2\right\|_{2,D_{s'}} = 0 \quad (36)$$

for $i = 1, 2, \ldots, 4(r - s)$. Since T_4 is an M-hyponormal operator, Lemma 3 implies that

$$\lim_{n\to\infty}\left\|T_2^{r-s-1}\overline{\partial}^i f_n^2\right\|_{2,D_{s+1}} = 0 \quad (37)$$

for $i = 1, 2, \ldots, 4(r - s - 1) + 2$, where $\sigma(T) \subsetneq D_{s+1} \subsetneq D'_s$. By repeating the process from (33) to (37), it holds for all $j = 0, 1, 2, \ldots, r - s$ that

$$\lim_{n\to\infty}\left\|T_2^{r-s-j}\overline{\partial}^i f_n^2\right\|_{2,D_{s+j}} = 0 \quad (38)$$

for $i = 1, 2, \ldots, 4(r - s - j) + 2$, where $\sigma(T) \subsetneq D_{r+1} \subsetneq D_r \subsetneq D_{r-1} \subsetneq \cdots \subsetneq D_{s+1} \subsetneq D_s$. In particular, let $r = s + j$,

$$\lim_{n\to\infty}\left\|\overline{\partial}^i f_n^2\right\|_{2,D_r} = 0 \quad (39)$$

for $i = 1, 2$. Hence, from the first equation of (27), we have

$$\lim_{n\to\infty}\left\|(T_1 - z)\,\overline{\partial}^i f_n^1\right\|_{2,D_r} = 0 \quad (40)$$

for $i = 1, 2$. Applying Corollary 2, we have

$$\lim_{n\to\infty}\left\|(I - P)\,f_n^1\right\|_{2,D'_r} = 0, \quad (41)$$

where P denotes the orthogonal projection of $L^2(D'_r, H)$ onto $A^2(D'_r, H)$ and $\sigma(T) \subsetneq D'_r \subsetneq D_r$. By combining (26) with (39) and (41), we obtain that

$$\lim_{n\to\infty}\left\|(z - T)\,Pf_n + 1 \otimes h_n\right\|_{2,D'_r} = 0, \quad (42)$$

where $Pf_n := \begin{pmatrix} Pf_n^1 \\ Pf_n^2 \end{pmatrix}$.

Let Γ be a curve in D'_r surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty}\left\|Pf_n(z) + (z - T)^{-1}\left(1 \otimes h_n\right)(z)\right\| = 0 \quad (43)$$

uniformly. Hence, by the Riesz functional calculus,

$$\lim_{n\to\infty}\left\|\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\,dz + h_n\right\| = 0. \quad (44)$$

But $(1/2\pi i)\int_\Gamma Pf_n(z)dz = 0$ by Cauchy's theorem. Hence, $\lim_{n\to\infty}h_n = 0$, and so V_{4r+2} is one-to-one and has closed range.

(ii) Set $m = 4k + 2$. By the hypothesis and (27), we have

$$\lim_{n\to\infty}\left\|(T_1 - z)\,\overline{\partial}^i f_n^1 + T_2\overline{\partial}^i f_n^2\right\|_{2,D} = 0,$$
$$\lim_{n\to\infty}\left\|T_2\overline{\partial}^i f_n^1 + (T_1 - z)\,\overline{\partial}^i f_n^2\right\|_{2,D} = 0 \quad (45)$$

for $i = 1, 2, \ldots, 4k+2$. Since T_2 is algebraic with order k, there exists a nonconstant polynomial $p(z) = (z-z_1)(z-z_2)\cdots(z-z_k)$ such that $p(T_2) = 0$. Set $q_j(z) = (z - z_{j+1})\cdots(z - z_k)$ for $j = 0, 1, 2, \ldots, k - 1$ and $q_k(z) = 1$.

Claim. For every $j = 0, 1, 2, \ldots, k$, the following equations hold:

$$\lim_{n\to\infty}\left\|q_j(T_2)\,\overline{\partial}^i f_n^1\right\|_{2,D_j} = 0,$$
$$\lim_{n\to\infty}\left\|q_j(T_2)\,\overline{\partial}^i f_n^2\right\|_{2,D_j} = 0 \quad (46)$$

for $i = 1, 2, \ldots, 4(k - j) + 2$, where $\sigma(T) \subsetneq D_k \subsetneq \cdots \subsetneq D_2 \subsetneq D_1 \subsetneq D$.

To prove the claim, we use the induction on j. Since $q_0(T_2) = p(T_2) = 0$, then when $j = 0$ the claim holds. Suppose that the claim is true for some $j = r$, where $0 \le r < k$. Multiplying (45) by $q_{r+1}(T_2)$, we obtain

$$\lim_{n\to\infty}\left\|(T_1 - z)\,q_{r+1}(T_2)\,\overline{\partial}^i f_n^1 + z_{r+1}q_{r+1}(T_2)\,\overline{\partial}^i f_n^2\right\|_{2,D_r} = 0,$$
$$\lim_{n\to\infty}\left\|z_{r+1}q_{r+1}(T_2)\,\overline{\partial}^i f_n^1 + (T_1 - z)\,q_{r+1}(T_2)\,\overline{\partial}^i f_n^2\right\|_{2,D_r} = 0 \quad (47)$$

for $i = 1, 2, \ldots, 4(k - r) + 2$. From (47), we derive

$$\lim_{n\to\infty}\left\|(T_1 - (z - z_{r+1}))\,q_{r+1}(T_2)\left(\overline{\partial}^i f_n^1 + \overline{\partial}^i f_n^2\right)\right\|_{2,D_r} = 0 \quad (48)$$

for $i = 1, 2, \ldots, 4(k - r) + 2$. Since T_1 is an M-hyponormal operator, from (48) and Lemma 3 we obtain

$$\lim_{n\to\infty}\left\|q_{r+1}(T_2)\left(\overline{\partial}^i f_n^1 + \overline{\partial}^i f_n^2\right)\right\|_{2,D'_r} = 0 \quad (49)$$

for $i = 1, 2, \ldots, 4(k - r)$, where $\sigma(T) \subsetneq D'_r \subsetneq D_r$. Combining (49) with the first equation of (47), we have

$$\lim_{n\to\infty}\left\|(T_1 - (z + z_{r+1}))\,q_{r+1}(T_2)\,\overline{\partial}^i f_n^1\right\|_{2,D'_r} = 0 \quad (50)$$

for $i = 1, 2, \ldots, 4(k - r)$. Since T_1 is an M-hyponormal operator, we obtain from Lemma 3 and (50) that

$$\lim_{n\to\infty}\left\|q_{r+1}(T_2)\,\overline{\partial}^i f_n^1\right\|_{2,D_{r+1}} = 0 \quad (51)$$

for $i = 1, 2, \ldots, 4(k-r-1)+2$, where $\sigma(T) \subsetneq D_{r+1} \subsetneq D'_r$. From (49) and (51), we obtain

$$\lim_{n\to\infty} \left\| q_{r+1}(T_2)\,\overline{\partial}^i f_n^2 \right\|_{2,D_{r+1}} = 0 \qquad (52)$$

for $i = 1, 2, \ldots, 4(k-r-1)+2$. Therefore, the proof of the claim is completed.

From the claim with $j = k$, we have

$$\lim_{n\to\infty} \left\| \overline{\partial}^i f_n^1 \right\|_{2,D_k} = \lim_{n\longrightarrow\infty} \left\| \overline{\partial}^i f_n^2 \right\|_{2,D_k} = 0 \qquad (53)$$

for $i = 1, 2$. From Lemma 1 we derive that

$$\lim_{n\to\infty} \left\| (I-P)\, f_n^1 \right\|_{2,D_k} = \lim_{n\longrightarrow\infty} \left\| (I-P)\, f_n^2 \right\|_{2,D_k} = 0, \qquad (54)$$

where P denotes the orthogonal projection of $L^2(D_k, H)$ onto $A^2(D_k, H)$. By applying the proof of (i), we obtain that V_{4k+2} is one-to-one and has closed range.

(iii) Set $m = 6$. By (27), we have

$$\begin{aligned} &\lim_{n\to\infty} \left\| (T_1T_3 - zT_3)\,\overline{\partial}^i f_n^1 + T_2T_3\overline{\partial}^i f_n^2 \right\|_{2,D} = 0, \\ &\lim_{n\to\infty} \left\| T_1T_3\overline{\partial}^i f_n^1 + (T_1T_4 - zT_1)\,\overline{\partial}^i f_n^2 \right\|_{2,D} = 0 \end{aligned} \qquad (55)$$

for $i = 1, 2, \ldots, 6$. Since $T_1T_4 = T_2T_3$, (55) implies that

$$\lim_{n\to\infty} \left\| z\left(T_1\overline{\partial}^i f_n^2 - T_3\overline{\partial}^i f_n^1\right) \right\|_{2,D} = 0 \qquad (56)$$

for $i = 1, 2, \ldots, 6$. Since the zero operator is hyponormal operator, it follows from Lemma 3 and (56) that

$$\lim_{n\to\infty} \left\| T_1\overline{\partial}^i f_n^2 - T_3\overline{\partial}^i f_n^1 \right\|_{2,D_1} = 0 \qquad (57)$$

for $i = 1, 2, \ldots, 4$, where $\sigma(T) \subsetneq D_1 \subsetneq D$. Using (57) and the second equation of (27), we obtain

$$\lim_{n\to\infty} \left\| (T_1 + T_4 - z)\,\overline{\partial}^i f_n^2 \right\|_{2,D_1} = 0 \qquad (58)$$

for $i = 1, 2, 3, 4$. Since $T_1 + T_4$ is an M-hyponormal operator, from Lemma 3 we have

$$\lim_{n\to\infty} \left\| \overline{\partial}^i f_n^2 \right\|_{2,D_2} = 0 \qquad (59)$$

for $i = 1, 2$, where $\sigma(T) \subsetneq D_2 \subsetneq D_1$. From (59) and the first equation of (27) we obtain that

$$\lim_{n\to\infty} \left\| (T_1 - z)\,\overline{\partial}^i f_n^1 \right\|_{2,D_2} = 0 \qquad (60)$$

for $i = 1, 2$. Since T_1 is an M-hyponormal operator, from Corollary 2 we get that

$$\lim_{n\to\infty} \left\| (I-P)\, f_n^1 \right\|_{2,D_3} = 0, \qquad (61)$$

where P denotes the orthogonal projection of $L^2(D_3, H)$ onto $A^2(D_3, H)$ and $\sigma(T) \subsetneq D_3 \subsetneq D_2$. By applying the similar way of (i), we obtain that V_6 is one-to-one and has closed range. □

Theorem 5. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(H \oplus H)$, where T_i are mutually commuting, and both T_1 and T_4 are M-hyponormal operators. If $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 4, then T is a subscalar operator of order m, where m is the appropriately chosen integer as in Lemma 4.*

Proof. Let D be a bounded disk in $\mathbb{C}$ containing $\sigma(T)$ and consider the quotient space

$$H(D) := \frac{W^m(D,H) \oplus W^m(D,H)}{(T-z)\,W^m(D,H) \oplus W^m(D,H)} \qquad (62)$$

endowed with the Hilbert space norm, where $m = 4N+2$, $N := \max\{r, s\}$ for (i), $m = 4k+2$ for (ii), and $m = 6$ for (iii) in Lemma 4. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}, \widetilde{S}$, respectively. Let M be the operator of multiplication by z on $W^m(D,H) \oplus W^m(D,H)$. Then M is a scalar operator of order m and has a spectral distribution Φ. Since $R(T-z)$ is invariant under M, $\widetilde{M}$ can be well defined. Moreover, consider the spectral distribution $\Phi : C_0^m(\mathbb{C}) \to B(W^m(D,H) \oplus W^m(D,H))$ defined by the following relation: for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(D,H) \oplus W^m(D,H)$, $\Phi(\phi)f = \phi f$. Then the spectral distribution Φ of M commutes with $T - z$, and so $\widetilde{M}$ is still a scalar operator of order m with $\widetilde{\Phi}$ as a spectral distribution. As in Lemma 4, if we define the map $V_m : H_1 \oplus H_2 \to H(D)$ by

$$\begin{aligned} V_m h = {} & 1 \otimes h \\ & + \overline{(T-z)\,W^m(D,H) \oplus W^m(D,H)}\left(= \widetilde{1 \otimes h}\right), \end{aligned} \qquad (63)$$

then V_m is one-to-one and has closed range. Since

$$V_m T h = \widetilde{1 \otimes Th} = \widetilde{z \otimes h} = \widetilde{M}\left(\widetilde{1 \otimes h}\right) = \widetilde{M} V_m h \qquad (64)$$

for all $h \in H \oplus H, V_m T = \widetilde{M} V_m$. In particular, $R(V_m)$ is invariant under $\widetilde{M}$ and $R(V_m)$ is closed; it is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{R(V_m)}$ and $\widetilde{M}$ is scalar of order m, T is a subscalar operator of order m. □

Corollary 6. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(H \oplus H)$, where T_i are mutually commuting, both T_1 and T_4 are M-hyponormal operators, and $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 4. Then T has property (β) and the single-valued extension property.*

Proof. From section one, we need only to prove that T has property (β). Since property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 5 to the case of a scalar operator. Since every scalar operator has property (β) (see [6]), T has property (β). □

Define the quasi-nilpotent part of $\lambda I - T$

$$H_0(\lambda I - T) := \left\{ x \in H : \lim_{n\to\infty} \left\| (\lambda I - T)^n x \right\|^{1/n} = 0 \right\}. \qquad (65)$$

Definition 7. An operator $T \in B(H)$ is said to belong to the class $H(p)$ if there exists a natural number $p := p(\lambda)$ such that

$$H_0(\lambda I - T) = N(\lambda I - T)^p \quad \forall \lambda \in \mathbb{C}. \tag{66}$$

Theorem 8 (see [13]). *Every subscalar operator $T \in B(H)$ is $H(p)$.*

Definition 9. An operator $T \in B(H)$ is said to be polaroid if every $\lambda \in \operatorname{iso}\sigma(T)$ is a pole of the resolvent of T.

Note that

$$T \text{ is polaroid } \Longleftrightarrow T^* \text{ is polaroid.} \tag{67}$$

The condition of being polaroid may be characterized by means of the quasi-nilpotent part.

Theorem 10 (see [14]). *An operator $T \in B(H)$ is polaroid if and only if there exists a natural number $p := p(\lambda)$ such that*

$$H_0(\lambda I - T) = N(\lambda I - T)^p \quad \forall \lambda \in iso\,\sigma(T). \tag{68}$$

Corollary 11. *Every $H(p)$ operator is polaroid.*

Since a subscalar operator is $H(p)$, we have the following.

Corollary 12. *Every subscalar operator is polaroid.*

Corollary 13. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(H \oplus H)$, where T_i are mutually commuting, both T_1 and T_4 are M-hyponormal operators, and $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 4. Then T is polaroid.*

If $T \in B(H)$ has SVEP, then T and T^* satisfy Browder's theorem. A sufficient condition for an operator T satisfying Browder's theorem to satisfy Weyl's theorem is that T is polaroid. Then we have the following result.

Corollary 14. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(H \oplus H)$, where T_i are mutually commuting, both T_1 and T_4 are M-hyponormal operators, and $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 4. Then Weyl's theorem holds for T and T^*.*

Observe that if $T \in B(H)$ has SVEP, then $\sigma(T) = \overline{\sigma_a(T^*)}$. Hence, if T has SVEP and is polaroid, then T^* satisfies a-Weyl's theorem [15, Theorem 3.10].

Corollary 15. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(H \oplus H)$, where T_i are mutually commuting, both T_1 and T_4 are M-hyponormal operators, and $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 4. Then a-Weyl's theorem holds for T^*.*

Proof. Since T is polaroid and has SVEP, then a-Weyl's theorem holds for T^*. □

In the following, f is an analytic function on $\sigma(T)$ and f is not constant on each connected component of the open set U containing $\sigma(T)$.

Corollary 16. *Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in B(H \oplus H)$, where T_i are mutually commuting, both T_1 and T_4 are M-hyponormal operators, and $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 4. Then the following assertions hold:*

(i) *Weyl's theorem holds for $f(T)$;*

(ii) *a-Weyl's theorem holds for $f(T^*)$.*

Proof. (i) Since T is polaroid and has SVEP, then $f(T)$ satisfies Weyl's theorem by [15, Theorem 3.14].

(ii) Since T is polaroid and has SVEP, then $f(T^*)$ satisfies a-Weyl's theorem by [15, Theorem 3.12]. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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