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*Research Article*

# Uncertainty Principles for Wigner-Ville Distribution Associated with the Linear Canonical Transforms

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The Heisenberg uncertainty principle of harmonic analysis plays an important role in modern applied mathematical applications, signal processing and physics community. The generalizations and extensions of the classical uncertainty principle to the novel transforms are becoming one of the most hottest research topics recently. In this paper, we firstly obtain the uncertainty principle for Wigner-Ville distribution and ambiguity function associate with the linear canonical transform, and then the $n$-dimensional cases are investigated in detail based on the proposed Heisenberg uncertainty principle of the $n$-dimensional linear canonical transform.

## 1. Introduction

The Heisenberg uncertainty principle, proposed by the German physicist Heisenberg in 1927, is a basic principle of quantum mechanics, and it means that the position and the momentum of a particle cannot be determined simultaneously in quantum mechanical systems. On the mathematical side, we can describe the Heisenberg uncertainty principle as the product of the variance of $f$ and $\mathscr{F}(f)$ (the Fourier transform of $f$) which cannot be infinitely small. We know that the variance of $f$ and $\mathscr{F}(f)$ represents, respectively, the temporal resolution and the frequency resolution of a signal; we can therefore obtain that the temporal resolution and the frequency resolution of any signal cannot be infinitely improved simultaneously in signal processing community.

The linear canonical transform (LCT) is the generalization of the traditional Fourier transform (FT) and the fractional Fourier transform (FRFT), which is used originally for solving differential equations and optical systems analysis [1]. With the rapid development of the fractional Fourier transform, the LCT has been paid more and more attention in applied mathematics and signal processing community. The filtering theory [2], the frame theory [3], the sampling theory [4–6], the discrete algorithms [7, 8], the Wigner-Ville distribution in the LCT domain (WDL) [9], and the ambiguity functions in the LCT domain [10] have been investigated recently. The LCT can be used to radar signal processing, communication signal processing, optical signal processing, image encryption, denoising, and so on.

The Heisenberg uncertainty principle associated with one-dimensional FT [11] plays an important role in modern applied mathematical community, and the other kinds of the uncertainty principles, such as the uncertainty principle associated with the classical WVD [11], are well investigated and studied. The Heisenberg uncertainty principle associated with the one-dimensional LCT for real signals is derived firstly in [12], and then Zhao et al. derived the similar results for complex signals [13]. In addition, in [14, 15], Xu et al. derived uncertainty principle of the LCT in three different forms. Recently, based on the relationship of the LCT and the FT, Heisenberg uncertainty principle for the windowed LCT [16] and the two-dimensional nonseparable LCT [17] have been obtained. On the other hand, with the increasing dimension, the calculation of the $n$-dimensional Heisenberg uncertainty principle of the LCT has not been well known.

In this paper, we investigate the uncertainty principle for the Wiger-Ville distribution associated with the linear canonical transform (WDL) in detail. Firstly, we obtain the uncertainty principle of the one-dimensional WDL based on the Moyal identical equation. Then, we derive the Heisenberg uncertainty principle of the $n$-dimensional LCT and obtain the uncertainty principle of the $n$-dimensional WDL. The paper is organized as follows. Section 2 introduces some general definitions and gives some classical Heisenberg uncertainty principles. In Section 3, we calculate the uncertainty principle of the WDL. In Section 4, we calculate the Heisenberg uncertainty principle of the $n$-dimensional LCT and obtain the uncertainty principle of the $n$-dimensional WDL.

## 2. Preliminaries

Before we proceed, some important definitions and results related to the LCT and the Heisenberg uncertainty principles are reviewed in this section.

*2.1. The Linear Canonical Transforms (LCT).* For each $2n \times 2n$ symplectic matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $M^{\tau} J M = J$, $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, the $n$-dimensional LCT [18, 19] is defined as follows:

$$\hat{f}(q) = [\mathscr{C}(M) f](q) = \int_{\mathbb{R}^n} C(M)(q, q') f(q') \, dq', \quad (1)$$

where

$$C(M)(q, q') = \frac{e^{(-in\pi/4)}}{(\sqrt{2\pi})^n \sqrt{\det(B)}} \cdot e^{i(q^T D B^{-1} q/2 - q^T B^{T^{-1}} q' + q'^T B^{-1} A q'/2)}. \quad (2)$$

And the inverse transform is

$$f(q') = [\mathscr{C}(M^{-1}) \hat{f}](q') = \int_{\mathbb{R}^n} C(M^{-1})^* (q, q') \hat{f}(q) \, dq. \quad (3)$$

We frequently use the one-dimensional LCT in signal processing [2] as

$$\begin{aligned} F_{a,b,c,d}(u) &= [\mathscr{C}(M) f(t)](u) \\ &= \begin{cases} \int_{-\infty}^{\infty} f(t) \sqrt{\frac{1}{i2\pi b}} e^{(i/2)((a/b)t^2 - (2/b)ut + (d/b)u^2)} dt, & b \neq 0 \\ \sqrt{d} e^{(i/2)cdu^2} f(du), & b = 0, \end{cases} \end{aligned} \quad (4)$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the parameter matrix of LCT satisfying $ad - bc = 1$; that is, $\det(M) = 1$.

The inverse transform of the one-dimensional LCT (ILCT) is given by the LCT having parameter $A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Hence, the original signal $x(t)$ can be derived from $\mathscr{C}(M)[f](u)$ via

$$\begin{aligned} f(t) &= \mathscr{C}(M^{-1}) [\mathscr{C}(M)[f](u)](t) \\ &= \sqrt{\frac{1}{i2\pi b}} e^{(-ia/2b)t^2} \\ &\quad \times \int_{-\infty}^{+\infty} \mathscr{C}(M)[f](u) e^{-i(d/2b)u^2} \times e^{i(1/b)ut} du. \end{aligned} \quad (5)$$

For more detailed definitions and properties of the LCT, one can refer to [20, 21].

*2.2. The Wigner-Ville Distributions (WVD).* The WVD and the ambiguity function (AF) are important tools for time-frequency analysis in the classical Fourier domain. The WVD of the signals $f(t)$ and $g(t)$ is defined as [11, 22]

$$W_{f,g}(t, u) = \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right) g^*\left(t - \frac{\tau}{2}\right) e^{-i2\pi u\tau} d\tau. \quad (6)$$

And the AF of the signals $f(t)$ and $g(t)$ is defined as

$$\mathrm{AF}_{f,g}(t, u) = \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right) g^*\left(t - \frac{\tau}{2}\right) e^{i2\pi ut} dt. \quad (7)$$

Based on the above definition, Pei and Ding [23] investigated the WVD and AF of the signal $F_{a,b,c,d}(u)$, and Zhao et al. [24] investigated the AF associated with LCT, proposed the following AF in the LCT domain, and gave the following definition:

$$\mathrm{AF}^{(2)}_{F_{a,b,c,d}}(t, u) = \int_{\mathbb{R}} F_{a,b,c,d}\left(t + \frac{\tau}{2}\right) F^*_{a,b,c,d}\left(t - \frac{\tau}{2}\right) e^{-i2\pi ut} dt. \quad (8)$$

Different from the above definition of the WVD associated with the LCT, Bai et al. [9] proposed another kind of definition named the WDL. We have the following definition:

$$\begin{aligned} W^f_M(t, u) &= \sqrt{\frac{1}{2i\pi b}} \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) e^{(i/2b)(du^2 - 2u\tau + a\tau^2)} d\tau, \end{aligned} \quad (9)$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad - bc = 1$. Then, the $n$-dimensional WDL is

$$W^f_M(t, u) = \int_{\mathbb{R}^n} f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) C(M)(u, \tau) \, d\tau, \quad (10)$$

where $C(M)(u, \tau)$ is the integral kernel of the $n$-dimensional LCT.

The AF associated with the linear canonical transform (AFL) [10] is

$$\begin{aligned} \mathrm{AFL}^f_M(t, u) &= \sqrt{\frac{1}{2i\pi b}} \int_{\mathbb{R}} f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) e^{(i/2b)(du^2 - 2ut + at^2)} dt, \end{aligned} \quad (11)$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad - bc = 1$.

For more knowledge of the WVD and the wavelet transforms, one can refer to [22, 25, 26].

*2.3. The Heisenberg Uncertainty Principles.* In this subsection, we review some Heisenberg uncertainty principles. First, the well-known Heisenberg uncertainty principle of the FT [11] is that the product of the variance of $f(t)$ and the variance of $\mathscr{F}(u) = (2\pi)^{-1/2}\int_{\mathbb{R}} f(t)e^{-itu}dt$ is not infinitely small. Suppose that

$$\Delta_t^2 = \frac{\left(\int_{\mathbb{R}}(t-t_0)^2|f(t)|^2dt\right)}{\left(\int_{\mathbb{R}}|f(t)|^2dt\right)}, \tag{12}$$

where $t_0 = (\int_{\mathbb{R}} t|f(t)|^2dt)/(\int_{\mathbb{R}}|f(t)|^2dt)$, and

$$\Delta_u^2 = \frac{\left(\int_{\mathbb{R}}(u-u_0)^2|\mathscr{F}(u)|^2du\right)}{\left(\int_{\mathbb{R}}|\mathscr{F}(u)|^2du\right)}, \tag{13}$$

where $u_0 = (\int_{\mathbb{R}} u|\mathscr{F}(u)|^2du)/(\int_{\mathbb{R}}|\mathscr{F}(u)|^2du)$.

Then we have

$$\Delta_t^2\cdot\Delta_u^2 \geq \frac{1}{4}. \tag{14}$$

The equality holds if and only if $f(t) = Ce^{-(t-t_0)^2/2}$ (where $C\in\mathbb{R}$). The Heisenberg uncertainty principle is useful to analyze the characteristics of a signal.

Based on the above results, in [11] the authors obtained the Heisenberg uncertainty principle of the WVD, and we have

$$\begin{aligned}\int_{\mathbb{R}^2}&\left(|x-a|^2+|u-a|^2\right)W_{f,f}(t,u)\,dt\,du\\&=\int_{\mathbb{R}}|x-a|^2|f(t)|^2dt+\int_{\mathbb{R}}|u-b|^2\left|\widehat{f}(u)\right|^2du\\&\geq\frac{\|f\|_2^2}{2\pi}.\end{aligned} \tag{15}$$

The equality holds if and only if $f(t) = Ce^{2\pi i u_0}e^{-(t-t_0)^2/2}$ (where $C\in\mathbb{R}$), and it means that $W_{f,f}(t,u)$ cannot be too sharply localized.

With the development of the LCT, the Heisenberg uncertainty principle is also extended to the one-dimensional LCT [15]. Suppose that

$$\Delta_t^2 = \frac{\left(\int_{\mathbb{R}}(t-t_0)^2|f(t)|^2dt\right)}{\left(\int_{\mathbb{R}}|f(t)|^2dt\right)}, \tag{16}$$

where $t_0 = (\int_{\mathbb{R}} t|f(t)|^2dt)/(\int_{\mathbb{R}}|f(t)|^2dt)$, and

$$\Delta_u^2 = \frac{\left(\int_{\mathbb{R}}(u-u_0)^2\left|\widehat{f}(u)\right|^2du\right)}{\left(\int_{\mathbb{R}}\left|\widehat{f}(u)\right|^2du\right)}, \tag{17}$$

where $u_0 = (\int_{\mathbb{R}} u|\widehat{f}(u)|^2du)/(\int_{\mathbb{R}}|\widehat{f}(u)|^2du)$.

Then we have

$$\Delta_t^2\cdot\Delta_u^2 \geq \frac{b^2}{4}. \tag{18}$$

Furthermore, if $f(t)$ is a real signal, the Heisenberg uncertainty principle of the LCT satisfies [12]

$$\Delta_t^2\cdot\Delta_u^2 \geq \frac{b^2}{4}+\left(a\Delta_t^2\right)^2. \tag{19}$$

In addition to the above uncertainty principles, there are the logarithm uncertainty principle and the entropy uncertainty principle, and one can find in [21].

## 3. The Main Results

*3.1. Uncertainty Principles for the WDL and the AFL.* It is shown in [9] that the WDL can be looked as the generalization of the classical WVD and can also be thought as the affine transform of the autocorrelation function of $f(x)$ in the time-frequency plane. The associated Moyal identical equations are obtained as [9]

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|W_M^f(t,u)\right|^2dt\,du = \|f\|_2^4\\&\int_{\mathbb{R}^2}\left|W_M^{\widehat{f}}(t,u)\right|^2dt\,du = \left\|\widehat{f}\right\|_2^4.\end{aligned} \tag{20}$$

We can regard $W_M^f(t,u)$ as a function of the time domain and $W_M^{\widehat{f}}(t,u)$ as a function of the frequency domain, and then based on the above equation we obtain the following.

**Theorem 1.** *Suppose that $f(t)\in L^2(\mathbb{R})$, $\widehat{f} = [\mathscr{C}(M)f](u)$, $M\in SL(2) = Sp(2,\mathbb{R})$, and $\|f\|_2 = 1$. Then the following inequality is satisfied:*

$$\begin{aligned}\int_{\mathbb{R}^2}&(t-t_0)^2\left|W_M^f(t,u)\right|^2dt\,du\\&+\int_{\mathbb{R}^2}(u-u_0)^2\left|W_M^{\widehat{f}}(t,u)\right|^2dt\,du\geq\frac{|b|}{4},\end{aligned} \tag{21}$$

*where $t_0 = \int_{\mathbb{R}} t|f(t)|^2dt$ and $u_0 = \int_{\mathbb{R}} u|\widehat{f}(u)|^2du$.*

*Proof.* Firstly, assume that $t_0 = 0$ and $u_0 = 0$; thus the inequality becomes

$$\int_{\mathbb{R}^2}t^2\left|W_M^f(t,u)\right|^2dt\,du+\int_{\mathbb{R}^2}u^2\left|W_M^{\widehat{f}}(t,u)\right|^2dt\,du\geq\frac{|b|}{4}. \tag{22}$$

Depending on the parameter $b$, the LCT has two different expressions. First, if $b\neq0$, then we have

$$\begin{aligned}\int_{\mathbb{R}^2}&t^2\left|W_M^f(t,u)\right|^2dt\,du+\int_{\mathbb{R}^2}u^2\left|W_M^{\widehat{f}}(t,u)\right|^2dt\,du\\&=\int_{\mathbb{R}^2}\frac{t^2}{2\pi|b|}\int_{\mathbb{R}}f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right)e^{(i/2b)(-2u\tau+a\tau^2)}d\tau\\&\quad\times\int_{\mathbb{R}}f^*\left(t+\frac{\tau'}{2}\right)f\left(t-\frac{\tau'}{2}\right)e^{(i/2b)(2u\tau'-a(\tau')^2)}d\tau'dt\,du\\&\quad+\int_{\mathbb{R}^2}\frac{u^2}{2\pi|b|}\int_{\mathbb{R}}\widehat{f}\left(u+\frac{\tau}{2}\right)\widehat{f}^*\left(u-\frac{\tau}{2}\right)e^{(i/2b)(-2t\tau+a\tau^2)}d\tau\\&\quad\times\int_{\mathbb{R}}\widehat{f}^*\left(u+\frac{\tau'}{2}\right)\widehat{f}\left(u-\frac{\tau'}{2}\right)e^{(i/2b)(2t\tau'-a(\tau')^2)}d\tau'dt\,du\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \frac{t^2}{2\pi |b|} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{(ia\tau^2/2b)} d\tau \\
&\qquad \times f^*\left(t+\frac{\tau'}{2}\right) f\left(t-\frac{\tau'}{2}\right) e^{-ia(\tau')^2/2b} d\tau' \\
&\quad \times \int_{\mathbb{R}} e^{iu(\tau'-\tau)/b} du\, dt \\
&\quad + \int_{\mathbb{R}^3} \frac{u^2}{2\pi |b|} \widehat{f}\left(u+\frac{\tau}{2}\right) \widehat{f}^*\left(u-\frac{\tau}{2}\right) e^{ia\tau^2/2b} d\tau \\
&\quad \times \widehat{f}^*\left(u+\frac{\tau'}{2}\right) \widehat{f}\left(u-\frac{\tau'}{2}\right) e^{-ia(\tau')^2/2b} d\tau' \\
&\quad \times \int_{\mathbb{R}} e^{it(\tau'-\tau)/b} dt\, du \\
&= \int_{\mathbb{R}^3} t^2 f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{ia\tau^2/2b} \\
&\qquad \times \delta\left(\tau-\tau'\right) f^*\left(t+\frac{\tau'}{2}\right) f\left(t-\frac{\tau'}{2}\right) \\
&\qquad \times e^{-ia(\tau')^2/2b} d\tau\, d\tau'\, dt \\
&\quad + \int_{\mathbb{R}^3} u^2 \widehat{f}\left(u+\frac{\tau}{2}\right) \widehat{f}^*\left(u-\frac{\tau}{2}\right) e^{ia\tau^2/2b} \\
&\qquad \times \delta\left(\tau-\tau'\right) \widehat{f}^*\left(u+\frac{\tau'}{2}\right) \widehat{f}\left(u-\frac{\tau'}{2}\right) \\
&\qquad \times e^{-ia(\tau')^2/2b} d\tau\, d\tau'\, du \\
&= \int_{\mathbb{R}^2} t^2 \left| f\left(t+\frac{\tau}{2}\right)\right|^2 \left| f^*\left(t-\frac{\tau}{2}\right)\right|^2 d\tau\, dt \\
&\quad + \int_{\mathbb{R}^2} u^2 \left| \widehat{f}\left(u+\frac{\tau}{2}\right)\right|^2 \left| \widehat{f}^*\left(u-\frac{\tau}{2}\right)\right|^2 d\tau\, du \\
&= \int_{\mathbb{R}^2} \left(\frac{t+\tau/2+t-\tau/2}{2}\right)^2 \left| f\left(t+\frac{\tau}{2}\right)\right|^2 \\
&\qquad \times \left| f^*\left(t-\frac{\tau}{2}\right)\right|^2 d\tau\, dt \\
&\quad + \int_{\mathbb{R}^2} \left(\frac{u+\tau/2+u-\tau/2}{2}\right)^2 \left| \widehat{f}\left(u+\frac{\tau}{2}\right)\right|^2 \\
&\qquad \times \left| \widehat{f}^*\left(u-\frac{\tau}{2}\right)\right|^2 d\tau\, du.
\end{aligned} \tag{23}
$$

Let $x = u + \tau/2$ and let $y = u - \tau/2$, and then we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} t^2 \left| W_M^f(t,u)\right|^2 dt\, du + \int_{\mathbb{R}^2} u^2 \left| W_M^{\widehat{f}}(t,u)\right|^2 dt\, du \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{x+y}{2}\right)^2 |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \left(\frac{w+v}{2}\right)^2 |\widehat{f}(w)|^2 |\widehat{f}(v)|^2 dw\, dv \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{x^2+y^2+2xy}{4}\right) |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \left(\frac{w^2+v^2+2wv}{4}\right) |\widehat{f}(w)|^2 |\widehat{f}(v)|^2 dw\, dv \\
&\quad = \int_{\mathbb{R}^2} \frac{x^2}{4} |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \frac{y^2}{4} |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \frac{xy}{2} |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \frac{w^2}{4} |\widehat{f}(w)\widehat{f}(v)|^2 dw\, dv \\
&\qquad + \int_{\mathbb{R}^2} \frac{v^2}{4} |\widehat{f}(w)\widehat{f}(v)|^2 dw\, dv \\
&\qquad + \int_{\mathbb{R}^2} \frac{wv}{2} |\widehat{f}(w)\widehat{f}(v)|^2 dw\, dv \\
&\quad = \int_{\mathbb{R}} \frac{x^2}{2} |f(x)|^2 dx + \int_{\mathbb{R}} \frac{w^2}{2} |\widehat{f}(w)|^2 dw \\
&\quad \geq \left(\int_{\mathbb{R}} \frac{x^2}{2} |f(x)|^2 dx \cdot \int_{\mathbb{R}} \frac{w^2}{2} |\widehat{f}(w)|^2 dw\right)^{1/2}.
\end{aligned} \tag{24}
$$

This is the uncertainty principle of the LCT, and we know that this inequality must be $\geq$ $|b|/4$. And the inequality achieves the minimum $|b|/4$ if and only if $f = e^{2\pi i\sqrt{bx}} e^{-\nu x^2}$, $\nu > 0$.

If $b = 0$, then $\widehat{f} = \sqrt{d} f(du) e^{i(cau^2/2)}$, and hence

$$
\begin{aligned}
&\int_{\mathbb{R}^2} t^2 \left| W_M^f(t,u)\right|^2 dt\, du + \int_{\mathbb{R}^2} u^2 \left| W_M^{\widehat{f}}(t,u)\right|^2 dt\, du \\
&\quad = \int_{\mathbb{R}^2} t^2 |d| \left| f\left(t+\frac{du}{2}\right)\right|^2 \left| f\left(t-\frac{du}{2}\right)\right|^2 dt\, du \\
&\qquad + \int_{\mathbb{R}^2} u^2 |d| \left| \widehat{f}\left(u+\frac{dt}{2}\right)\right|^2 \left| \widehat{f}\left(u-\frac{dt}{2}\right)\right|^2 dt\, du \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{x+y}{2}\right)^2 |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \left(\frac{w+v}{2}\right)^2 |\widehat{f}(w)|^2 |\widehat{f}(v)|^2 dw\, dv \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{x+y}{2}\right)^2 |f(x)|^2 |f(y)|^2 dx\, dy \\
&\qquad + \int_{\mathbb{R}^2} \left(\frac{w+v}{2}\right)^2 |d|^2 |f(d\cdot w)|^2 |f(d\cdot v)|^2 dw\, dv \\
&\quad = \frac{1}{2}\left(1+\frac{1}{|d|^2}\right) \int_{\mathbb{R}} t^2 |f(t)|^2 dt \geq 0.
\end{aligned} \tag{25}
$$

The inequality achieves the minimum 0 if and only if the variance of $f$ is zero.

Secondly, if $t_0 \neq 0$ and $u_0 \neq 0$, for $b \neq 0$, then we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (t-t_0)^2 \left|W_M^f(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} (u-u_0)^2 \left|W_M^{\hat{f}}(t,u)\right|^2 dt\,du \\
&= \int_{\mathbb{R}} \frac{(x-t_0)^2}{2} |f(x)|^2 dx + \int_{\mathbb{R}} \frac{(w-u_0)^2}{2} \left|\hat{f}(w)\right|^2 dw \\
&\geq \left( \int_{\mathbb{R}} \frac{(x-t_0)^2}{2} |f(x)|^2 dx \cdot \int_{\mathbb{R}} \frac{(w-u_0)^2}{2} \left|\hat{f}(w)\right|^2 dw \right)^{1/2} \\
&\geq \frac{|b|}{4}.
\end{aligned} \tag{26}$$

And, for $b = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (t-t_0)^2 \left|W_M^f(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} (u-u_0)^2 \left|W_M^{\hat{f}}(t,u)\right|^2 dt\,du \\
&= \int_{\mathbb{R}^2} \left(\frac{x-t_0+y-t_0}{2}\right)^2 |f(x)|^2 |f(y)|^2 dx\,dy \\
&\quad + \int_{\mathbb{R}^2} \left(\frac{w-u_0+v-u_0}{2}\right)^2 |d|^2 \\
&\quad \times |f(d\cdot w)|^2 |f(d\cdot v)|^2 dw\,dv \geq 0.
\end{aligned} \tag{27}$$

When $b \to 0$, in case of $b \neq 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} (t-t_0)^2 \left|W_M^f(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} (u-u_0)^2 \left|W_M^{\hat{f}}(t,u)\right|^2 dt\,du \geq 0.
\end{aligned} \tag{28}$$

Hence for both cases we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} (t-t_0)^2 \left|W_M^f(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} (u-u_0)^2 \left|W_M^{\hat{f}}(t,u)\right|^2 dt\,du \geq \frac{|b|}{4}.
\end{aligned} \tag{29}$$

This completes the proof of this theorem. □

When $M = \left(\begin{smallmatrix} 0 & 1/2\pi \\ -2\pi & 0 \end{smallmatrix}\right)$, then we have

$$W_M^f(t,u) = \sqrt{\frac{1}{i}} \int_{\mathbb{R}} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{-2\pi i u\tau} d\tau. \tag{30}$$

This is the WVD; hence we obtain a new uncertainty relation for the WVD.

**Corollary 2.** *Suppose that $f(t) \in L^2(\mathbb{R})$, $\|f\|_2 = 1$. Then the following inequality is satisfied:*

$$\begin{aligned}
&\int_{\mathbb{R}^2} (t-t_0)^2 \left|W_{f,f}(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} (u-u_0)^2 \left|W_{\mathcal{F}(f),\mathcal{F}(f)}(t,u)\right|^2 dt\,du \geq \frac{1}{8\pi},
\end{aligned} \tag{31}$$

*where $t_0 = \int_{\mathbb{R}} t|f(t)|^2 dt$ and $u_0 = \int_{\mathbb{R}} u\left|\hat{f}(u)\right|^2 du$.*

From the proof, one can find that the essence of this uncertainty principle is the Moyal identical equation, and the Moyal identical equations are also correct for the AF and the AFL [10]; hence we also obtain the uncertainty principle of the AFL as follows.

**Theorem 3.** *Suppose that $f \in L^2(\mathbb{R})$, $M \in SL(2)$, $\|f\|_2 = 1$, and both $t_0 = \int_{\mathbb{R}} t|f(t)|^2 dt$ and $u_0 = \int_{\mathbb{R}} u\left|\hat{f}(u)\right|^2 du$ exist. Then the following inequality is satisfied:*

$$\begin{aligned}
&\int_{\mathbb{R}^2} (t-t_0)^2 \left|AFL_M^f(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} (u-u_0)^2 \left|AFL_M^{\hat{f}}(t,u)\right|^2 dt\,du \geq \frac{|b|}{4}.
\end{aligned} \tag{32}$$

The proof is similar to Theorem 1. Denoting

$$\begin{aligned}
W_M^{f,g}(t,u) = {}& \sqrt{\frac{1}{2i\pi b}} \int_{\mathbb{R}} f\left(t+\frac{\tau}{2}\right) g^*\left(t-\frac{\tau}{2}\right) \\
&\times e^{(i/2b)(du^2-2u\tau+a\tau^2)} d\tau
\end{aligned} \tag{33}$$

we obtain the following.

**Theorem 4.** *Suppose that, if $f_1(t), f_2(t) \in L^2(\mathbb{R})$, $\hat{f}_i = [\mathscr{C}(M)f_i](u)$, $(i = 1,2)$, $M \in SL(2)$, $\|f_i\|_2 = 1$, and both $t_i = \int_{\mathbb{R}} t|f_i(t)|^2 dt$ and $u_i = \int_{\mathbb{R}} u\left|\hat{f}_i(u)\right|^2 du$ exist, the following inequality is satisfied:*

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(t-\frac{t_1+t_2}{2}\right)^2 \left|W_M^{f_1,f_2}(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} \left(u-\frac{u_1+u_2}{2}\right)^2 \left|W_M^{\hat{f}_1,\hat{f}_2}(t,u)\right|^2 dt\,du \geq \frac{|b|}{4}.
\end{aligned} \tag{34}$$

*Proof.* For the case of $b \neq 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(t-\frac{t_1+t_2}{2}\right)^2 \left|W_M^{f_1,f_2}(t,u)\right|^2 dt\,du \\
&\quad + \int_{\mathbb{R}^2} \left(u-\frac{u_1+u_2}{2}\right)^2 \left|W_M^{\hat{f}_1,\hat{f}_2}(t,u)\right|^2 dt\,du \\
&= \int_{\mathbb{R}^2} \left(\frac{t+\tau/2+t-\tau/2}{2} - \frac{t_1+t_2}{2}\right)^2 \\
&\quad \times \left|f_1\left(t+\frac{\tau}{2}\right)\right|^2 \left|f_2^*\left(t-\frac{\tau}{2}\right)\right|^2 d\tau dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^2}\left(\frac{u+\tau/2+u-\tau/2}{2}-\frac{u_1+u_2}{2}\right)^2 \\
&\quad\times\left|\hat{f}_1\left(u+\frac{\tau}{2}\right)\right|^2\left|\hat{f}_2^*\left(u-\frac{\tau}{2}\right)\right|^2 d\tau \\
&=\int_{\mathbb{R}}\frac{(x-t_1)^2}{4}\left|f_1(x)\right|^2+\frac{(x-t_2)^2}{4}\left|f(x)_2\right|^2 dx \\
&\quad+\int_{\mathbb{R}}\frac{(w-u_1)^2}{4}\left|\hat{f}_1(w)\right|^2+\frac{(w-u_2)^2}{4}\left|\hat{f}_2(w)\right|^2 dw \\
&\geq\frac{|b|}{4}.
\end{aligned}
\tag{35}
$$

And for the case of $b=0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(t-\frac{t_1+t_2}{2}\right)^2\left|W_M^{f_1,f_2}(t,u)\right|^2 dt\,du \\
&\quad+\int_{\mathbb{R}^2}\left(u-\frac{u_1+u_2}{2}\right)^2\left|W_M^{\hat{f}_1,\hat{f}_2}(t,u)\right|^2 dt\,du \\
&=\int_{\mathbb{R}^2}\left(\frac{x-t_1+y-t_2}{2}\right)^2\left|f_1(x)\right|^2\left|f_2(y)\right|^2 dx\,dy \\
&\quad+\int_{\mathbb{R}^2}\left(\frac{w-u_1+v-u_2}{2}\right)^2 d^2\left|f_1(d\cdot w)\right|^2\left|f_2(d\cdot v)\right|^2 dw\,dv \\
&\geq 0.
\end{aligned}
\tag{36}
$$

When $b\rightarrow 0$, in case of $b\neq 0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(t-\frac{t_1+t_2}{2}\right)^2\left|W_M^{f_1,f_2}(t,u)\right|^2 dt\,du \\
&\quad+\int_{\mathbb{R}^2}\left(u-\frac{u_1+u_2}{2}\right)^2\left|W_M^{\hat{f}_1,\hat{f}_2}(t,u)\right|^2 dt\,du\geq 0.
\end{aligned}
\tag{37}
$$

□

Therefore, we finish the proof of Theorem 4. From Theorem 4, we know that the lower bound of this uncertainty principle is only related to $M$.

Next, when we use $\hat{f}=[\mathscr{C}(M_1)f](u)$, $M_1=\left(\begin{smallmatrix}a_1 & b_1\\ c_1 & d_1\end{smallmatrix}\right)\neq M_2=\left(\begin{smallmatrix}a_2 & b_2\\ c_2 & d_2\end{smallmatrix}\right)$, $M_1,M_2\in SL(2)$, we obtain the following.

**Theorem 5.** *Suppose that, if $f(t)\in L^2(\mathbb{R})$, $\hat{f}=[\mathscr{C}(M_1)f](u)$, $\|f\|_2=1$, and both $t_0=\int_{\mathbb{R}}t|f(t)|^2dt$ and $u_0=\int_{\mathbb{R}}u|\hat{f}(u)|^2du$ exist, the following inequality is satisfied:*

$$
\begin{aligned}
&\int_{\mathbb{R}^2}(t-t_0)^2\left|W_{M_2}^{f}(t,u)\right|^2 dt\,du \\
&\quad+\int_{\mathbb{R}^2}(u-u_0)^2\left|W_{M_2}^{\hat{f}}(t,u)\right|^2 dt\,du\geq\frac{|b_1|}{4}.
\end{aligned}
\tag{38}
$$

The proof is similar to Theorem 4. Theorem 5 implies that the minimum of this inequality is determined only by $M_1$.

If let $f(x)\in L^2(\mathbb{R}^n)$, we can also obtain the similar result, but we need the Heisenberg uncertainty principles of the $n$-dimensional LCT. However, so far, there is no result about the Heisenberg uncertainty principles of the $n$-dimensional LCT; hence in the following subsection we calculate the Heisenberg uncertainty principles of the $n$-dimensional LCT.

*3.2. The Heisenberg Uncertainty Principles of the n-Dimensional LCT.* In this subsection, we calculate the Heisenberg uncertainty principle of the $n$-dimensional LCT. Our idea is to convert the LCT to the FT, and then we use the Heisenberg uncertainty principle of the one-dimensional FT to obtain the Heisenberg uncertainty principle of the $n$-dimensional LCT. Through calculating the Heisenberg uncertainty principle of the $n$-dimensional LCT, we see that the uncertainty principles of the $n$-dimensional LCT are essentially the uncertainty principles of the $n$-dimensional FT, since the decision effected in the LCT is the FT. We will obtain the following.

**Theorem 6.** *Suppose that $f(t_1,\ldots,t_n)\in L^2(\mathbb{R}^n)$ and $M\in Sp(2n,\mathbb{R})$. Then one has*

$$
\begin{aligned}
\Delta_t^2\cdot\Delta_u^2&=\frac{\int_{\mathbb{R}^n}(t-t_0)^{\mathsf{T}}(t-t_0)\left|f(t)\right|^2dt}{\int_{\mathbb{R}^n}\left|f(t)\right|^2dt} \\
&\quad\cdot\frac{\int_{\mathbb{R}^n}(u-u_0)^{\mathsf{T}}(u-u_0)\left|\hat{f}(u)\right|^2du}{\int_{\mathbb{R}^n}\left|\hat{f}(u)\right|^2du} \\
&\geq\left(\frac{\sqrt{\lambda_1}}{2}+\cdots+\frac{\sqrt{\lambda_n}}{2}\right)^2,
\end{aligned}
\tag{39}
$$

*where $t_0=(\int_{\mathbb{R}^n}t_1|f(t)|^2dt,\ldots,\int_{\mathbb{R}^n}t_n|f(t)|^2dt)^{\mathsf{T}}$, $u_0=(\int_{\mathbb{R}^n}u_1|\hat{f}(u)|^2du,\ldots,\int_{\mathbb{R}^n}u_n|\hat{f}(u)|^2du)^{\mathsf{T}}$, and $\lambda_i'$s are the eigenvalues of $B^{\mathsf{T}}B$.*

*Proof.* Here we assume that $\int_{\mathbb{R}^n}|f(t)|^2dt=1$; then by the Parseval identical equation, we have that $\|\hat{f}\|_2=\|f\|_2=1$. Because the LCT has the time shifting property, we only need to discuss $t_0=0$, $u_0=0$. Thus we only need to prove the following:

$$
\begin{aligned}
\Delta_t^2\cdot\Delta_u^2&=\int_{\mathbb{R}^n}t^{\mathsf{T}}t\left|f(t)\right|^2dt \\
&\quad\cdot\int_{\mathbb{R}^n}u^{\mathsf{T}}u\left|\hat{f}(u)\right|^2du\geq\left(\frac{\sqrt{\lambda_1}}{2}+\cdots+\frac{\sqrt{\lambda_n}}{2}\right)^2.
\end{aligned}
\tag{40}
$$

When selecting different $B$, we have different expressions of the LCT. Therefore, we need to discuss different cases. For the case of $\det(B)\neq 0$, we have

$$
\begin{aligned}
&\Delta_t^2\cdot\Delta_u^2 \\
&=\int_{\mathbb{R}^n}t^{\mathsf{T}}t\left|f(t)\right|^2dt\cdot\int_{\mathbb{R}^n}u^{\mathsf{T}}u\left|\hat{f}(u)\right|^2du \\
&=\int_{\mathbb{R}^n}x^{\mathsf{T}}x\left|f(x)\right|^2dx
\end{aligned}
$$

$$\cdot \int_{\mathbb{R}^n} u^\top u \left| \int_{\mathbb{R}^n} f(x) \frac{e^{(-in\pi/4)}}{(\sqrt{2\pi})^n \sqrt{\det(B)}} \times e^{(-iu^\top B^{\top -1} x + i(x^\top B^{-1} Ax/2))} dx \right|^2 du$$

$$= \int_{\mathbb{R}^n} x^\top x |f(x)|^2 dx \cdot \int_{\mathbb{R}^n} v^\top B^\top B v \frac{1}{(2\pi)^n} \times \left| \int_{\mathbb{R}^n} f(x) e^{(-iv^{-1}x + i(x^\top B^{-1} Ax/2))} dx \right|^2 dv$$

$$= \int_{\mathbb{R}^n} x^\top x \left| \widetilde{f(x)} \right|^2 dx \cdot \int_{\mathbb{R}^n} v^\top B^\top B v \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \widetilde{f(x)} e^{-iv^{-1}x} dx \right|^2 dv. \quad (41)$$

Notice that $B^\top B$ is symmetric; then there exists an orthogonal matrix $P$ so that $B^\top B = P^\top \Lambda P$, where $\lambda_i's$ are the eigenvalues of $B^\top B$ and $\lambda_i's$ are nonnegative. As a result, we have

$$\Delta_t^2 \cdot \Delta_u^2$$

$$= \int_{\mathbb{R}^n} x^\top x \left| \widetilde{f(x)} \right|^2 dx \cdot \int_{\mathbb{R}^n} v^\top P^\top \Lambda P v \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \widetilde{f(x)} e^{-iv^{-1}x} dx \right|^2 dv$$

$$= \int_{\mathbb{R}^n} x^\top x \left| \widetilde{f(x)} \right|^2 dx \cdot \int_{\mathbb{R}^n} \omega^\top \Lambda \omega \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \widetilde{f(x)} e^{-i\omega^\top P x} dx \right|^2 d\omega$$

$$= \int_{\mathbb{R}^n} y^\top y \left| \widetilde{f(P^\top y)} \right|^2 dy \cdot \int_{\mathbb{R}^n} \omega^\top \Lambda \omega \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \widetilde{f(P^\top y)} e^{-i\omega^\top y} dy \right|^2 d\omega$$

$$= \int_{\mathbb{R}^n} \left( y_1^2 + \cdots + y_n^2 \right) |h(y)|^2 dy \cdot \int_{\mathbb{R}^n} \left( \lambda_1 \omega_1^2 + \cdots + \lambda_n \omega_n^2 \right) \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} h(y) e^{-i\omega^\top y} dy \right|^2 d\omega$$

$$= \int_{\mathbb{R}^n} \left( y_1^2 + \cdots + y_n^2 \right) |h(y)|^2 dy \times \int_{\mathbb{R}^n} \left( \lambda_1 \left| \omega_1 \int_{\mathbb{R}^n} h(y) e^{-2\pi i\omega^\top y} dy \right|^2 + \cdots + \lambda_n \left| \omega_n \int_{\mathbb{R}^n} h(y) e^{-2\pi i\omega^\top y} dy \right|^2 \right) d\omega$$

$$= \int_{\mathbb{R}^n} \left( y_1^2 + \cdots + y_n^2 \right) |h(y)|^2 dy \times \int_{\mathbb{R}^n} \left( \lambda_1 \left| \int_{\mathbb{R}^n} h_1(y) e^{-i2\pi\omega^\top y} dy \right|^2 + \cdots + \lambda_n \left| \int_{\mathbb{R}^n} h_n(y) e^{-2\pi i\omega^\top y} dy \right|^2 \right) d\omega. \quad (42)$$

By using the Cauchy inequality, we get

$$\begin{aligned} \Delta_t^2 \cdot \Delta_u^2 &\geq \left( \int_{\mathbb{R}^n} \left( \left| y_1 h(y) \sqrt{\lambda_1} h_1^*(y) \right| + \cdots + \left| y_n h(y) \sqrt{\lambda_n} h_n^*(y) \right| \right) dy \right)^2 \\ &= \left( \sqrt{\lambda_1} \int_{\mathbb{R}^n} \left| y_1 h(y) h_1^*(y) \right| dy + \cdots + \sqrt{\lambda_n} \times \int_{\mathbb{R}^n} \left| y_n h(y) h_n^*(y) \right| dy \right)^2 \\ &= \left( \frac{\sqrt{\lambda_1}}{2} + \cdots + \frac{\sqrt{\lambda_n}}{2} \right)^2, \end{aligned} \quad (43)$$

where $\widetilde{f(x)} = f(x) e^{i(x^\top B^{-1} Ax/2)}$, $h(x) = \widetilde{f(P^\top x)}$, $h_i(y) = (\partial h(y)/\partial y_i)$. This inequality achieves the minimum $(\sqrt{\lambda_1}/2 + \cdots + \sqrt{\lambda_n}/2)^2$ if and only if $h = e^{-\sum \nu_i x_i^2}$ $(\nu > 0)$.

Here we have omitted some steps in the proof, and if one is familiar with the proof of the Heisenberg uncertainty principle of the FT, one can obviously see the result. Next, we discuss the case of $\det(B) = 0$. First, when $B = 0$, by using $\widehat{f}(u) = \sqrt{\det(D)} f(Du) e^{i(u^\top C(A^\top)^{-1} u/2)}$, we have

$$\begin{aligned} \Delta_x^2 \cdot \Delta_u^2 &= \int_{\mathbb{R}^n} x^\top x |f(x)|^2 dx \cdot \int_{\mathbb{R}^n} u^\top u \left| \sqrt{\det(D)} f(Du) e^{i(u^\top C(A^\top)^{-1} u/2)} \right|^2 du \\ &= \int_{\mathbb{R}^n} x^\top x |f(x)|^2 dx \cdot \int_{\mathbb{R}^n} u^\top u \left| \sqrt{\det(D)} f(Du) \right|^2 du \\ &\geq \lambda \left( \int_{\mathbb{R}^n} x^\top x |f(x)|^2 dx \right)^2 \geq 0, \end{aligned} \quad (44)$$

where $\lambda$ is the minimum eigenvalue of $(D^{-1})^\top D^{-1}$ and the inequality gets the minimum 0 if and only if the variance of $f$ is 0. The Heisenberg uncertainty principle can be zero; the reason is that the LCT is only a scaling transform.

For the case of $\det(B) = 0$ but $B \neq 0$, we see that $B = U \begin{pmatrix} \Lambda_m & 0 \\ 0 & 0 \end{pmatrix} V$, $U, V \in SO(n)$, $m = \operatorname{rank}(B)$. Similarly, as the proof of the case of $\det(B) \neq 0$, we have $\Delta_t^2 \cdot \Delta_u^2 \geq (\sqrt{\lambda_1}/2 + \cdots + \sqrt{\lambda_m}/2)^2$, where $\lambda_i's$ are the nonzero eigenvalues of $B^{\top}B$.

If $B \to 0$, then we have that $\Delta_t^2 \cdot \Delta_u^2 \geq 0$; hence we obtain

$$\Delta_t^2 \cdot \Delta_u^2 \geq \left(\frac{\sqrt{\lambda_1}}{2} + \cdots + \frac{\sqrt{\lambda_n}}{2}\right)^2. \quad (45)$$

When $n = 1$, we see that $\Delta_t^2 \cdot \Delta_u^2 \geq (\sqrt{\lambda}/2)^2$, where $\lambda = b^2$. This just is the Heisenberg uncertainty principle of the one-dimensional LCT. □

We have finished the Heisenberg uncertainty principle of the $n$-dimensional LCT, and this uncertainty principle is also called the Heisenberg-Weyl inequality.

*3.3. The Heisenberg Uncertainty Principles of the n-Dimensional WDL.* By Theorem 6, now we can obtain the Heisenberg uncertainty principles of the $n$-dimensional WDL.

**Theorem 7.** *Suppose that* $f(t) \in L^2(\mathbb{R}^n)$, $\hat{f} = [\mathscr{C}(M)f](u)$, $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $\det(B) \neq 0$, $\|f\|_2^2 = 1$, *and both* $t_0 = (\int_{\mathbb{R}} t_1 |f(t)|^2 dt, \ldots, \int_{\mathbb{R}} t_n |f(t)|^2 dt)^{\top}$ *and* $u_0 = (\int_{\mathbb{R}} u_1 |\hat{f}(u)|^2 du, \ldots, \int_{\mathbb{R}} u_n |\hat{f}(u)|^2 du)^{\top}$ *exist. Then the following inequality is satisfied:*

$$\int_{\mathbb{R}^{2n}} (t - t_0)^{\top} (t - t_0) \left|W_M^f(t,u)\right|^2 dt\, du + \int_{\mathbb{R}^{2n}} (u - u_0)^{\top} (u - u_0) \left|W_M^{\hat{f}}(t,u)\right|^2 dt\, du \geq \frac{\sqrt{\lambda_1}}{4} + \cdots + \frac{\sqrt{\lambda_n}}{4}, \quad (46)$$

*where* $\lambda_i's$ *are the eigenvalues of* $B^{\top}B$.

*Proof.* Because of $\det(B) \neq 0$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} (t - t_0)^{\top} (t - t_0) \left|W_M^f(t,u)\right|^2 dt\, du + \int_{\mathbb{R}^{2n}} (u - u_0)^{\top} (u - u_0) \left|W_M^{\hat{f}}(t,u)\right|^2 dt\, du \\
&= \int_{\mathbb{R}^{2n}} (t - t_0)^{\top} (t - t_0) \times \left|\int_{\mathbb{R}^n} f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) C(M)(u,\tau)\, d\tau\right|^2 dt\, du \\
&\quad + \int_{\mathbb{R}^{2n}} (u - u_0)^{\top} (u - u_0) \times \left|\int_{\mathbb{R}^n} \hat{f}\left(u + \frac{\tau}{2}\right) \hat{f}^*\left(u - \frac{\tau}{2}\right) C(M)(u,\tau)\, d\tau\right|^2 dt\, du \\
&= \int_{\mathbb{R}^{2n}} (t - t_0)^{\top} (t - t_0) \left|f\left(t + \frac{\tau}{2}\right)\right|^2 \left|f^*\left(t - \frac{\tau}{2}\right)\right|^2 dt\, du \\
&\quad + \int_{\mathbb{R}^{2n}} (u - u_0)^{\top} (u - u_0) \left|\hat{f}\left(u + \frac{\tau}{2}\right)\right|^2 \times \left|\hat{f}^*\left(u - \frac{\tau}{2}\right)\right|^2 dt\, du \\
&\geq \left(\int_{\mathbb{R}^{2n}} (t - t_0)^{\top} (t - t_0) \left|f\left(t + \frac{\tau}{2}\right)\right|^2 \times \left|f^*\left(t - \frac{\tau}{2}\right)\right|^2 dt\, du\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^{2n}} (u - u_0)^{\top} (u - u_0) \left|\hat{f}\left(u + \frac{\tau}{2}\right)\right|^2 \times \left|\hat{f}^*\left(u - \frac{\tau}{2}\right)\right|^2 dt\, du\right)^{1/2} \\
&\geq \frac{\sqrt{\lambda_1}}{4} + \cdots + \frac{\sqrt{\lambda_n}}{4}.
\end{aligned} \quad (47)$$

This uncertainty principle is based on the Moyal identical equation, which can be regarded as the inner product of the WDL, and it shows that the WDL of a signal in the time domain may be sharply localized. However, the WDL of its LCT in the frequency domain cannot be sharply localized simultaneously. □

## 4. Conclusion

In this paper, we first establish an uncertainty principle for the one-dimensional WDL, then we obtain the Heisenberg uncertainty principle of the $n$-dimensional LCT, and furthermore we obtain the uncertainty principle of the $n$-dimensional WDL. Although the $n$-dimensional WDL has $4n^2$ parameters, the lower bound of the uncertainty principle of the $n$-dimensional WDL only depends on $B$, and we also discuss the case of $B = 0$. The uncertainty principle of the WDL is different from the uncertainty principle of the WVD (18), while it reveals the uncertainty relations of $W_M^f(t,u)$ and $W_M^{\hat{f}}(t,u)$. The applications of the derived Heisenberg uncertainty principle of the WDL and the AFL will be studied in our future papers.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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