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Research Article

Geometric Singularities of the Stokes Problem

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When the domain is a polygon of $\mathbb{R}^2$, the solution of a partial differential equation is written as a sum of a regular part and a linear combination of singular functions. The purpose of this paper is to present explicitly the singular functions of Stokes problem. We prove the Kondratiev method in the case of the crack. We finish by giving some regularity results.

1. Introduction

The regularity of the solution of a partial differential equation depends on the geometry of the domain even when the data is smooth. Indeed, for each corner of the polygonal domain a countable family of singular functions can be defined, which depends only on the geometry of the domain. Then the solution of the equation can be written as the sum of a finite number of singular functions multiplied by appropriate coefficients and of a much more regular part. We refer to Kondratiev [1] and Grisvard [2] for their description.

The purpose of our work is to study the singularities of the Stokes equation and the behavior of the solution in the neighborhood of a corner of a polygonal domain of $\mathbb{R}^2$. We are interested to nonconvex domains; we assume that there exists an angle equal either to $3\pi/2$ or to 2π (case of the crack). Handling the singular function is local process, so that there is no restriction to suppose that the nonconvex corner is unique; see Dauge [3]. We deduce the singular function of the velocity from those of the bilaplacian problem with a homogenous boundary conditions by applying the curl operator. We prove the Kondratiev method in the case of the crack. The singularities of the pressure are done by integration from the singular functions of the velocity near the corner.

To approach these problems by a numerical method, we need to take into account the singular functions. Several numeric methods have been proposed in this context; see [4–9]. Since the singular functions are developed for the Stokes problem in this paper, we intend in future work to implement Strang and Fix algorithm, see [10], by the mortar spectral element method. It will be an extension of a work done on an elliptic operator [11, 12].

An outline of this paper is as follows. In Section 2, we present the geometry of the domain and the continuous problem. In Section 3, we give the singular functions and some regularity results. The Kondratiev method is described in Section 4. Section 5 is devoted to the conclusion.

2. The Continuous Problem

We suppose that Ω is a polygonal domain of $\mathbb{R}^2$ simply connected and has a connected boundary Γ. Γ is the union of vertex Γ_j for $j \in \{1, \ldots, J\}$; J is positive integer. Let a_j be the corner of Ω between Γ_j and Γ_{j+1}; ω_j is the measure of the angle on a_j. We consider the velocity-pressure formulation of the Stokes problem on the domain Ω.

Find the velocity $\mathbf{u}$ and the pressure p such that

$$\begin{gathered} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} = 0 \quad \text{on } \Gamma, \end{gathered} \tag{1}$$

where ν is the viscosity of the fluid that we suppose a positif constant and $\mathbf{f}$ is the data which represent a density of body forces. Then for $\mathbf{f}$ in $[H^{-1}(\Omega)]^2$, the functional spaces are $[H_0^1(\Omega)]^2$ for the velocity and $L_0^2(\Omega)$ for the pressure where

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q(x)\, dx = 0 \right\}. \tag{2}$$

The problem (1) is equivalent to the following variational formulation.

For $\mathbf{f}$ in $[H^{-1}(\Omega)]^2$, find $\mathbf{u}$ in $[H_0^1(\Omega)]^2$ and p in $L^2(\Omega)$ such that for all $\mathbf{v}$ in $[H_0^1(\Omega)]^2$ and for all q in $L^2(\Omega)$.

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \langle f,\mathbf{v}\rangle, \\ b(\mathbf{u},q) &= 0, \end{aligned} \tag{3}$$

where

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \nu \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\,dx, \\ b(\mathbf{u},q) &= -\int_\Omega (\operatorname{div}\mathbf{u})\,q\,dx, \end{aligned} \tag{4}$$

where the $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The bilinear form $a(\cdot,\cdot)$ is continuous on the space $[H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2$ and elliptic on $[H_0^1(\Omega)]^2$; also the bilinear form $b(\cdot,\cdot)$ is continuous and verifies the following inf-sup condition (see [13, 14]): there exists a nonnull positive constant β such that

$$\forall q \in L_0^2(\Omega), \quad \sup_{\mathbf{v}\in[H_0^1(\Omega)]^2} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{[H^1(\Omega)]^2}} \geq \beta\|q\|_{L^2(\Omega)}. \tag{5}$$

Then we conclude [15] that, for all $\mathbf{f}$ in the space $[H^{-1}(\Omega)]^2$, the problem (3) has a unique solution $(\mathbf{u},p)$ in $[H_0^1(\Omega)]^2 \times L_0^2(\Omega)$. This solution verifies the following stability condition:

$$\|\mathbf{u}\|_{[H^1(\Omega)]^2} + \beta\|p\|_{L^2(\Omega)} \leq C\|\mathbf{f}\|_{[H^{-1}(\Omega)]^2}, \tag{6}$$

where C is a positive constant.

We refer to the work of Pironneau [16] for the mathematical modeling of problems resulting from fluids mechanics and Girault and Raviart [17] for the mathematical analysis of the Navier-Stokes equations.

3. Singular Functions and Regularity Results

We recall that, in an open simply connected of $\mathbb{R}^2$, the condition of incompressibility $\operatorname{div}(\mathbf{u}) = 0$ induces the existence of a stream function φ in the space $H_0^2(\Omega)$ such that

$$\mathbf{u} = \operatorname{curl}(\varphi). \tag{7}$$

It thus brings to study the regularity of the function φ, solution of Dirichlet problem for bilaplacian:

$$\begin{aligned} -\nu\Delta^2\varphi &= \operatorname{curl}(\mathbf{f}), \quad \text{in } \Omega, \\ \varphi &= 0, \quad \text{on } \Gamma, \\ \frac{\partial\varphi}{\partial n} &= 0, \quad \text{on } \Gamma. \end{aligned} \tag{8}$$

The regularity of the solution of the problem is related to the geometry of the domain and its behavior is local. Let $g = \operatorname{curl}(\mathbf{f})$. We know, see Cattabriga [18] and Ladyzhenskaya [14], the following theorem.

Theorem 1. *For g in the space $H^s(\Omega)$, where $s \geq -2$, then*

$$\varphi \in H^{s+4}(\Omega \setminus \mathscr{V}), \tag{9}$$

where $\mathscr{V}$ is the union of neighborhoods V_j of a_j for $j \in \{1,\ldots,J\}$.

To study the function φ in the neighborhood of a fixed vertex a_j, $j \in 1,\ldots,J$, it is convenient to introduce the polar coordinates (r,θ), centered at a_j. We start by enunciating the characteristic equation of bilaplacian:

$$\sin^2\omega_j z = z^2\sin^2\omega_j. \tag{10}$$

Then we stated the following theorem. We refer to ([17], Chapter 7 Theorem 7.2.1.12) and Kondratiev [1] for the proof.

Theorem 2. *Let ω_j in $]0,2\pi[$. For all g in $H^s(\Omega)$, $s \geq -2$, the solution φ of the problem (8) is written as*

$$\varphi = \varphi_r + \varphi_s, \tag{11}$$

where φ_r is in the space $(H^{s+4}(\Omega) \cap H_0^2(\Omega))$.

We set

$$\begin{aligned} \tau_k(r,\theta) &= r^{1+z_k}\psi_k(\theta), \\ \mu_k(r,\theta) &= r^{1+\widehat{z}_k}\left(\sigma_k(\theta) + \log(r)\,\eta_k(\theta)\right). \end{aligned} \tag{12}$$

φ_s is given by

$$\varphi_s = \sum_{0<\operatorname{Re}(z_k)<s+2} \lambda_k\tau_k(r,\theta) + \sum_{0<\operatorname{Re}(\widehat{z}_k)<s+2} \widehat{\lambda}_k\mu_k(r,\theta), \tag{13}$$

where λ_k and $\widehat{\lambda}_k$ are real constants, ψ_k, σ_k, and η_k belong to the vectorial space $\mathscr{C}^\infty([0,\omega_j]) \cap H_0^2([0,\omega_j])$ of finite dimension, z_k and $\widehat{z}_k$ are, respectively, the simple and double roots of (10) in the band $0 < \operatorname{Re}(z_k) < s+2$, excepting 1 if $\omega_j \neq tg(\omega_j)$, without exception if $\omega_j = tg(\omega_j)$.

We called ω_e the unique solution of equation $\omega = tg(\omega)$, in the $]0,2\pi[$ ($\omega_e \simeq 1.430297\pi$). We prove, see ([19], chapter 3, § 3.3), that z is a double root of (10) if and only if $z = 0$ or $z = \pm\sqrt{(1/\sin\omega_j^2) - (1/\omega_j^2)}$. Hence the sufficient condition on the angle ω for a double root is

$$\sin\left(\sqrt{\frac{\omega_j^2}{(\sin\omega_j)^2} - 1}\right) = \pm\sqrt{\left(1 - \frac{(\sin\omega_j)^2}{\omega_j^2}\right)}, \qquad \omega_j \in\,]0,2\pi[\,. \tag{14}$$

Then, if ω_j is not a solution of (14), (10) takes the following simplified form:

$$\varphi_s(r,\theta) = \sum_{0<\operatorname{Re}(z_k)<s+2} \lambda_k\tau_k(r,\theta). \tag{15}$$

In the following we suppose that Ω has a unique vertex a where ω is equal to $3\pi/2$ or 2π and the other angles are $\pi/2$.

We introduce V as a neighborhood of a. All these angles are different of ω_e.

In the case where $\omega = 3\pi/2$ and $s = -1$, (10) has two real simple roots in the band $0 < \mathrm{Re}(z) < 1$. We apply the Newton method to approximate those roots $z_1 \simeq 0.544484$ and $z_2 \simeq 0.908529$. The functions τ_1 and τ_2 are written as follows:

$$\tau_1(r,\theta) = r^{1+z_1}\psi_1(\theta), \qquad \tau_2(r,\theta) = r^{1+z_2}\psi_2(\theta), \tag{16}$$

with

$$\psi_i(\theta) = \left((z_i-1)^{-1}\sin\left(\frac{3(z_i-1)\pi}{2}\right) - (z_i+1)^{-1}\times\sin\left(\frac{3(z_i+1)\pi}{2}\right)\right)\times(\cos((z_i-1)\theta) - \cos((z_i+1)\theta)) - \left((z_i-1)^{-1}\sin((z_i-1)\theta) - (z_i+1)^{-1}\sin((z_i+1)\theta)\right)\times\left(\cos\left(\frac{3(z_i-1)\pi}{2}\right) - \cos\left(\frac{3(z_i+1)\pi}{2}\right)\right). \tag{17}$$

Proposition 3. *For all $\epsilon \geq 0$, the functions τ_i, i in $\{1,2\}$, are belonging to the space $H^{(2+z_i)-\epsilon}(V)$ and are solutions of the problem:*

$$\Delta^2\tau_i = 0 \quad \text{in } V, \qquad \tau_i = \frac{\partial\tau_i}{\partial n} = 0 \quad \text{on } \Gamma\cap\partial V, \tag{18}$$

where Γ is the boundary of Ω.

Proof. Since ψ_i is in $\mathscr{C}^0(]0,3\pi/2[)$, we find s such that

$$r^{1+z_i}\psi_i(\theta) \in H^s(V). \tag{19}$$

Then we find (m,q) in $\mathbb{N}\times\mathbb{R}$, $m > s$, and $1 < q \leq 2$ such that:

$$r^{1+z_i}\psi_i(\theta) \in W_q^m(V) \subset H^s(V). \tag{20}$$

From the Sobolev injection theorem, we have $m-2 = s-1$. If $r^{1+z_i}\psi_i(\theta) \in W_q^m(V)$, then $1+z_i > m-2/q$. Since $1 < q \leq 2$, we can take only the value $E(3+z_i) = 3$ $(0 < z_i < 1)$. Consequently, $s < (2+z_i)$ for $i \in \{1,2\}$. And by construction, we obtain that τ_i, for $i \in \{1,2\}$, is the solution of problem (18). □

Corollary 4. *For all f in $[L^2(\Omega)]^2$, the velocity $\mathbf{u}$ is written as*

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_s, \tag{21}$$

where $\mathbf{u}_r$ is in $[H^2(\Omega)\cap H_0^1(\Omega)]^2$ and there exists a constant λ_i such that:

$$\mathbf{u}_s = \sum_{1\leq i\leq 2}\lambda_i s_i, \tag{22}$$

where $s_i(r,\theta) = \mathrm{curl}(\tau_i(r,\theta))$. The functions s_i belong to the space $[H^{(1+z_i)-\epsilon}(V)]^2$, i in $\{1,2\}$, for all $\epsilon > 0$.

4. Kondratiev's Method Case of the Crack

In the case of the crack there are no known results; we recall the method of kondratiev. We extend this method to the case of crack. We consider the polar coordinates and a truncation function χ with compact support that does not intersect with the boundary Γ. We define

$$G = \left\{re^{i\theta},\ 0 < \theta < 2\pi,\ r > 0\right\}. \tag{23}$$

Let $\psi = \chi\varphi$; then, ψ is in the space $H_0^2(G)$; with a compact support included in G, then ψ is the solution of the problem:

$$\nu\Delta^2\psi = g \quad \text{in } H^s(G), \qquad \psi \in H_0^2(G) \quad \text{with compact support.} \tag{24}$$

Definition 5. For $s \geq 0$, one defines the weight Sobolev space:

$$Z_2^s(G) = \left\{\psi \in L^2(G),\ r^{-s+|\alpha|}D^\alpha\psi \in L^2(G),\ |\alpha| \leq s\right\}$$
$$Z_2^{-1}(G) = \left\{\psi \in \mathscr{D}'(G),\ \psi = \frac{g_0}{r} + \partial_x g_1 + \partial_y g_2, \text{ such that } g_0, g_1 \text{ and } g_2 \in L^2(\Omega)\right\}. \tag{25}$$

Remark 6. We remark that $Z_2^0(G) = Z^2(G)$ and $H_0^s(G) \subset Z_2^s(G) \subset H^s(G)$.

In the following, we suppose that g is in the space $Z_2^s(G)$, for s in $\{-1,0\}$. The solution ψ of the problem:

$$\nu\Delta^2\psi = g \quad \text{in } H^s(G), \qquad u \in Z_2^2(G), \tag{26}$$

is with a compact support; then, there exists a real number R such that

$$\psi(r,\theta) = 0, \quad \text{for } r \geq R. \tag{27}$$

The Kondratiev's method consists to change the variable $r = e^t$; then we replace the domain G by the domain $B = \mathbb{R}\times]0,2\pi[$, and the weight Sobolev space by the ordinary one. We apply the Fourier transformation relative to the first variable of the problem (26).

Proposition 7. *If ψ is in the space $P_2^2(G)$, the function $v = e^{-t}\psi(e^t\cos\theta, e^t\sin\theta)$ belongs to $H^2(B)$.*

For $s \in \{-1,0\}$, if g belongs to $P_2^s(G)$, and let $h = e^{3t}\psi(e^t\cos\theta, e^t\sin\theta)$, then the function $e^{-(s+2)t}h$ is in the space $H^s(B)$.

Proof. Let $s = -1$.

Since $g \in P_2^{-1}(G)$, then $g = g_0/r + \partial_x g_1 + \partial_y g_2$ with g_0, g_1, and g_2 in $L^2(G)$.

We denote

$$\begin{aligned} F(t,\theta) &= g\left(e^t\cos\theta, e^t\sin\theta\right), \\ G_j(t,\theta) &= g_j\left(e^t\cos\theta, e^t\sin\theta\right), \quad \text{for } j \in \{0,1,2\}, \\ F(t,\theta) &= e^{-t}G_0 + \left(e^{-t}\cos\theta\frac{\partial G_1}{\partial t} - e^{-t}\sin\theta\frac{\partial G_1}{\partial\theta}\right) \\ &\quad + \left(e^{-t}\sin\theta\frac{\partial G_2}{\partial t} - e^{-t}\cos\theta\frac{\partial G_2}{\partial\theta}\right). \end{aligned} \tag{28}$$

We have

$$\begin{aligned} e^{-t}h = e^{2t}F(t,\theta) &= e^tG_0 + \partial_t\left(e^t\cos\theta G_1 + \sin\theta e^t G_2\right) \\ &\quad + \partial_\theta\left(-e^t\sin\theta G_1 + \cos\theta e^t G_2\right) \end{aligned} \tag{29}$$

and if

$$\begin{aligned} K_0 &= G_0, \\ K_1 &= \cos\theta G_1 + \sin\theta G_2, \\ K_2 &= -\sin\theta G_1 + \cos\theta G_2, \end{aligned} \tag{30}$$

then we conclude

$$e^{-t}h = e^tK_0 + \partial_t\left\{e^tK_1\right\} + \partial_\theta\left\{e^tK_2\right\}, \tag{31}$$

since $g_j \in L^2(G)$ et $e^tG_j \in L^2(B)$, then, $e^tK_j \in L^2(B)$ for $j \in \{0,1,2\}$. So

$$e^{-t}h \in H^{-1}(B). \tag{32}$$

We end the proof. □

As ψ is the solution of the problem (26), then for $h = e^{3t}g$ the function $v = e^{-t}\psi(e^t\cos\theta, e^t\sin\theta)$ is a solution of the problem:

$$\begin{gathered} \left(D_t^4 - D_t^2 + 1\right)v + 2\left(D_t^2 + 1\right)D_\theta^2 v + D_\theta^4 v = h \\ v(t,0) = v(t,2\pi) = \partial_t v(t,0) = \partial_\theta v(t,2\pi) = 0, \\ v \in H^2(B). \end{gathered} \tag{33}$$

We have $v(t,\theta) = 0$ for $t \geq \log(R) = t_0$.

The Fourier's transformation on the variable t of the function v is

$$\hat{v}(z,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-itz}v(t,z)\,dt. \tag{34}$$

z is a complex variable; we denote:

$$L(z,D_\theta) = \left(z^4 + 2z^2 + 1\right) + 2\left(1 - z^2\right)D_\theta^2 + D_\theta^4. \tag{35}$$

The function $\hat{v}$ becomes a solution of the problem:

$$\begin{gathered} L(z,D_\theta)\hat{v} = \hat{h}, \\ \hat{v}(t,0) = \hat{v}(t,2\pi) = \partial_t\hat{v}(t,0) = \partial_\theta\hat{v}(t,2\pi) = 0. \end{gathered} \tag{36}$$

For all z such that $\mathrm{Im}(z) \geq 0$, the operator $L(z,D_\theta)$ has

$$p^4 + 2\left(1 - z^2\right)p^2 + \left(z^4 + 2z^2 + 1\right) = 0 \tag{37}$$

as a characteristic polynomial. This polynomial function has two roots $p = \pm z$ and $p = \pm i$.

(i) If $z \neq 0$ and $z \neq \pm i$, then the fundamental solutions of the fourth order partial differential equation are

$$\sin\theta sh\theta z, \qquad \sin\theta chz\theta, \qquad \cos\theta ch\theta z, \qquad \text{et } \cos\theta shz\theta. \tag{38}$$

(ii) If $z = 0$, the solutions have the form

$$\sin\theta, \qquad \cos\theta, \qquad \theta\sin\theta, \qquad \text{et } \theta\cos\theta. \tag{39}$$

(iii) If $z = \pm i$, the solutions have the form

$$1, \quad \theta, \quad \sin 2\theta, \quad \text{et } \cos 2\theta. \tag{40}$$

Then, let the following homogeneous problem

$$\begin{gathered} L(z,D_\theta)\varphi = 0 \text{ in }]0,2\pi[, \\ \varphi(0) = \varphi(2\pi) = \partial_t\varphi(0) = \partial_\theta\varphi(2\pi) = 0. \end{gathered} \tag{41}$$

We note $E_z = \{\varphi_z \in H_0^2(]0,2\pi[), \varphi_z$ solution of the problem (41)$\}$.

Proposition 8. *The set E_z satisfies the following properties:*

(1) *if $z \neq 0$ and $z \neq \pm i$ where $sh2\pi z \neq 0$, then,*

$$E_z = \{0\}, \tag{42}$$

(2) *if $z \pm i$, then,*

$$E_z = \left\{\varphi_z \in H_0^2(]0,2\pi[), \varphi_z = \alpha_z\sin^2\theta, \alpha_z \in \mathbb{C}\right\}, \tag{43}$$

(3) *if $z \neq 0$, $z \neq \pm i$ and $sh2\pi z = 0$, which implies that z is equal to $-i(k/2)$ where $k \neq 0$ and $k \neq 2$, then,*

$$\begin{aligned} E_z &= \left\{\varphi_z \in H_0^2([0,2\pi]), \varphi_z = \alpha_k g_k(\theta) + \beta_k h_k(\theta), \right. \\ &\qquad \left.(\alpha_k,\beta_k) \in \mathbb{C}^2\right\}, \\ g_k(\theta) &= \sin\left(1 + \frac{k}{2}\right)\theta - \left(\frac{k+2}{k-2}\right)\sin\left(-1 + \frac{k}{2}\right)\theta, \\ h_k &= \cos\left(1 + \frac{k}{2}\right)\theta - \cos\left(-1 + \frac{k}{2}\right)\theta. \end{aligned} \tag{44}$$

Proof. If $z \neq 0$ and $z \neq \pm i$,

$$\varphi = \alpha\sin\partial shz\theta + \beta\sin\theta chz\theta + \gamma\cos\theta shz\theta + \delta\cos\theta chz\theta \tag{45}$$

is solution of the problem (41). Since $\varphi(0) = \varphi(2\pi) = \partial t\varphi(0) = \partial_\theta\varphi(2\pi) = 0$, we obtain the following system:

$$\begin{gathered} \delta = 0, \\ \gamma sh 2\pi z + \delta ch 2\pi z = 0, \\ \beta = 0, \\ \alpha sh 2\pi z + \beta ch 2\pi z + 2\pi\gamma sh 2\pi z + 2\pi\delta ch 2\pi z = 0, \end{gathered} \tag{46}$$

its determinant is equal to $sh^2 2\pi z$. Then, if $sh 2\pi z \neq 0$, $E_z = \{0\}$. Otherwise, the system has the following complexes roots: $z_k = -i(k/2)$ for $k \neq 0$ and $k \neq 2$, $k \in \mathbb{N}$. The space of solutions of the homogeneous equation is of dimension 2. Indeed, for $z = z_k$, we obtain

$$\varphi = \frac{\alpha}{2} i h_k(\theta) + i\gamma\left(\frac{k}{2} - 1\right) g_k(\theta). \tag{47}$$

We check that (h_k, g_k) are two independent solutions of the problem (41); then

$$\int_0^{2\pi} |h_k|\, d\theta = \int_0^{2\pi} |g_k|\, d\theta = 1. \tag{48}$$

This ends the proof of 1 and 3.

Now we prove property 2.

If $z = \pm i$,

$$\varphi = \alpha + \beta\theta + \gamma \sin 2\theta + \delta \cos 2\theta \tag{49}$$

is solution of the problem (41), which gives the following system:

$$\begin{gathered} \alpha + \delta = 0, \\ \alpha + 2\pi\beta + \delta = 0, \\ \beta + 2\gamma = 0. \end{gathered} \tag{50}$$

The determinant of this system is zero; then the dimension of E_z is equal to 1. Indeed,

$$\varphi = \alpha(1 - \cos 2\theta) = 2\alpha \sin^2 2\theta. \tag{51}$$

This completed the proof. □

Remark 9. If

$$D_k = \left\{ z_k = -i\frac{k}{2}, \text{ for } k \neq 0,\ k \neq 2,\ k \in \mathbb{N} \right\}. \tag{52}$$

D_k is the set of nonregular values. Since $\widehat{h}$ is analytic for $\operatorname{Im}(z) > -(s+2)$, then $\widehat{v}$ analytically extends across $\{\operatorname{Im}(z) > -(s+2)\} \cap D_k$.

Is then

$$d(z) = \frac{-sh^2 2\pi z}{z^2(1+z^2)}, \quad \text{for } z \neq 0,\ z \neq \pm i. \tag{53}$$

We verify that $d(z_k) = \partial_z d(z_k) = 0$ and $\partial_z^2 d(z_k) \neq 0$. Hence the multiplicity of z_k is 2 for $k \neq 0$ and $k \neq 2$. $d(\pm i) = 0$, then, $(\pm i)$ are simple solutions.

Theorem 10. *(1) The solution $\widehat{v}$ of the problem (36) can be written in the neighborhood of $-i$:*

$$\widehat{v}(z,\theta) = \frac{\varphi_1(\theta)}{(z+i)} + \widehat{w}_1(z,\theta), \tag{54}$$

where φ_1 is the solution of problem (41) with $z = (-i)$ and w_1 is an analytic function in the neighborhood of $-i$ with values in $H^2(]0,2\pi[)$.

(2) The solution $\widehat{v}$ of the problem (36) is written in the neighborhood of $(-i(k/2))$, $k = 1$ or $k = 3$, as

$$\widehat{v}(z,\theta) = \frac{\psi_k(\theta)}{(z + i(k/2))^2} + \frac{\varphi_k(\theta)}{(z + i(k/2))} + \widehat{w}_k(z,\theta), \tag{55}$$

where ψ_k and w_k satisfy the same properties as above and φ_k satisfies:

$$\begin{gathered} L(z, D_\theta)\varphi_k = 2ik\left(1 - \frac{k^2}{4}\right)\psi_k - 2ik\psi_k'', \\ \varphi_k(0) = \varphi_k(2\pi) = \partial_t\varphi_k(0) = \partial_\theta\varphi_k(2\pi) = 0. \end{gathered} \tag{56}$$

Proof. We verified that any solution to the problem (41) is a linear combination of

$$\begin{gathered} u_1(z,\theta) = \frac{\sin\theta sh z\theta}{z} \quad \text{for } z \neq 0, \\ u_1(0,\theta) = \theta\sin\theta, \\ u_2(z,\theta) = \frac{1}{z^2+1}\left(\frac{\cos\theta sh z\theta}{z} - \sin\theta ch z\theta\right), \\ \text{for } z \neq 0,\ z \neq \pm i, \\ u_2(0,\theta) = \theta\cos\theta - \sin\theta, \\ u_2(\pm i,\theta) = \frac{1}{2}\left(\frac{\sin 2\theta}{2} - \theta\right). \end{gathered} \tag{57}$$

In fact, if $z = -i(k/2)$ for $k = 1$ or $k = 3$, we have

$$\begin{gathered} u_1\left(-i\frac{k}{2},\theta\right) = -\frac{1}{k}h_k(\theta), \\ u_2\left(-i\frac{k}{2},\theta\right) = \frac{2}{k(k+2)}g_k(\theta). \end{gathered} \tag{58}$$

Therefore, any solution of the problem (41) is linear combination of u_1 and u_2. We also verified that u_1 and u_2 are whole solutions with respect to the variable z and if we set

$$d_z = u_1(z,2\pi) D_\theta u_2(z,2\pi) - u_2(z,2\pi) D_\theta u_1(z,2\pi), \tag{59}$$

we note that $d(z_k)\widehat{v}(z_k,\theta) = 0$. This implies that $\widehat{v}$ is analytical in the neighborhood of $z_k = -i(k/2)$ for k in $\{1,2,3\}$. $(-i)$ is a single root of d; then, $\widehat{v}$ admits a simple pole at $(-i)$. On the other side $-i/2$ and $(-3/2)i$ are roots of multiplicity two of d, and poles of multiplicity two of $\widehat{v}$.

In the neighborhood of $(-i)$, $\widehat{v}(z,\theta) = (\psi_1(\theta)/(z-i)) + \widehat{w}_1(z,\theta)$, and, for all z,

$$\begin{aligned} (z-i)\widehat{h} &= L(-i, D_\theta)\widehat{v}(z-i) \\ &= L(-i, D_\theta)\psi_1 + (z-i) L(-i, D_\theta)\widehat{w}_1. \end{aligned} \tag{60}$$

Then, for $z = -i$, we have

$$L(-i, D_\theta)\psi_1 = 0,$$
$$\psi_1(0) = (z - i)\widehat{\omega}_1(z, 0). \quad (61)$$

Then

$$\psi_1(0) = 0 \quad (62)$$

and even

$$\psi_1(2\pi) = 0,$$
$$\psi_1'(0) = \lim_{\theta\to 0}\frac{\psi(\theta)}{\theta} = \lim_{\theta\to 0}\frac{\widehat{v}(z,\theta) - (z+1)\widehat{\omega}_1(z,\theta)}{\theta} = 0, \quad (63)$$

and even for:

$$\psi_1'(2\pi) = 0. \quad (64)$$

Thus, ψ_1 is a solution of the problem (41) at $z = -i$.

The expression of $\widehat{v}$ in the neighborhood of $z_k = -i(k/2)$ for $k = 1$ and $k = 3$ is

$$\widehat{v}(z,\theta) = \frac{\psi_k(\theta)}{(z + i(k/2))^2} + \frac{\varphi_k(\theta)}{(z + i(k/2))} + \widehat{\omega}_k(z,\theta),$$
$$\text{for } k = 1,\ k = 3. \quad (65)$$

Then, for z in the neighborhood of z_k

$$\widehat{h}(z - z_k)^2 = L(z, D_\theta)\psi_k + (z - z_k)L(z, D_\theta)\varphi_k$$
$$+ (z - z_k)^2 L(z, D_\theta)\widehat{\omega}_k. \quad (66)$$

For $z = z_k$, we obtain that $L(z_k, D_\theta)\psi_k = 0$ and we verify that ψ_k is solution of the problem (41). And we have:

$$L(z, D_\theta)\varphi_k = -\frac{L(z, D\theta) + L(z_k, D_\theta)}{(z - z_k)}\psi_k$$
$$+ (z - z_k)\left(\widehat{h} - L(z, D\theta)\widehat{\omega}_k\right). \quad (67)$$

And if z tends to z_k

$$L(z, D_\theta)\varphi_k = 2ik\left(1 - \frac{k^2}{4}\right)\psi_k - 2ik\psi_k'' \quad (68)$$

and φ_k verifies the boundary conditions, which completes the proof. □

Remark 11. We remark that the operator $L(z, D_\theta)$ is self-adjoint:

$$\langle L\psi_k, \varphi_k\rangle = \langle \psi_k, L\varphi_k\rangle$$
$$= \int_0^{2\pi}\psi_k\left(2k\left(1 - \frac{k^2}{4}\right)\psi_k - 2ik\psi_k''\right)d\theta. \quad (69)$$

For $0 < k < 4$

$$\int_0^{2\pi}\psi_k^2 d\theta = \int_0^{2\pi}\psi_k'^2 d\theta = 0. \quad (70)$$

This implies that $\psi_k = 0$; thus

$$\widehat{v}(z,\theta) = \frac{\varphi_k(\theta)}{(z + i(k/2))} + \widehat{\omega}_k(z,\theta),$$
$$\text{for } k = 1 \text{ or } k = 3, \quad (71)$$

where φ_k is solution of problem (41).

Let η be in $\mathbb{R}$ such that $0 < \eta < 1/2$. We note

$$w_s(t,\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{itz}\widehat{v}(z - i(s+2) + i\eta, \theta)\,dz,$$
$$\mu(\theta) = i\sqrt{2\pi}e^{itz}\widehat{v}(z,\theta) \quad \text{for fixed } t. \quad (72)$$

We prove by Cauchy's theorem that

$$v(t,\theta) = e^{(s+2-\epsilon)t}w_s(t,\theta) + 2i\pi\sum_{-(s+2)<\operatorname{Im}(z)<0} R_s. \quad (73)$$

μ is an analytic function on D_k; R_s is its residue on the poles z_k. Then,

(i) for $s = -1$

$$v(t,\theta) = e^{(1-\eta)t}w_{-1}(t,\theta) + i\sqrt{2\pi}\psi_1(\theta), \quad (74)$$

(ii) for $s = 0$

$$v(t,\theta) = e^{(2-\eta)t}w_0(t,\theta)$$
$$+ i\sqrt{2\pi}\left(e^{3t/2}\psi_3(\theta) + e^{t/2}\psi_1(\theta) + e^t\psi_2(\theta)\right), \quad (75)$$

where ψ_i are solutions of the problem (36).

Notation. Using the variables (r,θ), we introduce the following notations.

For $s = -1$, we define the two singular functions:

$$\tau_1(r,\theta) = r^{3/2}\left(\sin\frac{3}{2}\theta - 3\sin\frac{\theta}{2}\right),$$
$$\mu_1(r,\theta) = r^{3/2}\left(\cos\frac{3}{2}\theta - \cos\frac{\theta}{2}\right). \quad (76)$$

For $s = 0$, we define four singular functions:

$$\tau_1(r,\theta) = r^{3/2}\left(\sin\frac{3}{2}\theta - 3\sin\frac{\theta}{2}\right),$$
$$\mu_1(r,\theta) = r^{3/2}\left(\cos\frac{3}{2}\theta - \cos\frac{\theta}{2}\right),$$
$$\tau_2(r,\theta) = r^{5/2}\left(\sin\frac{5}{2}\theta - 5\sin\frac{\theta}{2}\right),$$
$$\mu_2(r,\theta) = r^{5/2}\left(\cos\frac{5}{2}\theta - \cos\frac{\theta}{2}\right). \quad (77)$$

Proposition 12. *For $\epsilon > 0$, τ_1 and μ_1 belong to the space $H^{5/2-\epsilon}(V)$, τ_2 and μ_2 belong to the space $H^{7/2-\epsilon}(V)$ and They are solutions to the problem:*

$$\Delta^2\varphi = 0 \quad in\ V,$$
$$\varphi = \frac{\partial\varphi}{\partial n} = 0 \quad on\ \Gamma \cap \partial V, \tag{78}$$

where Γ is the boundary of Ω.

Proof. It is the same prove that in the case $\omega = 3\pi/2$. □

For the Stokes problem we handle the case $s = -1$; thus $\mathbf{f}$ belongs to the space $[L^2(\Omega)]^2$.

Corollary 13. *For $\mathbf{f}$ in $[L^2(\Omega)]^2$, the velocity is written in the form*

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_s, \tag{79}$$

where $\mathbf{u}_r$ is in $[H^2(\Omega) \cap H_0^1(\Omega)]^2$ and there exist two real constants λ and $\tilde{\lambda}$ such that

$$\mathbf{u}_s = \lambda s_1 + \tilde{\lambda}\tilde{s}_1 \tag{80}$$

with

$$s_1(r,\theta) = r^{1/2}\left(3\sin\theta\sin\frac{\theta}{2}, 3(1-\cos\theta)\sin\frac{\theta}{2}\right),$$
$$\tilde{s}_1(r,\theta) = r^{1/2}\left(2\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2}, (1-\cos\theta)\cos\frac{\theta}{2}\right). \tag{81}$$

s_1 and $\tilde{s}_1$ belong to the $[H^{3/2-\epsilon}(V)]^2$, for all ϵ positive.

For handling the singularities of the pressure we define

$$\eta(\omega) = \inf\{\operatorname{Re}(z),\ z \text{ is solution of } (10)\}. \tag{82}$$

Indeed for $\omega \in]0, 2\pi]$, using the Newton method, we can obtain a good approximation of roots of (10). See [20] for the approximation of $\eta(\omega)$. Thus

$$\text{if } \pi < \omega < 2\pi, \quad 1 - \frac{\pi}{\omega} < \eta(\omega) < \frac{\pi}{\omega}, \quad \eta(\omega) > \frac{1}{2}. \tag{83}$$

To find the pressure, we note that $\mathbf{f} - \nu\Delta\mathbf{u}$ belongs to $[H^{-1}(\Omega)]^2 \cap W^{\perp}$, where

$$W = \left\{\mathbf{u} \in \left[H_0^1(\Omega)\right]^2, \text{ such that, } \operatorname{div}(\mathbf{u}) = 0\right\}. \tag{84}$$

Then, see [15], there exists a unique function p in $L^2(\Omega)$ defined such that:

$$\nabla p = \mathbf{f} + \nu\Delta\mathbf{u} \quad \text{in } D'(\Omega). \tag{85}$$

From this equality we determine the singularities of the pressure from the singularities of the velocity.

Then from a regular data $\mathbf{f}$

$$p = p_r + p_s, \quad p_r \in H^1(\Omega), \quad p_s = \beta S_{p1}, \tag{86}$$

where β is first singular coefficient. The linearity of (85) gives us

$$\nabla S_{p1} = \nu\Delta\left(\operatorname{curl}\left(r^{1+\eta(\omega)}\psi(\theta)\right)\right) \tag{87}$$

and since we explicitly know the singularities of the velocity we can deduce, by simple integration, the singularities of the pressure.

Consider

$$S_{p1}(r,\theta) = \frac{-r^{\eta(\omega)-1}\left((1+\eta(\omega))^2(\partial\psi(\theta)/\partial\theta) + \left(\partial^3\psi(\theta)/\partial\theta\right)\right)}{(1+\eta(\omega))} \tag{88}$$

in the case of crack

$$\psi(\theta) = 3\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3}{2}\theta\right), \qquad \eta(\omega) = \frac{1}{2}, \tag{89}$$

in the case of $\omega = 3\pi/2$

$$\psi(\theta) = \frac{\sin((1+\eta(\omega))\theta)\cos(\eta(\omega)\omega)}{(1+\eta(\omega))} - \cos((1+\eta(\omega))\theta) + \frac{\sin((\eta(\omega)-1)\theta)\cos(\eta(\omega)\omega)}{(1-\eta(\omega))} + \cos((\eta(\omega)-1)\theta), \tag{90}$$

and $\eta(\omega) = 0.544484$.

Proposition 14. *For $\epsilon > 0$, S_{p1} belongs to $H^{0.544484}(V)$ when $\omega = 3\pi/2$ and S_{p1} belongs to $H^{0.5}(V)$ when $\omega = 2\pi$.*

Proof. It is the same proof that in the case of Proposition 3. □

5. Conclusion

We summarize below the regularity results for the Stokes problem previously obtained.

(1) If $\mathbf{f}$ belongs to $[H^{s-2}(\Omega)]^2$, where $s > 0$, then $(\mathbf{u}, p)$ belongs to the space $[H^s(\Omega)]^2 \times H^{s-1}(\Omega)$, if $s < 1 + \eta(\omega)$. This means

$$[H^s(\Omega)]^2 \times H^{s-1}(\Omega), \quad \text{for } s < 1.544484 \text{ when } \omega = \frac{3\pi}{2},$$
$$[H^s(\Omega)]^2 \times H^{s-1}(\Omega), \quad \text{for } s < 1.5 \text{ when } \omega = 2\pi. \tag{91}$$

$(\mathbf{u}, p)$ satisfies the following stability condition:

$$\|\mathbf{u}\|_{[H^s(\Omega)]^2} + \|p\|_{H^{s-1}(\Omega)} \leq C\|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2}. \tag{92}$$

(2) We know, from Corollaries 4 and 13 and formula (86), that if $\omega \neq 2\pi$, $(\mathbf{u}, p)$ is written

$$\mathbf{u} = \mathbf{u}_r + \lambda S_1, \qquad p = p_r + \beta S_{p1} \tag{93}$$

and if $\omega = 2\pi$

$$\mathbf{u} = \mathbf{u}_r + \lambda_1 S_1 + \tilde{\lambda}_1 \tilde{S}_1, \qquad p = p_r + \beta_1 S_{p1} + \tilde{\beta}_1 \tilde{S}_{p1}. \tag{94}$$

If $\mathbf{f}$ belongs to $[H^{s-2}(\Omega)]^2$, $(\mathbf{u}_r, p_r)$ belongs to $[H^s(\Omega)]^2 \times H^{s-1}(\Omega)$ for $s < 1 + \eta_1(\omega)$, where $\eta_1(\omega)$ is the second real solution of (10), in the band $0 < \mathrm{Re}(z) < s$; hence

$$\begin{gathered}
[H^s(\Omega)]^2 \times H^{s-1}(\Omega), \quad \text{for } s < 1.908529 \text{ when } \omega = \frac{3\pi}{2}, \\
[H^s(\Omega)]^2 \times H^{s-1}(\Omega), \quad \text{for } s < 2.5 \text{ when } \omega = 2\pi.
\end{gathered} \tag{95}$$

We have the following stability condition:

$$\|\mathbf{u}_r\|_{[H^s(\Omega)]^2} + \|p_r\|_{H^{s-1}(\Omega)} + |\lambda_1| + |\beta_1| \leq C\|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2}. \tag{96}$$

(3) If $\mathbf{f}$ belongs to $[H^{s-2}(\Omega)]^2$, we can further decompose the regular part $(\mathbf{u}_r, p_r)$ of the solution as follows:

$$\mathbf{u}_r = \tilde{\mathbf{u}}_r + \lambda_2 S_2, \qquad p_r = \tilde{p}_r + \beta_2 \tilde{S}_{p2} \tag{97}$$

$(\tilde{\mathbf{u}}_r, \tilde{p}) \in [H^s(\Omega)]^2 \times H^{s-1}(\Omega)$ for $s < 1 + \eta_2(\omega)$ where $\eta_2(\omega)$ is the third reel solution of (10), in the band $0 < \mathrm{Re}(z) < s$; then

$$\begin{gathered}
[H^s(\Omega)]^2 \times H^{s-1}(\Omega), \quad \text{for } s < \eta_2\left(\frac{3\pi}{2}\right) \text{ when } \omega = \frac{3\pi}{2}, \\
[H^s(\Omega)]^2 \times H^{s-1}(\Omega), \quad \text{for } s < 3,5 \text{ when } \omega = 2\pi.
\end{gathered} \tag{98}$$

(λ_2, β_2) is the singular coefficient associated to the second singular function $(S_2, \tilde{S}_{p2})$; then

$$\begin{aligned}
&\|\tilde{\mathbf{u}}_r\|_{[H^s(\Omega)]^2} + \|\tilde{p}_r\|_{H^s(\Omega)} + |\lambda_1| + |\lambda_2| + |\beta_1| + |\beta_2| \\
&\quad \leq C\|\mathbf{f}\|_{[H^{s-2}(\Omega)]^2}.
\end{aligned} \tag{99}$$

In general we can decompose $(\mathbf{u}, p)$ as

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_r + \lambda_1 S_1 + \lambda_2 S_2 + \cdots + \lambda_k S_k, \\
p &= p_r + \beta_1 S_{p1} + \beta_2 S_{p2} + \cdots + \beta_k S_{pk},
\end{aligned} \tag{100}$$

where k is an integer number.

If $\mathbf{f}$ belongs to $[H^{s-2}(\Omega)]^2$, $(\mathbf{u}_r, p_r)$ belongs to $[H^s(\Omega)]^2 \times H^{s-1}(\Omega)$ for $s < 1 + \eta_k(\omega)$ where $\eta_k(\omega)$, k-ème reel solution of (10), in the band $0 < \mathrm{Re}(z) < s$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] V. A. Kondratiev, "Boundary value problems for elliptic equations in domains with conical or angular points," *Transactions of the Moscow Mathematical Society*, vol. 16, pp. 227–313, 1967.

[2] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, vol. 24, Pitman, Boston, Mass, USA, 1985.

[3] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, vol. 1341 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1988.

[4] M. Amara, C. Bernardi, and M. A. Moussaoui, "Handling corner singularities by the mortar spectral element method," *Applicable Analysis*, vol. 46, no. 1-2, pp. 25–44, 1992.

[5] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions," *Communications on Pure and Applied Mathematics*, vol. 12, pp. 623–727, 1959.

[6] M. Amara and M. Moussaoui, "Approximation de coefficient de singularités," *Comptes Rendus de l'Académie des Sciences I*, vol. 313, no. 5, pp. 335–338, 1991.

[7] I. Babuška, "Finite element method for domains with corners," *Computing*, vol. 6, no. 3-4, pp. 264–273, 1970.

[8] J. Lelièvre, "Sur les éléments finis singuliers," *Comptes Rendus de l'Académie des Sciences A*, vol. 283, no. 15, pp. 1029–1032, 1967.

[9] M. Suri, "The p-version of the finite element method for elliptic equations of order 2l," *ESAIM*, vol. 24, no. 2, pp. 265–304, 1990.

[10] G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1973.

[11] N. Chorfi, "Handling geometric singularities by the mortar spectral element method: case of the Laplace equation," *SIAM Journal of Scientific Computing*, vol. 18, no. 1, pp. 25–48, 2003.

[12] F. Ben Belgacem, C. Bernardi, N. Chorfi, and Y. Maday, "Inf-sup conditions for the mortar spectral element discretization of the Stokes problem," *Numerische Mathematik*, vol. 85, no. 2, pp. 257–281, 2000.

[13] I. Babuška, "The finite element method with Lagrangian multipliers," *Numerische Mathematik*, vol. 20, no. 3, pp. 179–192, 1973.

[14] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers, New York, NY, USA, 1962.

[15] R. Temam, *Theory and Numerical Analysis of the Navier-Stokes Equations*, North-Holland, Amsterdam, The Netherlands, 1977.

[16] O. Pironneau, *Méthode des éléments finis pour les fluides*, Masson, Paris, France, 1988.

[17] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, vol. 5, Springer, New York, NY, USA, 1986.

[18] L. Cattabriga, "Su un problema al contorno relativo al sistema di equazioni di Stokes," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 31, pp. 1–33, 1961.

[19] P. Grisvard, *Singularities in Boundary Value Problems*, vol. 22, Springer, Amesterdam, The Netherlands, 1992.

[20] C. Bernardi and G. Raugel, "Méthodes d'éléments finis mixtes pour les équations de Stokes et de Navier-Stokes dans un polygone non convexe," *Calcolo*, vol. 18, no. 3, pp. 255–291, 1981.