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Research Article

Necessary and Sufficient Conditions of Optimality for a Damped Hyperbolic Equation in One-Space Dimension

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The present paper deals with the necessary optimality condition for a class of distributed parameter systems in which the system is modeled in one-space dimension by a hyperbolic partial differential equation subject to the damping and mixed constraints on state and controls. Pontryagin maximum principle is derived to be a necessary condition for the controls of such systems to be optimal. With the aid of some convexity assumptions on the constraint functions, it is obtained that the maximum principle is also a sufficient condition for the optimality.

1. Introduction

It is well known that many processes in science and engineering are modeled by partial differential equations. The problems concerning the control of vibrating systems are generally governed by hyperbolic partial differential equations which are obtained by using conservation laws as a description of the distributed parameter system. In order to control these systems, the derivation of necessary conditions in the form of a maximum principle has been studied since the 1960s.

The Maximum principle is introduced for the first time by Pontryagin and his students as a necessary condition for the optimality of a mechanical system which is defined by ordinary differential or difference-differential equations [1]. Pontryagin's maximum principle is given in the form of a Hamiltonian that is defined in terms of an adjoint variable and the control function. The first application of maximum principle was the maximization of the terminal velocity of a rocket. Following this initial work, it is shown that the maximum principle is a necessary condition, if the value of the Hamiltonian maximized over the controls is concave in the state variables and the sufficient condition with appropriate transversality conditions [2]. Necessary conditions of optimality for distributed parameter systems described by boundary-value problems for hyperbolic and parabolic equations are studied in [3] where the completeness assumption on the class of admissible controls is imposed. In [4], Necessary condition for optimality is derived in a form similar to Pontryagin's maximum principle without admissible control function set which is bounded or closed. These necessary conditions are expressed in terms of certain "generalized Jocabians." For a general control problem formulated in terms of a differential inclusion, maximum principle is given by weak pseudo-Lipschitz behavior that is postulated on the underlying multifunction [5]. Necessary condition in the form of Pontryagin maximum principle by adopting the Dubovitskiy-Milyutin functional analytical approach is derived in [6]. In [7], Barnes presents a maximum principle as a necessary condition for optimal control of vibrating system that is modeled by second order linear hyperbolic PDE where completeness assumption is dropped and a regular point is used. Furthermore, Barnes shows that, by certain convexity assumptions on the constraint functions, the maximum principle is also a sufficient condition for the optimality of distributed parameters systems. Other related theoretical studies about the maximum principle in the literature are available such as [8–25]. In [1–7], the optimal necessary and sufficient conditions are derived for a single partial differential equation without damping term subject to the homogeneous boundary conditions in one space dimension.

The systems that involve only one control function are studied in [1–25], while studies in [26–30] examine the systems with multiple control functions due to the size of the structures or to increase the control efficiency.

In the present paper, inspired by [7], the necessary optimality condition is given for a hyperbolic partial differential equation in one-space dimensional system. The system involves a damping and several control functions and is subject to the mixed integral constraints on state and control functions. The partial differential equation under consideration involves spatial derivatives of at most order four. The main goal of the control problem interested here is to minimize the performance index of the control problem in a given period of time with the control functions and state variable subject to the constraints in the form of integral equality or/and inequality. Under proper convexity assumptions on the constraint functions, the maximum principle is also a sufficient condition for a general class of hyperbolic partial differential equations in one-space dimensional system.

2. Mathematical Formulation of the Problem

Let us consider the following partial differential equation [31]:

$$\mu(x) w_{tt} + N(x)[w] + M(x)[w_t] = E(x,t) + \sum_{j=1}^{N} f_j(x,t) \quad \text{on } \Omega = (0,\ell) \times (0,t_f), \tag{1}$$

where $w(x,t)$ is the transversal displacement at $(x,t) \in \overline{\Omega} = \{(x,t) : x \in [0,\ell], t \in [0,t_f]\}$, x is the space variable, t is the time variable, $E(x,t)$ is the external excitation, $\mu(x) > 0$ is the mass per unit length of the beam, t_f is predetermined terminal time, and $f_j(x,t)$, $j = 1,\ldots,N$ are the control functions,

$$\begin{aligned} M(x)[w] &= \sum_{j=0}^{4} m_j(x) \left(\frac{\partial^j w}{\partial x^j}\right), \\ N(x)[w] &= \sum_{j=0}^{4} n_j(x) \left(\frac{\partial^j w}{\partial x^j}\right), \end{aligned} \tag{2}$$

in which $m_j(x)$ and $n_j(x)$ are continuous functions on Ω, $M(x)$ and $N(x)$ are positive-definite operators, and $M(x)$ is the damping operator. Equation (1) is subject to the following boundary conditions:

$$a_j^k(x,t) \left.\frac{\partial^j w}{\partial x^j}\right|_{x=0} = T_{j+k}(t), \quad k = 1,2, \; j = 0,1,2,3 \tag{3a}$$

$$a_j^k(x,t) \left.\frac{\partial^j w}{\partial x^j}\right|_{x=\ell} = S_{j+k}(t), \quad k = 3,4, \; j = 0,1,2,3, \tag{3b}$$

where $a_j^k(x,t)$, $S_{j+k}(t)$, and $T_{j+k}(t)$ are continuous functions on Ω and the initial conditions

$$\begin{aligned} w(x,0) &= w_0(x) \in H^1(0,\ell), \\ w_t(x,0) &= w_1(x) \in L^2(0,\ell) \end{aligned} \tag{4}$$

in which

$$H^1(0,\ell) = \left\{ w_0(x) \in L^2(0,\ell) : \frac{\partial w_0(x)}{\partial x} \in L^2(0,\ell) \right\}. \tag{5}$$

The following assumptions are made:

(A1) $\partial^j w/\partial x^j$, $\partial^{j+1} w/\partial x^j \partial t$, $\partial^2 w/\partial t^2 \in L^2(\overline{\Omega})$, $j = 0, 1, \ldots, 4$, $\overline{\Omega}$ is the closure of Ω;

(A2) $a_j^k(x,t)$, $T_{j+k}(t)$, $S_{j+k}(t) \in L^2(\Omega)$ $k = 1,\ldots,4$, $j = 0,\ldots,3$;

(A3) the set of admissible control functions is given by

$$\begin{aligned} U_{\text{ad}} = \{ & f_k(x,t) \mid f_k(x,t) \in L^2(\Omega), \\ & |f_k(x,t)| \le M_k < \infty, \; k = 1,\ldots,N \}. \end{aligned} \tag{6}$$

in which $L^2(\Omega)$ denotes the Hilbert space of real-valued square-integrable functions on the domain Ω in the Lebesgue sense with the usual inner product and norm defined by

$$\begin{aligned} \langle f, g \rangle_\Omega &= \int_\Omega f(x,t) g(x,t) \, d\Omega, \\ \|f\|^2 &= \langle f, f \rangle, \end{aligned} \tag{7}$$

respectively. Under these assumptions, the system equations (1)–(4) have a solution [32].

3. Formulation of the Control Problem

The optimal control problem aims to determine optimal control functions $f_i^\circ(x,t) \in U_{\text{ad}}$, $i = 1,\ldots,N$ that minimizes the performance index at the terminal time t_f:

$$\begin{aligned} &\mathcal{J}_0(f_1(x,t), f_2(x,t), \ldots, f_N(x,t)) \\ &= \int_0^\ell \left[\mathcal{G}_1(x, w(x,t_f)) + \mathcal{G}_2(x, w_t(x,t_f))\right] dx \\ &\quad + \int_0^{t_f} \int_0^\ell \mathcal{G}_0(x, t, w(x,t), f_1(x,t), \ldots, f_N(x,t)) \, dx \, dt. \end{aligned} \tag{8}$$

The first two terms in the right hand side of (8) denote the modified energy of the system and the last term represents the control effort spent in control duration. kth admissible control function $f_k^\circ(x,t)$ subject to (1)–(4) and the following constraints:

$$\int_0^\ell h_{2k}\left(x, w_t\left(x, t_f\right)\right) dx + \int_0^{t_f}\int_0^\ell \mathcal{G}_{-2k}\left(x, t, w\left(x,t\right), f_k\left(x,t\right)\right) dx\, dt = c_{-2k} \tag{9a}$$

$$\int_0^\ell h_{1k}\left(x, w\left(x, t_f\right)\right) dx + \int_0^{t_f}\int_0^\ell \mathcal{G}_{-1k}\left(x, t, w\left(x,t\right), f_k\left(x,t\right)\right) dx\, dt = c_{-1k} \tag{9b}$$

$$\int_0^{t_f}\int_0^\ell \mathcal{G}_{ik}\left(x, t, w\left(x,t\right), f_k\left(x,t\right)\right) dx\, dt \le c_{ik}, \quad 1 \le i \le m \tag{9c}$$

$$\int_0^{t_f}\int_0^\ell \mathcal{G}_{ik}\left(x, t, w\left(x,t\right), f_k\left(x,t\right)\right) dx\, dt = c_{ik}, \quad m < i \le M, \tag{9d}$$

in which $h_{1k}, h_{2k}, \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$, and $\mathcal{G}_{ik}$, for $k = 1, 2, \ldots, N$, $i = -2, -1, 1, \ldots, M$, are continuous functions of their parameters. Also, $h_{1k}, \mathcal{G}_0, \mathcal{G}_1$, and $\mathcal{G}_{ik}$, for $k = 1, 2, \ldots, N$, $i = -2, -1, 1, \ldots, M$, have the continuous derivatives with respect to w and $h_{2k}, \mathcal{G}_2$ have the continuous derivatives with respect to w_t.

4. Necessary and Sufficient Conditions for Optimality

In this section, necessary condition of optimality is derived in the form of the maximum principle. Also, under proper convexity assumptions on the constraint functions, it is shown that maximum principle is also sufficient condition of optimality. For convenience, let us assume that

$$\mathcal{G}_0\left(x, t, w, f_1\left(x,t\right), \ldots, f_N\left(x,t\right)\right) = \sum_{k=1}^{N} \mathcal{G}_{0k}\left(x, t, w, f_k\left(x,t\right)\right), \tag{10}$$

where $\mathcal{G}_{0k}(x,t,w,f_k(x,t))$ is the related term with kth control function in $\mathcal{G}_0$. In order to achieve the maximum principle, an adjoint variable $v(x,t)$ along with the adjoint operator is introduced. The adjoint variable $v(x,t)$ satisfies the following equation:

$$\mu\left(x\right) v_{tt} + N^*\left(x\right)\left[v\right] - M^*\left(x\right)\left[v_t\right] = \sum_{k=1}^{N}\sum_{i=-2}^{M} \lambda_{ik} \frac{\partial \mathcal{G}_{ik}}{\partial w}\left(x, t, w, f_k\left(x,t\right)\right), \quad \lambda_{0k} = \lambda_0, \tag{11}$$

where we introduced the Lagrange multiplier $\lambda_{ik} \le 0$ and

$$M^*\left(x\right)\left[v\right] = \sum_{j=0}^{4} (-1)^j \left(\frac{\partial^j}{\partial x^j}\right)\left(v m_j\left(x\right)\right),$$
$$N^*\left(x\right)\left[v\right] = \sum_{j=0}^{4} (-1)^j \left(\frac{\partial^j}{\partial x^j}\right)\left(v n_j\left(x\right)\right). \tag{12}$$

Equation (11) is subject to the following homogeneous boundary conditions:

$$a_j^k\left(x,t\right) \left.\frac{\partial^j v}{\partial x^j}\right|_{x=0} = 0, \quad k = 1, 2,\ j = 0, 1, 2, 3 \tag{13a}$$

$$a_j^k\left(x,t\right) \left.\frac{\partial^j v}{\partial x^j}\right|_{x=\ell} = 0, \quad k = 3, 4,\ j = 0, 1, 2, 3 \tag{13b}$$

and the terminal conditions

$$v\left(x, t_f\right) = \frac{1}{\mu\left(x\right)} \lambda_0 \frac{\partial \mathcal{G}_2}{\partial w_t}\left(x, w_t\left(x, t_f\right)\right) + \frac{1}{\mu\left(x\right)} \sum_{k=1}^{N} \lambda_{-2k} \frac{\partial h_{2k}}{\partial w_t}\left(x, w_t\left(x, t_f\right)\right), \tag{14a}$$

$$v_t\left(x, t_f\right) - M^*\left(x\right)\left[v\right]\left(x, t_f\right) = -\frac{1}{\mu\left(x\right)} \sum_{k=1}^{N} \lambda_{-1k} \frac{\partial h_{1k}}{\partial w}\left(x, w\left(x, t_f\right)\right) - \frac{1}{\mu\left(x\right)} \lambda_0 \frac{\partial \mathcal{G}_1}{\partial w}\left(x, w\left(x, t_f\right)\right). \tag{14b}$$

Let us introduce a special perturbation of the optimal control problem and three lemmas to derive the maximum principle. Suppose that $f_k^\circ(x,t)$, $k = 1, \ldots, N$ are optimal control functions corresponding to optimal displacement w°. Let $(x_1, t_1), \ldots, (x_P, t_P)$ be P arbitrary points in the open region Ω and let $f_{kj}(x,t)$, $k = 1, \ldots, N$, $j = 1, \ldots, P$ be P arbitrary subfunctions for every admissible control function $f_k \in U_{\text{ad}}$, $k = 1, 2, \ldots, N$. Also, let us assume that $x_1 \le x_2 \le \cdots \le x_P$. Choose $\varsigma > 0$ such that $x_i + P\varsigma < x_j$, if $x_i < x_j$, $x_P + P\varsigma < \ell$ and $t_i + \varsigma < t_f$, for each $0 \le i \le P$. Let $\varepsilon_1, \ldots, \varepsilon_P$ be real parameters satisfying $0 \le \varepsilon_j \le \varsigma^2$. Let $X_1 = x_1$ and $X_j = x_j + \sqrt{\varepsilon_1} + \cdots + \sqrt{\varepsilon_{j-1}}$ be for $1 < j \le P$. Hence, the intervals $X_j \le x \le X_j + \sqrt{\varepsilon_j}$ and the rectangles $R_j : [X_j, X_j + \sqrt{\varepsilon_j}] \times [t_j, t_j + \sqrt{\varepsilon_j}]$ do not have any intersection for $1 \le j \le P$, respectively. ε denotes the vector $(\varepsilon_1, \ldots, \varepsilon_P) \in \mathbb{R}^P$; $\mathbb{R}^P$ is P-dimensional Euclidean space, and the norm of ε is given by $|\varepsilon| = \varepsilon_1 + \cdots + \varepsilon_P$. The admissible controls $f_{k\varepsilon}(x,t) \in \overline{\Omega}$ are defined by

$$f_{k\varepsilon}\left(x,t\right) = \begin{cases} f_k^\circ\left(x,t\right), & \text{if } \left(x,t\right) \notin \bigcup_{j=1}^{P} R_j, \\ f_{kj}\left(x,t\right), & \text{if } \left(x,t\right) \in R_j,\ j = 1, \ldots, P, \end{cases} \tag{15}$$

for $k = 1, 2, \ldots, N$.

Lemma 1. *Let w_ε satisfy the system given by* (1)–(4) *corresponding to the controls $f_{1\varepsilon}(x,t), f_{2\varepsilon}(x,t), \ldots, f_{N\varepsilon}(x,t)$. Consider the following difference functions,*

$$\Delta F\left(x,t\right) = F_\varepsilon\left(x,t\right) - F^\circ\left(x,t\right),$$
$$\Delta w\left(x,t\right) = w_\varepsilon\left(x,t\right) - w^\circ\left(x,t\right), \tag{16}$$

in which

$$F(x,t) = \sum_{k=1}^{N} f_k(x,t). \quad (17)$$

Note that $\Delta w(x,t)$ satisfies following equation:

$$\mu(x)\Delta w_{tt} + N(x)[\Delta w] + M(x)[\Delta w_t] = \Delta F(x,t), \quad (18)$$

and the following homogeneous boundary conditions:

$$b_j^k(x,t)\left.\frac{\partial^j \Delta w}{\partial x^j}\right|_{x=0} = 0, \quad k = 1,2,\ j = 0,1,2,3 \quad (19a)$$

$$b_j^k(x,t)\left.\frac{\partial^j \Delta w}{\partial x^j}\right|_{x=\ell} = 0, \quad k = 3,4,\ j = 0,1,2,3 \quad (19b)$$

and also, the zero initial conditions:

$$\Delta w(x,0) = 0, \qquad \Delta w_t(x,0) = 0. \quad (20)$$

Then,

$$\begin{gathered}\int_0^\ell \Delta w^2(x,t_f)\,dx = o(\varepsilon), \qquad \int_0^\ell \Delta w_t^2(x,t_f)\,dx = o(\varepsilon),\\ \int_0^{t_f}\int_0^\ell \Delta w^2(x,t)\,dx\,dt = o(\varepsilon).\end{gathered} \quad (21)$$

$o(\varepsilon)$ is a quantity such that

$$\lim_{\varepsilon\to 0}\left(\frac{o(\varepsilon)}{|\varepsilon|}\right) = 0. \quad (22)$$

Proof. We start the proof by examining the following energy integral:

$$E(t) = \frac{1}{2}\int_0^\ell \left\{(\mu(x)\Delta w_t)^2 + \frac{1}{2}N(x)[\Delta w]^2\right\}dx; \quad (23)$$

then, it can be rewritten in [33] as

$$\begin{aligned}E(t) &= \int_0^t \frac{dE(\tau)}{d\tau}d\tau\\ &= \int_0^t \frac{1}{2}\int_0^\ell \Big\{2\mu(x)\Delta w_t\Delta w_{tt}\\ &\qquad + \frac{1}{2}N(x)[2\Delta w\Delta w_t]\Big\}\,dx\,d\tau.\end{aligned} \quad (24)$$

With integration by parts and using homogeneous boundary conditions given by (19a) and (19b), (24) becomes

$$\begin{aligned}E(t) &= \int_0^t\int_0^\ell \{\mu(x)\Delta w_{tt} + N(x)[\Delta w]\}\Delta w_t dx\,d\tau\\ &= \int_0^t\int_0^\ell \{\Delta F(x,\tau)\Delta w_t - \Delta w_t M(x)[\Delta w_t]\}\,dx\,d\tau\\ &\le \int_0^t\int_0^\ell \Delta F(x,\tau)\Delta w_t(x,\tau)\,dx\,d\tau.\end{aligned} \quad (25)$$

Note that if $N(x)$ is nonself adjoint operator, the foregoing inequality is also satisfied. By applying the Cauchy-Schwartz inequality to the space integral, we obtain

$$\begin{aligned}E(t) &\le \int_0^t \left[\int_0^\ell (\Delta w_t)^2 dx\right]^{1/2}\left[\int_0^\ell (\Delta F(x,\tau))^2 dx\right]^{1/2} d\tau\\ &\le \int_0^t E^{1/2}(\tau)\left[\int_0^\ell (\Delta F(x,\tau))^2 dx\right]^{1/2} d\tau.\end{aligned} \quad (26)$$

Taking the sup of both sides of (26) leads to

$$\begin{aligned}\sup E(t) &\le \sup E^{1/2}(t)\int_0^t \left[\int_0^\ell (\Delta F(x,\tau))^2 dx\right]^{1/2} d\tau\\ &= \sup E^{\frac{1}{2}}(t)\sum_{i=1}^{P} o\left(\varepsilon_i^{5/4}\right),\end{aligned} \quad (27)$$

where $o(r)$ is a quantity such that

$$\lim_{r\to 0^+}\left(\frac{o(r)}{r}\right) = \text{constant}. \quad (28)$$

By means of (27), the following inequality is observed for each $t \in [0,t_f]$:

$$0 \le E^{1/2}(t) \le o\left(\varepsilon^{5/4}\right). \quad (29)$$

Because $5/4 > 0$ [34], the following equality is obtained:

$$E(t) = o(\varepsilon). \quad (30)$$

Because the coefficients of (1) are bounded away from zero, the conclusion of Lemma 1 is obtained from (30). It is concluded from Lemma 1 that

$$\lim_{\Delta F(x,t)\to 0} \Delta w(x,t) = 0. \quad (31)$$

Namely, (1)–(4) have a unique solution. □

Let us define the differential operator $\mathscr{L}$ and its adjoint operator $\mathscr{M}$ as follows:

$$\begin{aligned}\mathscr{L}[w] &\equiv \mu(x)w_{tt} + N(x)[w] + M(x)[w_t],\\ \mathscr{M}[v] &\equiv \mu(x)v_{tt} + N^*(x)[v] - M^*(x)[v_t].\end{aligned} \quad (32)$$

Lemma 2. *Let v and $\Delta w(x,t) = w(x,t) - w^\circ(x,t)$ be two functions which are defined in $L^2(\Omega)$. Also, let us assume that v and $\Delta w(x,t)$ satisfy conditions* (13a) *and* (13b) *and* (19a) *and* (19b)-(20), *respectively. Then,*

$$\begin{aligned}&\int_0^{t_f}\int_0^\ell \{v\mathscr{L}[\Delta w] - \Delta w\mathscr{M}[v]\}\,dx\,dt\\ &\quad = \int_0^\ell \mu(x)\Big\{v(x,t_f)\Delta w_t(x,t_f) - \Delta w(x,t_f)\\ &\qquad\qquad \times \left\{v_t(x,t_f) - M^*(x)\left[v(x,t_f)\right]\right\}\Big\}\,dx.\end{aligned} \quad (33)$$

Proof. The reader is referred to [31]. □

Definition 3. For the arbitrary constants, $\lambda_{-2k}, \lambda_{-1k}, \lambda_0, \ldots, \lambda_{Mk}, k = 1, \ldots, N, v$ is the solution of the system equations (11)–(14a) and (14b). Let w° be the response function corresponding to optimal control functions $f_k^\circ \in U_{\text{ad}}$, $k = 1, \ldots, N$, u be any of the following functions:

$$u(x,t) = \mathcal{G}_{ik}(x,t,w,f_k(x,t)), \quad f_k \in U_{\text{ad}} \text{ is fixed}$$
$$u(x,t) = \mathcal{G}_{ik}(x,t,w^\circ,f_k^\circ(x,t)),$$
$$u(x,t) = \sum_{k=1}^{N} v f_k^\circ, \quad f_k^\circ \in U_{\text{ad}}, \tag{34}$$
$$u(x,t) = \sum_{k=1}^{N} v f_k, \quad f_k \in U_{\text{ad}} \text{ is fixed.}$$

A point $(\bar{x}, \bar{t})$ is called a regular point, for each f_k, $k = 1, \ldots, N \in U_{\text{ad}}$, if it satisfies the following equality for any sufficiently small $\varepsilon > 0$:

$$\int_{\bar{t}}^{\bar{t}+\sqrt{\varepsilon}} \int_{\bar{x}}^{\bar{x}+\sqrt{\varepsilon}} u(x,t)\,dx\,dt = \varepsilon u(\bar{x},\bar{t}) + o(\varepsilon). \tag{35}$$

It can be concluded from [35] that all points of $[0,\ell] \times [0,t_f]$ are regular for each f_k, $k = 1, \ldots, N \in U_{\text{ad}}$.

Let $\mathcal{J}_k$ and Z denote the vector valued functional $(\mathcal{J}_{-2k}, \mathcal{J}_{-1k}, \mathcal{J}_{0k}, \ldots, \mathcal{J}_{Mk})$, for $k = 1, \ldots, N$, and the set

$$Z = \{\mathcal{J}_k(f_k) : f_k \in U_{\text{ad}}\} \subset \mathbb{R}^{M+3}. \tag{36}$$

Definition 4. If there exists a surface in the following form:

$$\mathcal{J}_{k\varepsilon} = \mathcal{J}_k(f_k^\circ) + \sum_{j=1}^{P} d_{kj}\varepsilon_j + o(\varepsilon) \tag{37}$$

in Z for sufficiently small ε_j and $d_{k1}, \ldots, d_{kP}$ are any finite collection of vectors from D, then the set D is called as a derived set of the set Z at $\mathcal{J}_k(f_k^\circ)$ [36].

Lemma 5. *Assume that the points (x_i, t_i) are regular points in Ω, for $i = 1, 2, \ldots, P$, and $\overline{\mathcal{J}}_k(f_k)$ is introduced as follows, for any $f_k \in U_{ad}$, $k = 1, \ldots, N$:*

$$\begin{aligned}\overline{\mathcal{J}}_k(f_k) = \int_0^\ell \Big\{ &\left(\frac{\lambda_0}{N}\right) \\ &\times \left[\mathcal{G}_1(x, w(x,t_f)) + \mathcal{G}_2(x, w_t(x,t_f))\right] \\ &+ \left[\lambda_{-2k} h_{2k}(x, w_t(x,t_f)) \right. \\ &\left. + \lambda_{-1k} h_{1k}(x, w(x,t_f))\right] \Big\}\,dx \\ &+ \int_0^{t_f}\int_0^\ell \sum_{i=-2}^{M} \left[\lambda_{ik}\mathcal{G}_{ik}(x,t,w,f_k)\right]dx\,dt.\end{aligned} \tag{38}$$

If $P = 1$, for $k = 1, \ldots, N$ $f_k \in U_{ad}$, there exist constants $\lambda_{-2k}, \lambda_{-1k}, \lambda_0, \ldots, \lambda_{Mk}$ (not all zero) such that

$$\begin{gathered}\lambda_0 \le 0, \quad \lambda_{ik} \le 0 \quad (0 \le i \le m), \\ \lim_{\varepsilon \to 0^+} \frac{\overline{\mathcal{J}}_k(f_{k\varepsilon}) - \overline{\mathcal{J}}_k(f_k^\circ)}{\varepsilon} \le 0,\end{gathered} \tag{39}$$

in which f_k°'s and $f_{\varepsilon k}$'s are functions which are defined in (15).

Proof. Define the functionals $\mathcal{J}_{-2k}, \mathcal{J}_{-1k}, \mathcal{J}_{1k}, \ldots, \mathcal{J}_{Mk}$ on the class of admissible controls by

$$\begin{aligned}\mathcal{J}_{-2k}(f_k) &= \int_0^\ell h_{2k}(x, w_t(x,t_f))\,dx \\ &\quad + \int_0^{t_f}\int_0^\ell G_{-2k}(x,t,w,f_k)\,dx\,dt, \\ \mathcal{J}_{-1k}(f_k) &= \int_0^\ell h_{1k}(x, w_t(x,t_f))\,dx \\ &\quad + \int_0^{t_f}\int_0^\ell G_{-1k}(x,t,w,f_k)\,dx\,dt, \\ \mathcal{J}_{ik}(f_k) &= \int_0^{t_f}\int_0^\ell G_{ik}(x,t,w,f_k)\,dx\,dt, \quad i = 1, \ldots, M.\end{aligned} \tag{40}$$

In order to use the Lagrange multiplier rule, we should construct a derived set D for the set Z at $\mathcal{J}_k(f_k^\circ)$ [36]. To this end, functions v_{jk} are introduced, for $k = 1, 2, \ldots, N$, $j = -2, -1, 1, \ldots, M$, satisfying

$$\begin{aligned}\mu(x)\frac{\partial^2 v_{jk}}{\partial t^2} + N^*(x)\left[v_{jk}\right] - M^*(x)\left[\frac{\partial v_{jk}}{\partial t}\right] \\ = \frac{\partial \mathcal{G}_{jk}}{\partial w}(x,t,w,f_k), \quad 0 \le x \le \ell,\ 0 \le t \le t_f,\end{aligned} \tag{41a}$$

$$\begin{gathered}a_i^\alpha(x,t)\left.\frac{\partial^i v_{jk}}{\partial x^i}\right|_{x=0} = 0, \\ \alpha = 1,2, \quad i = 0,1,2,3 \quad -2 \le j \le M, \quad k = 1, \ldots, N,\end{gathered} \tag{41b}$$

$$\begin{gathered}a_i^\alpha(x,t)\left.\frac{\partial^i v_{jk}}{\partial x^i}\right|_{x=\ell} = 0, \\ \alpha = 3,4, \quad i = 0,1,2,3 \quad -2 \le j \le M, \quad k = 1, \ldots, N,\end{gathered} \tag{41c}$$

$$\begin{gathered}v_{-2k}(x,t_f) = \frac{1}{\mu(x)}\frac{\partial h_{2k}}{\partial w_t}(x, w_t(x,t_f)), \\ \frac{\partial v_{-2k}}{\partial t}(x,t_f) = 0,\end{gathered} \tag{41d}$$

$$\begin{gathered}v_{-1k}(x,t_f) = 0, \\ \frac{\partial v_{-1k}}{\partial t}(x,t_f) = \frac{-1}{\mu(x)}\frac{\partial h_{1k}}{\partial w}(x, w(x,t_f)),\end{gathered} \tag{41e}$$

$$\begin{gathered}v_{jk}(x,t_f) = 0, \qquad \frac{\partial v_{jk}}{\partial t}(x,t_f) = 0, \\ j = 1, \ldots, M, \quad k = 1, \ldots, N\end{gathered} \tag{41f}$$

and, for $k = 1, \ldots, N$ and $j = 0$, satisfying

$$\mu(x)\frac{\partial^2 v_{0k}}{\partial t^2} + N^*(x)[v_{0k}] - M^*(x)\left[\frac{\partial v_{0k}}{\partial t}\right] = \frac{1}{N}\frac{\partial \mathcal{G}_0}{\partial w}(x,t,w,f_k), \tag{42a}$$

$$v_{0k}(x,t_f) = \frac{1}{N\mu(x)}\frac{\partial \mathcal{G}_2}{\partial w_t}(x,w_t(x,t_f)) \tag{42b}$$

$$\frac{\partial v_{0k}}{\partial t}(x,t_f) - M^*(x)[v_{0k}](x,t_f) = \frac{-1}{N\mu(x)}\frac{\partial \mathcal{G}_1}{\partial w}(x,w(x,t_f)). \tag{42c}$$

For each point, $(x,t) \in (0,\ell)\times(0,t_f)$, $i = -2,-1,0,\ldots,M$, $k = 1,\ldots,N$, $d_k^i(x,t,\overline{f}_k)$ is defined as follows:

$$d_k^i(x,t,\overline{f}_k) = v_{ik}(x,t)(\overline{f}_k - f_k^\circ) + \mathcal{G}_{ik}(x,t,w^\circ(x,t),\overline{f}_k) - \mathcal{G}_{ik}(x,t,w^\circ(x,t),f_k^\circ). \tag{43}$$

To show that

$$D = \{d_k \mid d_k = (d_k^{-2}(x,t,\overline{f}_k),\ldots,d_k^M(x,t,\overline{f}_k)),\ (x,t) \text{ a regular point of } f_k^\circ, \overline{f}_k \in U_{\text{ad}}\} \tag{44}$$

is a derived set for Z at $\mathcal{J}_k(f_k^\circ)$, let $d_{k1}, d_{k2}, \ldots, d_{kP}$ be an arbitrary finite collection of vectors from D. We must show that there exist points $\mathcal{J}_{\varepsilon k} \in Z$, which are subject to the vector parameter $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_P)$ for all sufficiently small positive values of ε such that

$$\mathcal{J}_{\varepsilon k} = \mathcal{J}_k(f_k^\circ) + \sum_{j=1}^{P} d_{kj}\varepsilon_j + o(\varepsilon). \tag{45}$$

Since $d_{kj} \in D$, $j = 1,\ldots,P$ $k = 1,\ldots,N$, there exist $(x_1,t_1),\ldots,(x_P,t_P)$ regularity points of f_k° and subfunctions $f_{k1},\ldots,f_{kP} \in U_{\text{ad}}$ such that

$$d_{kj} = (d_k^{-2}(x_j,t_j,f_{kj}),\ldots,d_k^M(x_j,t_j,f_{kj})),\quad j = 1,\ldots,P,\quad k = 1,\ldots,N. \tag{46}$$

To show that $\mathcal{J}_{\varepsilon k}$ can be written as $\mathcal{J}_{\varepsilon k} = \mathcal{J}_k(f_{k\varepsilon})$ where $f_{k\varepsilon}$ is the admissible control given by (15), for $i = 1,\ldots,M$, we observe that

$$\begin{aligned}
&\mathcal{J}_{ik}(f_{k\varepsilon}) - \mathcal{J}_{ik}(f_k^\circ) \\
&= \int_0^{t_f}\int_0^{\ell}[\mathcal{G}_{ik}(x,t,w_\varepsilon(x,t),f_{k\varepsilon}(x,t)) - \mathcal{G}_{ik}(x,t,w^\circ(x,t),f_k^\circ(x,t))]\,dx\,dt \\
&= \int_0^{t_f}\int_0^{\ell}[\mathcal{G}_{ik}(x,t,w_\varepsilon(x,t),f_{k\varepsilon}(x,t)) - \mathcal{G}_{ik}(x,t,w^\circ(x,t),f_{k\varepsilon}(x,t))]\,dx\,dt \\
&\quad + \int_0^{t_f}\int_0^{\ell}[\mathcal{G}_{ik}(x,t,w^\circ(x,t),f_{k\varepsilon}(x,t)) - \mathcal{G}_{ik}(x,t,w^\circ(x,t),f_k^\circ(x,t))]\,dx\,dt \\
&= \int_0^{t_f}\int_0^{\ell}\frac{\partial \mathcal{G}_{ik}}{\partial w}(x,t,w^\circ,f_k^\circ(x,t))\,\Delta w(x,t)\,dx\,dt \\
&\quad + \sum_{j=1}^{P}\varepsilon_j[\mathcal{G}_{ik}(x_j,t_j,w^\circ(x_j,t_j),f_{kj}) - \mathcal{G}_{ik}(x_j,t_j,w^\circ(x_j,t_j),f_{kj}^\circ(x_j,t_j))] \\
&\quad + \sum_{j=1}^{P}o(\varepsilon_j).
\end{aligned} \tag{47}$$

In order to obtain (47), we use that f_k° is regular at each of the points (x_j,t_j) and we use the conclusion of Lemma 1 . If the following equality is substituted in (33)

$$\mathcal{M}v_{ki} = \frac{\partial \mathcal{G}_{ik}}{\partial w}(x,t,w^\circ(x,t),f_k^\circ),\quad i = 1,\ldots,M,\quad k = 1,\ldots,N, \tag{48}$$

it is observed that

$$\begin{aligned}
&\int_0^{\ell}\int_0^{t_f}\Delta w(x,t)\,\mathcal{M}v_{ki}\,dx\,dt \\
&= \int_0^{\ell}\int_0^{t_f}v_{ki}(x,t)(f_{k\varepsilon}(x,t) - f_k^\circ(x,t)) \\
&= \sum_{j=1}^{P}\varepsilon_j v_{ki}(x_j,t_j)(f_{kj} - f_{kj}^\circ(x_j,t_j)) + o(\varepsilon).
\end{aligned} \tag{49}$$

By means of (43) and (47), we can write

$$\mathcal{J}_{ik}(f_{k\varepsilon}) = \mathcal{J}_{ik}(f_k^\circ) + \sum_{j=1}^{P}d_{kj}^i\varepsilon_j + o(\varepsilon),\quad i = 1,\ldots,M,\quad k = 1,\ldots,N, \tag{50}$$

where d_{kj}^i denotes the ith component of d_{kj}. For $i = 0$, we have

$$\begin{aligned}
&\mathcal{J}_{0k}(f_{k\varepsilon}) - \mathcal{J}_{0k}(f_k^\circ) \\
&= \int_0^{\ell}\left[\frac{1}{N}\frac{\partial \mathcal{G}_1}{\partial w}(x,w^\circ(x,t_f))\,\Delta w(x,t_f) + \frac{1}{N}\frac{\partial \mathcal{G}_2}{\partial w_t}(x,w_t^\circ(x,t_f))\,\Delta w_t(x,t_f)\right]dx \\
&\quad + \sum_{j=1}^{P}\varepsilon_j[\mathcal{G}_{0k}(x_j,t_j,w_\varepsilon(x_j,t_j),f_{kj}) - \mathcal{G}_{0k}(x_j,t_j,w_\varepsilon(x_j,t_j),f_{kj}^\circ)] \\
&\quad + \int_0^{\ell}\int_0^{t_f}(\mathcal{M}v_{0k})\,\Delta w(x,t)\,dx\,dt + o(\varepsilon),
\end{aligned} \tag{51}$$

in which $\mathcal{J}_{0k} = (1/N)\int_0^{\ell}[\mathcal{G}_1(x, w(x,t_f)) + \mathcal{G}_2(x, w_t(x, t_f))]dx + \int_0^{t_f}\int_0^{\ell}\mathcal{G}_{0k}(x,t,w,f_k)dx\,dt$. Considering (33) and (41a), (41b), (41c), (41d), (41e), and (41f), it is observed that

$$\begin{aligned}\int_0^{t_f}\int_0^{\ell}\Delta w(x,t)\,\mathcal{M}v_{0k}dx\,dt &= \int_0^{t_f}\int_0^{\ell}v_{0k}(x,t)\left(f_{k\varepsilon}(x,t)-f_k^{\circ}(x,t)\right)dx\,dt\\ &\quad-\int_0^{\ell}\left[\frac{1}{N}\frac{\partial\mathcal{G}_1}{\partial w}\left(x,w^{\circ}\left(x,t_f\right)\right)\Delta w\left(x,t_f\right)\right.\\ &\qquad\left.+\frac{1}{N}\frac{\partial\mathcal{G}_2}{\partial w_t}\left(x,w_t\left(x,t_f\right)\right)\Delta w_t\left(x,t_f\right)\right].\end{aligned} \tag{52}$$

If (52) is substituted into (51), (50) is obtained for $i = 0$, $k = 1,\ldots,N$. For $i = -2, -1$, (50) can be obtained by using (33)–(41a), (41b), (41c), (41d), (41e), and (41f). By the definition $\mathcal{J}_k$, one obtains

$$\mathcal{J}_k\left(f_{k\varepsilon}\right)-\mathcal{J}_k\left(f_k^{\circ}\right)=\sum_{j=1}^{P}d_{kj}\varepsilon_j+o(\varepsilon), \tag{53}$$

where $\mathcal{J}_{k\varepsilon}$ is taken as $\mathcal{J}_k(f_{k\varepsilon})$. This completes the proof of D being a derived set for Z at $\mathcal{J}_k(f_k^{\circ})$. Then, there exist non-positive Lagrange multipliers [36], not zero simultaneously, satisfying

$$\sum_{i=-2}^{M}\lambda_{ik}d_k^i \le 0, \qquad \lambda_{0k}=\lambda_0, \tag{54}$$

for any vector $d_k = (d_k^{-2}, d_k^{-1}, d_k^0, \ldots, d_k^M)$ in D. Let us take $P = 1$ in the foregoing discussion to obtain the conclusion of Lemma 5 and put

$$\overline{\mathcal{J}}_k=\sum_{i=-2}^{M}\lambda_{ik}\mathcal{J}_{ik}. \tag{55}$$

By (50), it follows that

$$\overline{\mathcal{J}}_k\left(f_{k\varepsilon}\right)-\overline{\mathcal{J}}_k\left(f_k^{\circ}\right)=\varepsilon\sum_{i=-2}^{M}\lambda_{ik}d_k^i+o(\varepsilon), \tag{56}$$

for any $d_k = (d_k^{-2}, d_k^{-1}, d_k^0, \ldots, d_k^M)$ in D. Then, we obtain the proof of Lemma 5 as follows:

$$\lim_{\varepsilon\to 0^+}\frac{\overline{\mathcal{J}}_k\left(f_{k\varepsilon}\right)-\overline{\mathcal{J}}_k\left(f_k^{\circ}\right)}{\varepsilon}=\sum_{i=-2}^{M}\lambda_{ik}d_k^i\le 0. \tag{57}$$

□

Theorem 6 (maximum principle). *For the optimal control functions $f_1^{\circ}(x,t),\ldots,f_N^{\circ}(x,t) \in U_{ad}$, the corresponding optimal state and adjoint variables are $w^{\circ}(x,t) = w(x,t,f_1^{\circ}(x,t), f_2^{\circ}(x,t),\ldots,f_N^{\circ}(x,t))$ satisfying* (1)–(4) *and $v^{\circ}(x,t) = v(x,t,f_1^{\circ}(x,t), f_2^{\circ}(x,t),\ldots,f_N^{\circ}(x,t))$ satisfying* (11), *boundary conditions* (13a) *and* (13b), *and terminal conditions* (14a) *and* (14b), *respectively. The maximum principle states that if*

$$\begin{aligned}&\mathcal{H}\left[x,t,v^{\circ},f_1^{\circ}(x,t),\ldots,f_N^{\circ}(x,t)\right]\\ &\quad=\max_{f_k\in U_{ad}}\mathcal{H}\left[x,t,v,f_1(x,t),\ldots,f_N(x,t)\right], \quad k=1,\ldots,N,\end{aligned} \tag{58}$$

where the Hamiltonian is given by

$$\begin{aligned}\mathcal{H}\left[x,t,v,f_1,\ldots,f_N\right]&=\sum_{k=1}^{N}v(x,t)\,f_k\\ &\quad+\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\mathcal{G}_{ik}\left(x,t,w(x,t),f_k\right),\end{aligned} \tag{59}$$

then the performance index equation (8) *is minimized; that is,*

$$\mathcal{J}_0\left[f_1^{\circ}(x,t),\ldots,f_N^{\circ}(x,t)\right]\le\mathcal{J}_0\left[f_1(x,t),\ldots,f_N(x,t)\right] \quad \text{for any } f_k\in U_{ad}. \tag{60}$$

Proof. Let (x,t) be a regular point for admissible optimal control functions f_k°, $k = 1,\ldots,N$. By Lemma 5, for $0 \le i \le m$ and some $\lambda_{ik} \ne 0$, there exists Lagrange multipliers $\lambda_{-2k}, \lambda_{-1k}, \lambda_0, \ldots, \lambda_{NM}$ independent of (x,t) with $\lambda_{ik} \le 0$ such that

$$\begin{aligned}\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}&\left[v_{ik}(x,t)\left(f_k-f_k^{\circ}(x,t)\right)+\mathcal{G}_{ik}\left(x,t,w(x,t),f_k\right)\right.\\ &\left.-\mathcal{G}_{ik}\left(x,t,w^{\circ}(x,t),f_k^{\circ}(x,t)\right)\right]\le 0,\end{aligned} \tag{61}$$

for any function $f_k \in U_{\text{ad}}$. Note that the term

$$\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}v_{ik}(x,t)\,f_k+\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\mathcal{G}_{ik}\left(x,t,w(x,t),f_k\right) \quad \text{in (61)} \tag{62}$$

reaches its maximum value at $f_k = f_k^{\circ}(x,t) \in U_{\text{ad}}$, for $k = 1,\ldots,N$. Let us consider the first term in (62),

$$\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}v_{ik}(x,t)\,f_k, \tag{63}$$

that can be rewritten in the form

$$\sum_{k=1}^{N}\sum_{i=-2}^{M}\sum_{r=1}^{N}\lambda_{ik}v_{ik}(x,t)\,f_r \tag{64}$$

to be subject to

$$\lambda_{ik}v_{ik}(x,t)\,f_r=\begin{cases}\lambda_{ik}v_{ik}(x,t)\,f_k, & r=k,\\ 0, & r\ne k.\end{cases} \tag{65}$$

If we define $v = \sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}v_{ik}(x,t)$, we obtain

$$\sum_{k=1}^{N}v(x,t)\,f_k+\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\mathcal{G}_{ik}\left(x,t,w(x,t),f_k\right), \tag{66}$$

the conclusion of the maximum principle. This completes the proof of Theorem 6. □

Theorem 7. *Consider the control system equations* (1)–(4) *and* (8)-(9a), (9b), (9c), *and* (9d). *Let the functions* G_{ik} *be in the form*

$$G_{ik}(x,t,w,f_k) = \mathscr{G}_k^i(x,t,w) + H_k^i(x,t,f_k), \quad i=-2,-1,0,\ldots,M, \quad k=1,\ldots,N \tag{67}$$

and let v *satisfying* (13a) *and* (13b)-(14a) *and* (14b) *be the nonzero solution of*

$$\mathscr{M}v = \sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\frac{\partial G_k^i(x,t,w^\circ(x,t))}{\partial w}. \tag{68}$$

Assume that there exist admissible control functions $f_1^\circ, f_2^\circ,\ldots,f_N^\circ$ *and the constants* λ_0, λ_{ik}, $i = -2,-1,1,\ldots,M, k = 1,\ldots,N$ *that satisfy the maximum principle equations* (58). *Let us assume that following assumptions are satisfied:*

(a) *for* $k = 1,\ldots,N$, $\mathscr{G}_1, h_{1k}, \mathscr{G}_{ik},\ldots,\mathscr{G}_{mk}$ *are convex functions of* w *and* $\mathscr{G}_2$, h_{2k} *are convex functions of* w_t;

(b) $\lambda_0 < 0$, $\lambda_{ik} \le 0$, *for* $i = -1,\ldots,m$, $k = 1,\ldots,N$;

(c) *the constraints equation* (9a), (9b), (9c), *and* (9d) *are satisfied by* $f_1^\circ, f_2^\circ,\ldots,f_N^\circ$;

(d) *if the strict inequality holds in* (9a), (9b), (9c), *and* (9d), *the corresponding Lagrange multiplier* $\lambda_{ik} = 0$;

(e) $-\lambda_{ik}G_k^i$, $-\lambda_{-1k}h_{1k}$ *are convex functions of* w *and* $-\lambda_{-2k}h_{2k}$ *is convex function of* w_t, *for* $k = 1,\ldots, N$, $m < i < M$.

Under these assumptions, the maximum principle given by (58) *is also sufficient condition for the admissible control functions* $f_1^\circ, f_2^\circ,\ldots,f_N^\circ$ *to be optimal. The condition (d) is proved in [36]. If the* h_{1k}, G_{ik}, *and* h_{2k} *are linear functions of* w *and* w_t, *respectively, the condition (e) is satisfied, for* $k = 1,\ldots,N$, $m < i \le M$.

Proof. If f_k's and w satisfy (9a), (9b), (9c), and (9d), then, by the condition (d),

$$\int_0^{t_f}\int_0^{\ell}\lambda_{ik}\left[G_k^i(x,t,w) - G_k^i(x,t,w^\circ)\right]dx\,dt + \int_0^{t_f}\int_0^{\ell}\lambda_{ik}\left[H_k^i(x,t,f_k) - H_k^i(x,t,f_k^\circ)\right]dx\,dt = 0, \tag{69}$$

for $i = -2,-1,\ldots,M$, $k = 1,\ldots,N$. Then, we can write the following inequality

$$\begin{aligned}
&-\lambda_0\left[\mathscr{J}_0(f_1,f_2,\ldots,f_N) - \mathscr{J}_0(f_1^\circ,f_2^\circ,\ldots,f_N^\circ)\right]\\
&\quad\ge -\int_0^{\ell}\lambda_0\left[\mathscr{G}_2\left(x,w_t\left(x,t_f\right)\right) - \mathscr{G}_2\left(x,w_t^\circ\left(x,t_f\right)\right) + \mathscr{G}_1\left(x,w\left(x,t_f\right)\right) - \mathscr{G}_1\left(x,w^\circ\left(x,t_f\right)\right)\right]dx\\
&\quad -\int_0^{t_f}\int_0^{\ell}\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\left\{G_k^i(x,t,w(x,t)) - G_k^i(x,t,w^\circ(x,t)) - \left[H_k^i(x,t,f_k(x,t)) - H_k^i(x,t,f_k^\circ(x,t))\right]\right\}dx\,dt\\
&\quad -\int_0^{\ell}\sum_{k=1}^{N}\lambda_{-1k}\left[h_{1k}\left(x,w\left(x,t_f\right)\right) - h_{1k}\left(x,w^\circ\left(x,t_f\right)\right)\right]dx\\
&\quad -\int_0^{\ell}\sum_{k=1}^{N}\lambda_{-2k}\left[h_{2k}\left(x,w_t\left(x,t_f\right)\right) - h_{2k}\left(x,w_t^\circ\left(x,t_f\right)\right)\right]dx
\end{aligned} \tag{70}$$

By using the convexity assumption (e),

$$\begin{aligned}
&-\lambda_0\left[\mathscr{J}_0(f_1,f_2,\ldots,f_N) - \mathscr{J}_0(f_1^\circ,f_2^\circ,\ldots,f_N^\circ)\right]\\
&\quad\ge -\int_0^{\ell}\lambda_0\left[\frac{\partial\mathscr{G}_1}{\partial w}\left(x,w^\circ\left(x,t_f\right)\Delta w\left(x,t_f\right)\right) + \frac{\partial\mathscr{G}_2}{\partial w_t}\left(x,w_t^\circ\left(x,t_f\right)\Delta w_t\left(x,t_f\right)\right)\right]dx\\
&\quad -\int_0^{t_f}\int_0^{\ell}\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\frac{\partial G_k^i}{\partial w}(x,t,w^\circ(x,t))\,\Delta w(x,t)\,dx\,dt\\
&\quad +\int_0^{t_f}\int_0^{\ell}\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\left[H_k^i(x,t,f_k^\circ(x,t)) - H_k^i(x,t,f_k(x,t))\right]dx\,dt\\
&\quad -\int_0^{\ell}\sum_{k=1}^{N}\lambda_{-1k}\frac{\partial h_{1k}}{\partial w}\left(x,w^\circ\left(x,t_f\right)\right)\Delta w\left(x,t_f\right)dx\\
&\quad -\int_0^{\ell}\sum_{k=1}^{N}\lambda_{-2k}\frac{\partial h_{2k}}{\partial w_t}\left(x,w_t^\circ\left(x,t_f\right)\right)\Delta w_t\left(x,t_f\right)dx\\
&\quad = -\int_0^{\ell}\lambda_0\left[\frac{\partial\mathscr{G}_1}{\partial w}\left(x,w^\circ\left(x,t_f\right)\right)\Delta w\left(x,t_f\right) + \frac{\partial\mathscr{G}_2}{\partial w_t}\left(x,w_t^\circ\left(x,t_f\right)\right)\Delta w_t\left(x,t_f\right)\right]dx\\
&\quad -\int_0^{t_f}\int_0^{\ell}(\mathscr{M}v)\,\Delta w(x,t)\,dx\,dt\\
&\quad +\int_0^{t_f}\int_0^{\ell}\sum_{k=1}^{N}\sum_{i=-2}^{M}\lambda_{ik}\left[H_k^i(x,t,f_k^\circ(x,t)) - H_k^i(x,t,f_k(x,t))\right]dx\,dt\\
&\quad -\int_0^{\ell}\sum_{k=1}^{N}\left[\lambda_{-1k}\frac{\partial h_{1k}}{\partial w}\left(x,w^\circ\left(x,t_f\right)\right)\Delta w\left(x,t_f\right) + \lambda_{-2k}\frac{\partial h_{2k}}{\partial w_t}\left(x,w_t^\circ\left(x,t_f\right)\right)\Delta w_t\left(x,t_f\right)\right]dx.
\end{aligned} \tag{71}$$

And finally by applying Lemma 2 and the conditions (13a) and (13b)-(14a) and (14b), we obtain

$$-\lambda_0 \left[\mathcal{J}_0 \left(f_1, f_2, \ldots, f_N\right) - \mathcal{J}_0 \left(f_1^{\circ}, f_2^{\circ}, \ldots, f_N^{\circ}\right)\right] \geq \int_0^{t_f} \int_0^{\ell} \left\{ \sum_{k=1}^{N} \nu(x,t) \left[f_k^{\circ}(x,t) - f_k(x,t)\right] + \sum_{k=1}^{N} \sum_{i=-2}^{M} \lambda_{ik} \left[H_k^i\left(x,t,f_k^{\circ}(x,t)\right) - H_k^i\left(x,t,f_k(x,t)\right)\right] \right\} dx\, dt. \quad (72)$$

Note that the right-hand side of (72) is nonnegative due to the condition (b). Then, we obtain

$$\mathcal{J}_0 \left(f_1(x,t), f_2(x,t), \ldots, f_N(x,t)\right) - \mathcal{J}_0 \left(f_1^{\circ}(x,t), f_2^{\circ}(x,t), \ldots, f_N^{\circ}(x,t)\right) \geq 0. \quad (73)$$

It follows that the maximum principle is also a sufficient condition for a global minimum of the performance index (8). It completes the proof of Theorem 7. □

5. Conclusion

A necessary optimality condition for a general class of damped hyperbolic partial differential equation in one-space dimension is derived in the form of a maximum principle by using the derived set and regular point concepts. Under proper convexity assumption on state variable, it is proved that-the maximum principle is also sufficient condition for the controls to be optimal . In [1–7], the systems under consideration include only one control function. Also, the necessary and sufficient conditions for optimality are given for a single hyperbolic differential equation without damping term subject to the homogeneous boundary conditions in one-space dimension. But, in the present paper, as an original contribution to literature, the necessary and sufficient optimality conditions obtained in [1–7] are generalized for a general class of damped hyperbolic equation involving damping and several control functions subject to nonhomogeneous boundary conditions in one space dimension.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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