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Research Article

Strong Convergence Theorems for Solutions of Equilibrium Problems and Common Fixed Points of a Finite Family of Asymptotically Nonextensive Nonself Mappings

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An iterative algorithm for finding a common element of the set of common fixed points of a finite family of asymptotically nonextensive nonself mappings and the set of solutions for equilibrium problems is discussed. A strong convergence theorem of common element is established in a uniformly smooth and uniformly convex Banach space.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual of E, and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Suppose that C is a nonempty, closed convex subset of E. Let f be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $f : C \times C \to R$ is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $p \in EP(f)$ if and only if $\langle Tp, y - p\rangle \geq 0$ for all $y \in C$; that is, p is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). Some methods have been proposed to solve the equilibrium problems; see [1–5].

Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\| \|f\|, \|x\| = \|f\|\} \tag{2}$$

for all $x \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Let C be a nonempty closed convex subset of a Hilbert space H and let $P_C : H \to C$ be the metric projection of H onto C; then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently it is not available in more general Banach spaces. In this connection, Alber [6] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{3}$$

Observe that, in a Hilbert space H, (3) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \tag{4}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(y,x) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E, \tag{5}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{6}$$

$$\begin{aligned}\phi(x,y) &= \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \\ &\le \|x\| \|Jx - Jy\| + \|y - x\| \|y\|, \quad \forall x, y, z \in E.\end{aligned} \tag{7}$$

Let C be a nonempty subset of E and let $T : C \to E$ be a mapping. The set of fixed points of T is denoted by $F(T)$. $T : C \to E$ is called asymptotically nonextensive if and only if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\phi\left(T(\Pi_C T)^{n-1}x, T(\Pi_C T)^{n-1}y\right) \le k_n \phi(x,y), \quad \forall x, y \in C, \quad n \ge 1. \tag{8}$$

Asymptotically nonextensive mappings coincide with asymptotically nonexpansive mappings in Hilbert spaces.

In [7], Chidume et al. studied the fixed point problem of an asymptotically nonextensive nonself mapping and obtained weak convergence theorem. Recently, in [8], liu introduced the following iterative scheme for approximating a common fixed point of two asymptotically nonextensive nonself mappings in a uniformly smooth and uniformly convex Banach space:

$$\begin{aligned} y_n &= \Pi_C\left(J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JS(\Pi_C S)^{n-1} x_n\right)\right), \\ x_{n+1} &= \Pi_C\left(J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT(\Pi_C T)^{n-1} y_n\right)\right). \end{aligned} \tag{9}$$

Liu obtained strong convergence theorem.

Inspired and motivated by the facts above, the purpose of this paper is to prove a strong convergence theorem for finding a common element of the set of common fixed points of a finite family of asymptotically nonextensive nonself mappings and the set of solutions for equilibrium problems in a uniformly smooth and uniformly convex Banach space.

2. Preliminaries

Let E be a real Banach space. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. E is said to have the Kadec-Klee property if and only if for a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property.

A mapping $T : C \to C$ is said to be closed; if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Lemma 1. *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec-Klee property, let C be a nonempty, closed, and convex subset of E, and let $T : C \to E$ be an asymptotically nonextensive nonself mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that T is closed. Then $F(T)$ is closed and convex.*

Proof. Take $x, y \in F(T)$, $t \in (0,1)$. Put $z := tx + (1-t)y$. Using the same argument presented in the proof of [9, Theorem 2.1, page 854-855], we can obtain that $\lim_{n\to\infty} T(\Pi_C T)^{n-1} z = z$. By the continuity of Π_C, we have

$$\lim_{n\to\infty} (\Pi_C T)^n z = z. \tag{10}$$

Therefore,

$$\lim_{n\to\infty} \left((\Pi_C T)^n z - T(\Pi_C T)^n z\right) = 0. \tag{11}$$

By (10), (11) and the closedness of T, we have $z \in F(T)$ which implies that $F(T)$ is convex.

Let $x_n \in F(T)$ and $x_n \to q$; then, we have $x_n - Tx_n \to 0$. It follows from the closedness of T that $q \in F(T)$. This implies that $F(T)$ is closed. □

Lemma 2 (see [6]). *Let E be a reflexive, strictly convex, and smooth Banach space; let C be a nonempty, closed, and convex subset of E. Then the following conclusions hold:*

(1) $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y,x)$, *for all $y \in C$, and $x \in E$;*

(2) *if $x \in E$ and $z \in C$, then $z = \Pi_C x$ if and only if $\langle z - y, Jx - Jz\rangle \ge 0$, for all $y \in C$;*

(3) *for $x, y \in E$, $\phi(y,x) = 0$ if and only if $x = y$.*

Lemma 3 (see [10]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 4 (see [11]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|) \tag{12}$$

for all $x, y \in B_r$ and $t \in [0,1]$, where $B_r = \{z \in E : \|z\| \le r\}$.

For solving the equilibrium problem, let us assume that a bifunction $f : C \times C \to R$ satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone; that is, $f(x,y) + f(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\to 0} f(tz + (1-t)x, y) \le f(x,y); \tag{13}$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Lemma 5 (see [12]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C\times C$ to R satisfying (A1)–(A4), and for $r>0$ and $x\in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}. \tag{14}$$

Then the following conclusions hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive; that is, for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{15}$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex;*

(5) *$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$, for all $q \in F(T_r)$.*

3. Main Results

Theorem 6. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let N be some positive integer. Let $S_i : C \to E$ be a closed asymptotically nonextensive nonself mapping with sequence $\{k_{n,i}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_{n,i}-1) < \infty$ for every $1 \leq i \leq N$. Suppose that $\Omega = \bigcap_{i=1}^{N} F(S_i) \cap EP(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{aligned}
&x_0 \in E, \qquad C_1 = C,\\
&x_1 = \Pi_{C_1} x_0,\\
&y_n = J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J S_i (\Pi_C S_i)^{n-1} x_n\right),\\
&u_n = T_{r_n} y_n,\\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_0,
\end{aligned} \tag{16}$$

where $\theta_n = (k_n - 1)\sup_{z\in\Omega}\phi(z, x_n)$, $k_n = \max\{k_{n,i}\}$. $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ for every $0 \leq i \leq N$, $\{r_n\}$ is a real number sequence in $[a,\infty)$, where a is some positive real number. Assume that $\sum_{i=0}^{N}\alpha_{n,i} = 1$ and $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$ for every $1 \leq i \leq N$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

Proof. First, we show that C_n is closed and convex. From the definitions of C_n, it is obvious C_n is closed. Moreover, since $\phi(z,u_n) \leq \phi(z,x_n) + \theta_n$ is equivalent to $2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2 + \theta_n$, it follows that C_n is convex. From Lemmas 1 and 5, we have that Ω is closed and convex. Then $\{x_n\}$ is well defined.

Next, we prove $\Omega \subset C_n$ for all $n \geq 1$. $\Omega \subset C_1 = C$ is obvious. Suppose that $\Omega \subset C_n$ for some $n \geq 2$; for each $z \in \Omega$, from Lemma 5, we have

$$\begin{aligned}
&\phi(z, u_n)\\
&\quad = \phi(z, T_{r_n} y_n) \leq \phi(z, y_n)\\
&\quad = \|z\|^2 - 2\left\langle z, \alpha_{n,0} J x_n + \sum_{i=1}^{N}\alpha_{n,i} J S_i (\Pi_C S_i)^{n-1} x_n \right\rangle\\
&\quad + \left\| \alpha_{n,0} J x_n + \sum_{i=1}^{N}\alpha_{n,i} J S_i (\Pi_C S_i)^{n-1} x_n \right\|^2\\
&\quad \leq \|z\|^2 - 2\alpha_{n,0}\langle z, Jx_n\rangle - 2\sum_{i=1}^{N}\alpha_{n,i}\langle z, JS_i(\Pi_C S_i)^{n-1}x_n\rangle\\
&\quad + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{N}\alpha_{n,i}\left\|S_i(\Pi_C S_i)^{n-1}x_n\right\|^2\\
&\quad = \alpha_{n,0}\phi(z, x_n) + \sum_{i=1}^{N}\alpha_{n,i}\phi\left(z, S_i(\Pi_C S_i)^{n-1}x_n\right)\\
&\quad \leq \alpha_{n,0}\phi(z, x_n) + \sum_{i=1}^{N}\alpha_{n,i}k_{n,i}\phi(z, x_n)\\
&\quad \leq \alpha_{n,0}\phi(z, x_n) + \sum_{i=1}^{N}\alpha_{n,i}k_n\phi(z, x_n)\\
&\quad = \phi(z, x_n) + (1-\alpha_{n,0})(k_n - 1)\phi(z, x_n)\\
&\quad \leq \phi(z, x_n) + \theta_n.
\end{aligned} \tag{17}$$

This implies that $z \in C_{n+1}$, and so $\Omega \subset C_{n+1}$. From $x_n = \Pi_{C_n} x_0$, one sees

$$\langle x_n - u, Jx_0 - Jx_n\rangle \geq 0, \quad \forall u \in C_n. \tag{18}$$

Since $\Omega \subset C_{n+1}$, we arrive at

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in \Omega. \tag{19}$$

Next we show that the sequence $\{x_n\}$ is bounded. From Lemma 2, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(z, x_0) - \phi(z, x_n) \leq \phi(z, x_0), \tag{20}$$

for each $z \in \Omega \subset C_n$ and for all $n \geq 1$. Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (5) that the sequence $\{x_n\}$ is also bounded. By the assumption, we have

$$\lim_{n\to\infty}\theta_n = 0. \tag{21}$$

On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \tag{22}$$

for all $n \geq 1$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the definition of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{23}$$

Letting $m, n \to \infty$ in (23), we have $\phi(x_m, x_n) \to 0$. It follows from Lemma 3 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is a closed and convex, one can assume that $x_n \to \overline{x} \in C$ as $n \to \infty$.

Next we show that $\overline{x} \in \bigcap_{i=1}^N F(S_i)$. By taking $m = 1$ in (23), we have that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{24}$$

From Lemma 3, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{25}$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n. \tag{26}$$

It follows from (21) and (24) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{27}$$

From Lemma 3, we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{28}$$

Combining (25) with (28), we obtain that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{29}$$

It follows from $x_n \to \overline{x}$ as $n \to \infty$ that $u_n \to \overline{x}$, *as* $n \to \infty$. Since J is uniformly norm-to-norm continuous on each bounded set, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{30}$$

On the other hand, we have

$$\begin{aligned}&\phi(z, x_n) - \phi(z, u_n) \\ &\quad = \|x_n\|^2 - \|u_n\|^2 - 2\langle z, Jx_n - Ju_n\rangle \\ &\quad \leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|z\| \|Jx_n - Ju_n\|.\end{aligned} \tag{31}$$

We obtain that

$$\lim_{n\to\infty} (\phi(z, x_n) - \phi(z, u_n)) = 0. \tag{32}$$

Since E is a uniformly smooth Banach space, we know that E^* is a uniformly convex Banach space. From Lemma 4, we find that

$$\begin{aligned}&\phi(z, u_n) \\ &\quad = \phi(z, T_{r_n} y_n) \leq \phi(z, y_n) \\ &\quad = \|z\|^2 - 2\left\langle z, \alpha_{n,0} Jx_n + \sum_{i=1}^N \alpha_{n,i} JS_i (\Pi_C S_i)^{n-1} x_n \right\rangle \\ &\qquad + \left\| \alpha_{n,0} Jx_n + \sum_{i=1}^N \alpha_{n,i} JS_i (\Pi_C S_i)^{n-1} x_n \right\|^2 \\ &\quad \leq \|z\|^2 - 2\alpha_{n,0} \langle z, Jx_n\rangle - 2\sum_{i=1}^N \alpha_{n,i} \left\langle z, JS_i (\Pi_C S_i)^{n-1} x_n \right\rangle \\ &\qquad + \alpha_{n,0} \|x_n\|^2 + \sum_{i=1}^N \alpha_{n,i} \left\| S_i (\Pi_C S_i)^{n-1} x_n \right\|^2 \\ &\qquad - \alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) \\ &\quad = \alpha_{n,0} \phi(z, x_n) + \sum_{i=1}^N \alpha_{n,i} \phi\left(z, S_i (\Pi_C S_i)^{n-1} x_n\right) \\ &\qquad - \alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) \\ &\quad \leq \alpha_{n,0} \phi(z, x_n) + \sum_{i=1}^N \alpha_{n,i} k_{n,i} \phi(z, x_n) \\ &\qquad - \alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) \\ &\quad \leq \alpha_{n,0} \phi(z, x_n) + \sum_{i=1}^N \alpha_{n,i} k_n \phi(z, x_n) \\ &\qquad - \alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) \\ &\quad = \phi(z, x_n) + (1 - \alpha_{n,0})(k_n - 1)\phi(z, x_n) \\ &\qquad - \alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) \\ &\quad \leq \phi(z, x_n) + \theta_n - \alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right).\end{aligned} \tag{33}$$

Therefore we have

$$\begin{aligned}&\alpha_{n,0}\alpha_{n,1} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) \\ &\quad \leq \phi(z, x_n) - \phi(z, u_n) + \theta_n.\end{aligned} \tag{34}$$

From $\liminf_{n\to} \alpha_{n,0}\alpha_{n,1} > 0$ and (21), (32), we have

$$\lim_{n\to\infty} g\left(\left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\|\right) = 0. \tag{35}$$

Therefore, from the property of g we have

$$\lim_{n\to\infty} \left\| Jx_n - JS_1 (\Pi_C S_1)^{n-1} x_n \right\| = 0. \tag{36}$$

Since J^{-1} is also uniformly norm-to-norm continuous on each bounded set, we have

$$\lim_{n\to\infty} \left\| x_n - S_1(\Pi_C S_1)^{n-1} x_n \right\| = 0. \quad (37)$$

Using (7), (34), and (36), we have

$$\lim_{n\to\infty} \phi\left(x_n, S_1(\Pi_C S_1)^{n-1} x_n\right) = 0. \quad (38)$$

By (6), we obtain

$$\begin{aligned}\phi(x_n, S_1 x_n) &= \phi(x_n, x_{n+1}) + \phi(x_{n+1}, S_1 x_n) \\ &\quad + 2\langle x_n - x_{n+1}, Jx_{n+1} - JS_1 x_n\rangle \\ &= \phi(x_n, x_{n+1}) + \phi\left(x_{n+1}, S_1(\Pi_C S_1)^n x_{n+1}\right) \\ &\quad + \phi\left(S_1(\Pi_C S_1)^n x_{n+1}, S_1(\Pi_C S_1)^n x_n\right) \\ &\quad + \phi\left(S_1(\Pi_C S_1)^n x_n, S_1 x_n\right) \\ &\quad + 2\left\langle S_1(\Pi_C S_1)^n x_{n+1} \right. \\ &\qquad \left. - S_1(\Pi_C S_1)^n x_n, JS_1(\Pi_C S_1)^n x_n - JS_1 x_n\right\rangle \\ &\quad + 2\left\langle x_{n+1} - S_1(\Pi_C S_1)^n x_{n+1}, JS_1(\Pi_C S_1)^n x_{n+1} \right. \\ &\qquad \left. - JS_1 x_n\right\rangle \\ &\quad + 2\langle x_n - x_{n+1}, Jx_{n+1} - JS_1 x_n\rangle.\end{aligned} \quad (39)$$

Since $\phi(x_n, (\Pi_C S_1)^n x_n) \le \phi(x_n, S_1(\Pi_C S_1)^{n-1} x_n)$, from (38), we have $\lim_{n\to\infty}\phi(x_n, (\Pi_C S_1)^n x_n) = 0$. Since $\phi(S_1(\Pi_C S_1)^n x_n, S_1 x_n) \le k_1 \phi((\Pi_C S_1)^n x_n, x_n)$, then

$$\lim_{n\to\infty} \phi\left(S_1(\Pi_C S_1)^n x_n, S_1 x_n\right) = 0. \quad (40)$$

Applying (24), (38), (40), the definition of S_1, and Lemma 3 to (39), we obtain that

$$\lim_{n\to\infty} \phi(x_n, S_1 x_n) = 0. \quad (41)$$

From Lemma 3, we obtain that

$$\lim_{n\to\infty} \|x_n - S_1 x_n\| = 0. \quad (42)$$

In the same way, we can obtain

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0, \quad 2 \le i \le N. \quad (43)$$

From the closedness of S_i, $1 \le i \le N$, we have $\overline{x} \in \bigcap_{i=1}^N F(S_i)$.

Next, we show $\overline{x} \in EP(f)$. From Lemma 5, we have

$$\begin{aligned}\phi(u_n, y_n) &= \phi(Tr_n y_n, y_n) \\ &\le \phi(z, y_n) - \phi(z, Tr_n y_n) \\ &\le \phi(z, y_n) - \phi(z, u_n) \\ &= \phi(z, x_n) + \theta_n - \phi(z, u_n).\end{aligned} \quad (44)$$

It follows from (21) and (32) that $\lim_{n\to\infty}\phi(u_n, y_n) = 0$. From Lemma 3, we see that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \quad (45)$$

Since J is uniformly norm-to-norm continuous on each bounded set, we have

$$\lim_{n\to\infty} \|Jy_n - Ju_n\| = 0. \quad (46)$$

From $r_n \ge a$, we have

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \quad (47)$$

By $u_n = T_{r_n} y_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C. \quad (48)$$

From (A2), we have

$$\begin{aligned}\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\ge -f(u_n, y) \ge f(y, u_n), \quad \forall y \in C.\end{aligned} \quad (49)$$

Letting $n \to \infty$, we have from (A4), (47) and $u_n \to \overline{x}$, as $n \to \infty$ that

$$f(y, \overline{x}) \le 0, \quad \forall y \in C. \quad (50)$$

For $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)\overline{x}$. Since $y \in C$ and $\overline{x} \in C$, we have $y_t \in C$ and hence $f(y_t, \overline{x}) \le 0$. So, from (A1) and (A4) we have

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t) f(y_t, \overline{x}) \le tf(y_t, y). \quad (51)$$

Dividing by t, we have

$$f(y_t, y) \ge 0, \quad \forall y \in C. \quad (52)$$

Letting $t \to 0$, from (A3), we have $f(\overline{x}, y) \ge 0$, for all $y \in C$. Therefore, $\overline{x} \in EP(f)$.

Finally, we show $\overline{x} = \Pi_\Omega x_0$. By taking limit in (19), we have

$$\langle \overline{x} - z, Jx_0 - J\overline{x}\rangle \ge 0, \quad \forall z \in \Omega. \quad (53)$$

At this point, in view of Lemma 2, we have that $\overline{x} = \Pi_\Omega x_0$. This completes the proof. □

Remark 7. Theorem 6 improves the main theorem in [8] in the following senses.

(1) Theorem 6 generalizes this theorem from two asymptotically nonextensive operators to a finite family of asymptotically nonextensive operators.

(2) Theorem 6 removes the condition that S_i is completely continuous or semicompact.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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