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Research Article

Robust Synchronization of Fractional-Order Hyperchaotic Systems Subjected to Input Nonlinearity and Unmatched External Perturbations

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This paper investigates the robust synchronization problem for a class of fractional-order hyperchaotic systems subjected to unmatched uncertainties and input nonlinearity. Based on the sliding mode control (SMC) technique, this approach only uses a single controller to achieve chaos synchronization, which reduces the cost and complexity for synchronization control implementation. As expected, the error states can be driven to zero or into predictable bounds for matched and unmatched perturbations, respectively, even with input nonlinearity.

1. Introduction

Synchronization, which means "designing a system whose behavior mimics that of another chaotic system," has become more and more interesting topic to engineering and science communities [1]. Fractional calculus as an extension of ordinary calculus has a 300-year-old mathematical topic; the applications of the fractional calculus to physics and engineering are just a recent focus of interest [2, 3]. It has been recognized that many dynamical systems can be more precisely modeled by using the means of the fractional calculus, such as mechanics [4, 5], image processing [6], viscoelastic materials [7], electrical circuits [8], and population models [9]. Meanwhile, it has been demonstrated that some dynamics of fractional-order systems can behave chaotically or hyperchaotically [10, 11]. Due to the potential applications in physics and engineering, many methods have been presented to achieve synchronization for fractional-order chaotic systems such as sliding mode control [12, 13], H^{∞} control method [14], and active control [15], among many others [16, 17]. Unfortunately, all synchronization schemes in the above-mentioned papers for fractional-order chaotic systems are derived on the basis of the ideal assumption of control input or matched external perturbations. As well known, the control schemes for robust chaos synchronization can be realized by electronic components such as operational amplifier (OPA), resistor, and capacitor. However, in practice, there always exists nonlinearity in the control input including saturation, backlash, and dead zone in OPA or electromechanical devices. Therefore the implementation of control inputs of practical systems is frequently subjected to nonlinearity as a result of physical limitations. It has been shown that input nonlinearity might cause a serious degradation of the system performance, a reduced rate of response, and, in a worst-case scenario, system failure if the controller is not well designed [18, 19]. Therefore, its effect cannot be ignored in analysis of control design and realization for chaos synchronization. On the other hand, for designing a robust control, sliding mode control is frequently adopted due to its inherent advantages of easy realization, fast response, good transient performance, and being insensitive to variation in plant parameters or external disturbances [20, 21]. However, the property of robustness to external perturbations is just for the case of matched condition. The dynamics of controlled systems in the sliding manifold is still influenced by unmatched perturbations.

Therefore, it still needs to discuss the effect of unmatched external perturbations for fractional-order chaotic systems in the sliding mode.

Motivated by the above discussions, this paper considers the robust synchronization problem for robust synchronization of fractional-order hyperchaotic systems subjected to input nonlinearity and unmatched external perturbations. To achieve this goal, a new fraction-integer integral (FII) switching surface is newly proposed such that it becomes easy to analyze the stability of the closed-loop nonlinear chaotic systems. Having established the fractional switching surface, a sliding mode controller is designed. This controller is robust to the nonlinear input and guarantees the occurrence of sliding motion of the controlled fractional-order chaotic system. In our design, a single controller is used enough to realize synchronization, which reduces the cost and complexity for synchronization control implementation. As expected, the synchronization error states can be driven to zero with the matched perturbations or into predictable bounds with unmatched perturbations.

This paper is organized as follows. Section 2 describes the problem formulation, FII switching surface, and the sliding mode controller design; a numerical example to demonstrate the effectiveness of the proposed method is included in Section 3. In Section 4, we draw conclusions on the new results. Throughout this paper, it is noted that notation M^T is used to denote the transpose for a square matrix M, while, for $x \in \mathfrak{R}^n$, $\|x\| = (x^T x)^{1/2}$ denotes the Euclidean norm of the vector. Note that, for a scalar σ, sign(σ) is the sign function of σ; when $\sigma > 0$, sign(σ) $= 1$; when $\sigma = 0$, sign(σ) $= 0$; when $\sigma < 0$, sign(σ) $= -1$.

2. System Description and Problem Formulation

Consider a four-dimensional fractional-order hyperchaotic system; the dynamics is described by the following equations [22]:

$$\begin{aligned} D^q x &= ax - y, \\ D^q y &= x - yz^2, \\ D^q z &= -b_1 y - b_2 z - b_3 w, \\ D^q w &= z + cw, \end{aligned} \tag{1}$$

where a, b_1, b_2, b_3, c are system parameters. D^q denotes the Riemann-Liouville fractional derivative of order $q \in R$ defined as follows [23]:

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^q} d\tau. \tag{2}$$

Also $\Gamma(q)$ is the Euler Gamma function given as

$$\Gamma(q) = \int_0^\infty v^{q-1} e^v dv. \tag{3}$$

The order denoted by q is subject to $0 < q < 1$.

In this paper, we focus on system (1) since it is a hyperchaotic system with more complicated dynamical behavior. Also, methods developed herein are also applicable to other fractional chaotic systems. System (1) generates chaotic oscillations when the system parameters and initial condition are set as $a = 0.56$, $b_1 = 1.0$, $b_2 = 1.0$, $b_3 = 6.0$, $c = 0.8$, and $q = 0.95$ and initial condition $[x(0) \;\; y(0) \;\; z(0) \;\; w(0)] = [0.5 \;\; 0.3 \;\; -0.1 \;\; 0.1]$. Figure 1 shows the typical chaotic attractors. This paper aims to design a robust synchronization controller such that the response system, even with unmatched external perturbations and input nonlinearity, is able to mimic the behavior of the drive chaotic system. Let the drive system and response system be defined below, respectively.

Drive system is

$$\begin{aligned} D^q x_m &= ax_m - y_m, \\ D^q y_m &= x_m - y_m z_m^2, \\ D^q z_m &= -b_1 y_m - b_2 z_m - b_3 w_m, \\ D^q w_m &= z_m + cw_m, \end{aligned} \tag{4}$$

and response system is

$$\begin{aligned} D^q x_s &= ax_s - y_s + d_1, \\ D^q y_s &= x_s - y_s z_s^2 + d_2 + \phi(u), \\ D^q z_s &= -b_1 y_s - b_2 z_s - b_3 w_s + d_3, \\ D^q w_s &= z_s + cw_s + d_4, \end{aligned} \tag{5}$$

where $u(t) \in R$ is the control input. $\phi(u(t))$ is a continuous nonlinear function and $\phi(0) = 0$, where $\phi : R \to R$ with the law $u(t) \to \phi(u(t))$ and inside sector $[\beta_1 \;\; \beta_2]$; that is,

$$\beta_2 u^2(t) \geq u(t)\phi(u(t)) \geq \beta_1 u^2(t), \tag{6}$$

where β_1 and β_2 are nonzero positive constants [19]. A nonlinear function $\phi(u(t))$ is illustrated in Figure 2. Also $d_i(t)$, $i = 1, 2, 3, 4$, are the unavoidable external perturbations in practical systems and assumed bounded; that is,

$$|d_i(t)| \leq \alpha_i, \quad i = 1, 2, 3, 4, \tag{7}$$

where $\alpha_i > 0$ are given. Generally, d_2 is called the matched perturbation and d_i, $i = 1, 3, 4$, are the unmatched perturbations. Now define the synchronization error as $e_1 = x_s - x_m$, $e_2 = y_s - y_m$, $e_3 = z_s - z_m$, $e_4 = w_s - w_m$, respectively. Then yield the following error system:

$$\begin{aligned} D^q e_1 &= ae_1 - e_2 + d_1, \\ D^q e_2 &= e_1 - y_s z_s^2 + y_m z_m^2 + d_2 + \phi(u), \\ D^q e_3 &= -b_1 e_2 - b_2 e_3 - b_3 e_4 + d_3, \\ D^q e_4 &= e_3 + ce_4 + d_4. \end{aligned} \tag{8}$$

Obviously, the aim of this work is to propose a sliding mode control law $u(t)$ subjected to input nonlinearity

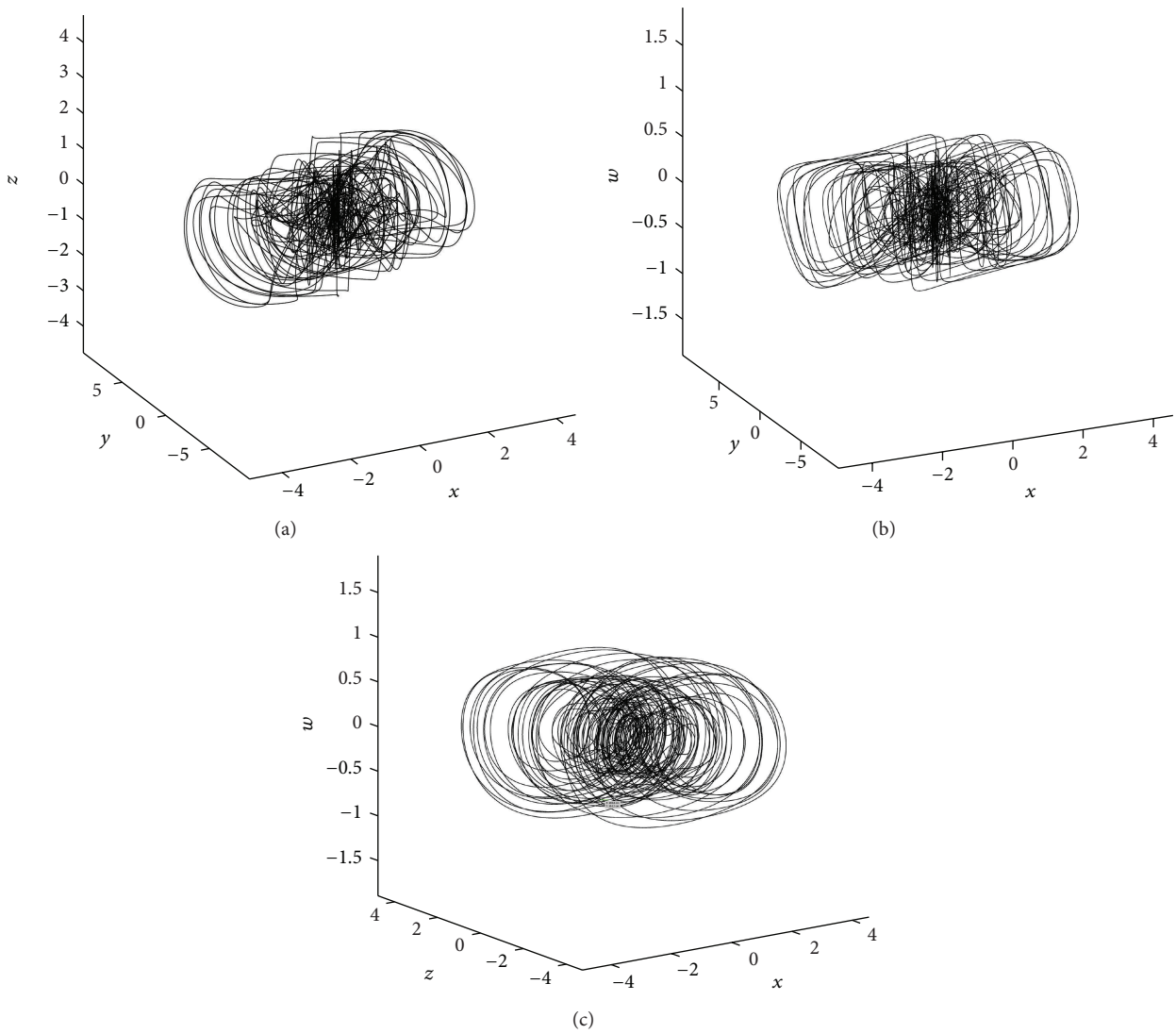

Figure 1: Attractors of the considered fractional-order system: (a) three-dimensional view ($x-y-z$); (b) three-dimensional view ($x-y-w$); (c) three-dimensional view ($x-z-w$).

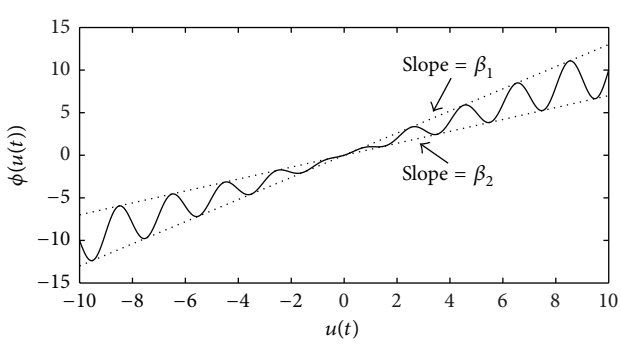

Figure 2: A scalar nonlinear function $\phi(u)$ inside sector $[\beta_1 \quad \beta_2]$.

specified by (6), such that the resulting tracking error state vector $E = [e_1 \;\; e_2 \;\; e_3 \;\; e_4]$ can be forced to zero or into a predictable bound when unmatched external perturbations are present. Accordingly, to achieve the control goal by using the SMC technique, there exist two basic steps for the design procedure. The first step is to construct an appropriate switching surface such that the sliding motion can result in $\lim_{t\to\infty}\|E(t)\| \leq \rho$ and $\rho \geq 0$ are a predictable constant depending on the external perturbations, which will be explained later. The second step is to establish a SMC law which can guarantee the attraction of the sliding manifold even with the input nonlinearity (6).

2.1. Switching Surface Design of Chaos Synchronization. To complete the design steps above, we firstly propose a novel type of FII switching surface as

$$\sigma(t) = I^{1-q}e_2(t) + \int_0^t KE(\tau)\,d\tau, \tag{9}$$

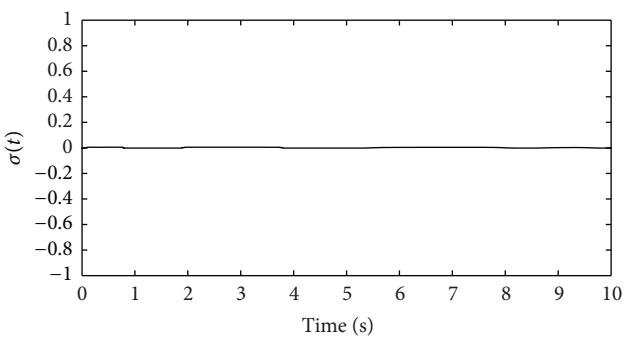

Figure 3: The time response of switching surface $\sigma(t)$.

where $I^{1-q}e_2(t)$ is the Riemann-Liouville fractional integral of order $1-q$ given by

$$I^{1-q}e_2(t) = \frac{1}{\Gamma(1-q)}\int_0^t \frac{e_2(\tau)}{(t-\tau)^q}d\tau. \tag{10}$$

Obviously, when the system operates in the sliding mode, the controlled system satisfies the following conditions [20, 21]:

$$\sigma(t) = 0; \qquad \dot{\sigma}(t) = 0. \tag{11}$$

Then, based on (8)–(11), one can deduce the following result:

$$D^q E = (A - BK)E + D, \tag{12}$$

where

$$E = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}; \qquad A = \begin{bmatrix} a & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_2 & -b_3 \\ 0 & 0 & 1 & c \end{bmatrix};$$
$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \qquad K = [k_1 \;\; k_2 \;\; k_3 \;\; k_4]; \qquad D = \begin{bmatrix} d_1 \\ 0 \\ d_3 \\ d_4 \end{bmatrix}. \tag{13}$$

When the system enters into the sliding mode, the system dynamics is governed by (12). It has been shown that system (12) without external perturbations is asymptotically stable if the eigenvalues of the matrix $A - BK$ satisfy the following argument stability criterion [24]:

$$\min_i \left|\arg \lambda_i (A - BK)\right| > q\frac{\pi}{2}, \quad i = 1, 2, \ldots, n. \tag{14}$$

By (13), obviously (A, B) is controllable. Therefore, a parameter vector K does exist such that the maximum real part eigenvalue of $A - BK$ is negative and (14) is satisfied. Furthermore, we can easily assign the system performance in the sliding mode just by selecting an appropriate matrix K using any pole assignment method.

The solution of the dynamics (12) can be obtained as follows [23]:

$$E(t) = e_q^{(A-BK)(t-t_1)}E(t_1) + \int_0^{t-t_1} e_q^{(A-BK)(t-t_1-s)}D(s)\,ds, \tag{15}$$

where

$$e_q^{(A-BK)(t-t_1)} = (t-t_1)^{q-1}\sum_{k=0}^{\infty}\left((A-BK)^k \frac{(t-t_1)^{kq}}{\Gamma((k+1)q)}\right) \tag{16}$$

is the q-exponential function and represents the transition matrix of system (12).

From (15), we have

$$\begin{aligned} \|E(t)\| &= \left\| e_q^{(A-BK)(t-t_1)}E(t_1) + \int_0^{t-t_1} e_q^{(A-BK)(t-t_1-s)}D(s)\,ds \right\| \\ &\le \left\| e_q^{(A-BK)(t-t_1)} \right\| \|E(t_1)\| \\ &\quad + \|D(s)\| \left\| \int_0^{t-t_1} e_q^{(A-BK)(t-t_1-s)}ds \right\| \\ &\le \left\| e_q^{(A-BK)(t-t_1)} \right\| \|E(t_1)\| \\ &\quad + \sum_{i=1,3,4}(\alpha_i) \left\| \int_0^{t-t_1} e_q^{(A-BK)(t-t_1-s)}ds \right\|. \end{aligned} \tag{17}$$

Furthermore,

$$\begin{aligned} &\int_0^{t-t_1} e_q^{(A-BK)(t-t_1-s)}ds \\ &\quad = (A-BK)^{-1}\left(E_{q,1}\left((A-BK)(t-t_1)^q\right) - I_n\right), \end{aligned} \tag{18}$$

where $E_{q,1}((A-BK)(t-t_1)^q)$ denotes the Mittag-Leffler function defined as [23]

$$E_{q,1}\left((A-BK)(t-t_1)^q\right) = \sum_{k=0}^{\infty}(A-BK)^k \frac{(t-t_1)^{kq}}{\Gamma(kq+1)}. \tag{19}$$

Since we assign an appropriate matrix K such that the argument stability criterion (14) is satisfied, then $\lim_{t\to\infty}E_{q,1}((A-BK)(t-t_1)^q) = 0$ and $\lim_{t\to\infty}e_q^{(A-BK)t} = 0$. Therefore, from (17), (18), and (19), we have

$$\begin{aligned} \lim_{t\to\infty}\|E(t)\| &\le \lim_{t\to\infty}\left\| e_q^{(A-BK)(t-t_1)} \right\| \|E(t_1)\| \\ &\quad + \sum_{i=1,3,4}(\alpha_i)\lim_{t\to\infty}\int_0^{t-t_1}\left\| e_q^{(A-BK)(t-t_1-s)} \right\| ds \\ &\le \rho = \sum_{i=1,3,4}(\alpha_i)\left\|(A-BK)^{-1}\right\|. \end{aligned} \tag{20}$$

According to the discussion above, we can conclude that when the fractional-order system is in the sliding manifold, the tracking error $\|E\|$ can converge to a predictable bound ρ relative to $\sum_{i=1,3,4}\alpha_i$ and parameter matrix K chosen in the switching surface (9).

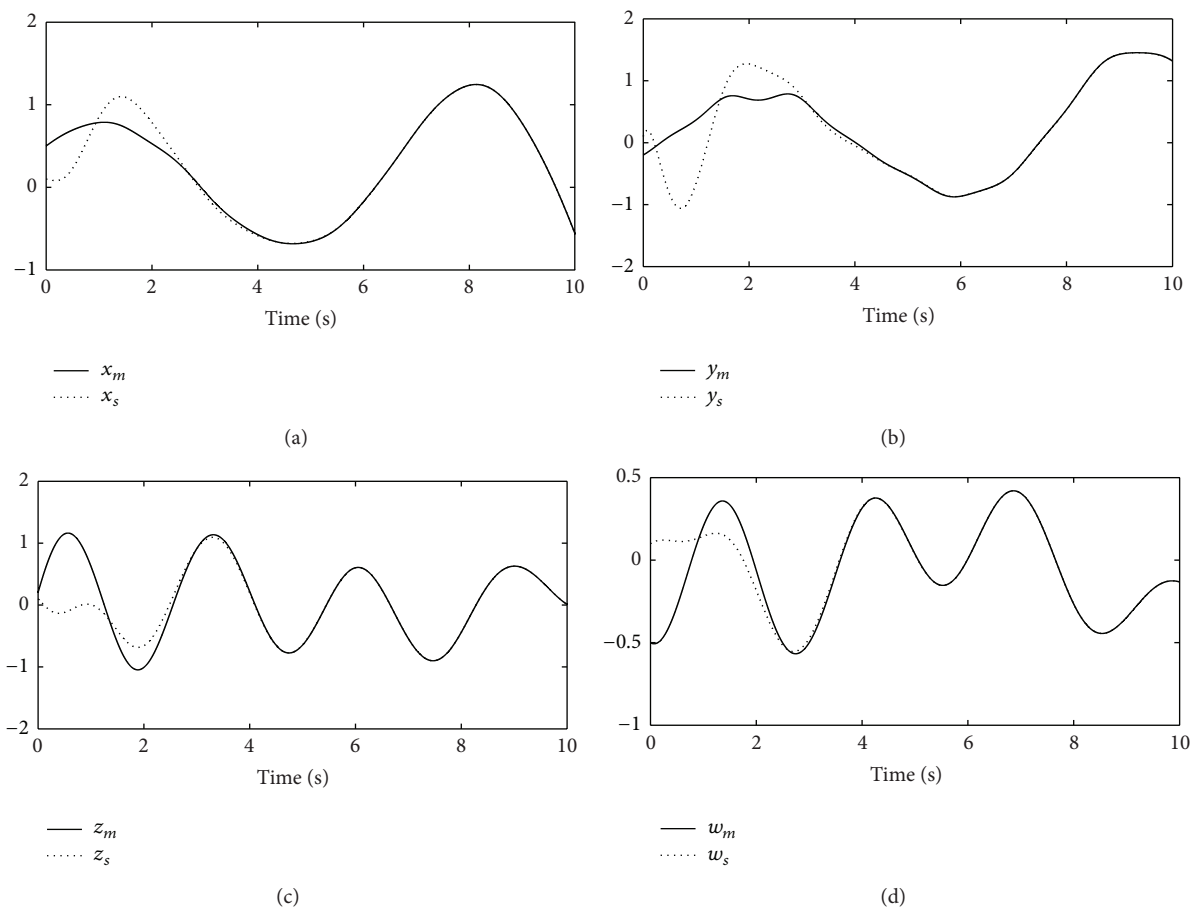

FIGURE 4: State responses of controlled drive and response systems: (a) state response of x_m, x_s; (b) state response of y_m, y_s; (c) state response of z_m, z_s; (d) state response of w_m, w_s.

2.2. Design of Sliding Mode Controller with Input Nonlinearity. In order to guarantee the occurrence of sliding manifold even with the input nonlinearity, we choose a sliding mode control of the form

$$u(t) = -\zeta\eta \operatorname{sign}(\sigma(t)), \quad \zeta > \frac{1}{\beta_1}, \tag{21}$$

where $\eta = |e_1 - y_s z_s^2 + y_m z_m^2 + KE| + \alpha_2$.

In the following, the proposed scheme (21) will be proved to be able to derive the uncertain error dynamics (8) onto the sliding mode $\sigma(t) = 0$.

Theorem 1. *If the control $u(t)$ is given by* (21)*, the reaching condition of expression $\sigma(t)\dot{\sigma}(t) < 0$ of the sliding mode is satisfied in spite of the input nonlinearity.*

Proof. Substituting (9), (10), and (21) into $\sigma(t)\dot{\sigma}(t)$, we obtain

$$\begin{aligned}
&\sigma(t)\dot{\sigma}(t) \\
&= \sigma(t)\left[D^q e_2(t) + KE\right] \\
&= \sigma(t)\left[e_1(t) - y_s(t) z_s^2(t) + y_m(t) z_m^2(t) \right. \\
&\qquad \left. + d_2 + \phi(u(t)) + KE\right] \\
&= \sigma(t)\left[e_1(t) - y_s(t) z_s^2(t) + y_m(t) z_m^2(t) + KE + d_2\right] \\
&\quad + \sigma(t)\phi(u(t)) \\
&\le \left|e_1(t) - y_s(t) z_s^2(t) + y_m(t) z_m^2(t) + KE\right| |\sigma(t)| \\
&\quad + \alpha_2 |\sigma(t)| + \sigma(t)\phi(u(t)) \\
&= \eta |\sigma(t)| + \sigma(t)\phi(u(t)).
\end{aligned} \tag{22}$$

Furthermore, from (6), we have

$$\begin{aligned}
\beta_2\zeta^2\eta^2[\operatorname{sign}(\sigma(t))]^2 &\ge -\zeta\eta[\operatorname{sign}(\sigma(t))]\phi(u(t)) \\
&\ge \beta_1\zeta^2\eta^2[\operatorname{sign}(\sigma(t))]^2.
\end{aligned} \tag{23}$$

Since $\sigma^2(t) \ge 0$, we get

$$\begin{aligned}
&\beta_2\zeta^2\eta^2[\operatorname{sign}(\sigma(t))]^2\sigma^2(t) \\
&\quad \ge -\zeta\eta[\operatorname{sign}(\sigma(t))]\phi(u(t))\sigma^2(t) \\
&\quad \ge \beta_1\zeta^2\eta^2[\operatorname{sign}(\sigma(t))]^2\sigma^2(t)
\end{aligned}$$

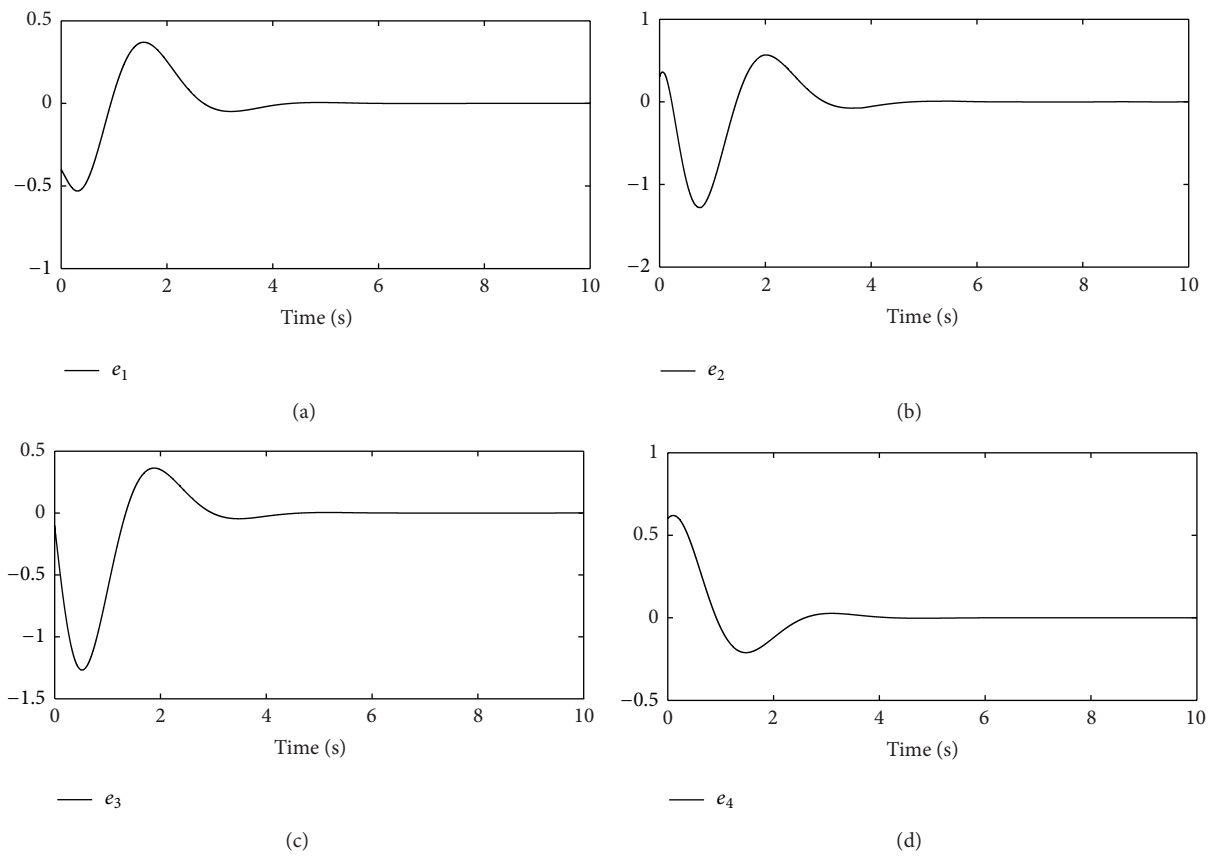

FIGURE 5: Error states of controlled drive and response systems: (a) error state of e_1; (b) error state of e_2; (c) error state of e_3; (d) error state of e_4.

$$\begin{aligned}
&\Longrightarrow \beta_2\zeta^2\eta^2|\sigma(t)|^2 \geq -\zeta\eta|\sigma(t)|^2\phi(u(t)) \geq \beta_1\zeta^2\eta^2|\sigma(t)|^2 \\
&\Longrightarrow \beta_2\zeta\eta\,|\sigma(t)| \geq -|\sigma(t)|\,\phi(u(t)) \geq \beta_1\zeta\eta\,|\sigma(t)| \\
&\Longrightarrow -\beta_2\zeta\eta\,|\sigma(t)| \leq \sigma(t)\,\phi(u(t)) \leq -\beta_1\zeta\eta\,|\sigma(t)|\,.
\end{aligned} \tag{24}$$

By placing (24) into (22), we get

$$\begin{aligned}
\sigma(t)\,\dot{\sigma}(t) &\leq -\beta_1\zeta\eta\,|\sigma(t)| + \eta\,|\sigma(t)| \\
&\leq (1-\beta_1\zeta)\,\eta\,|\sigma(t)|\,.
\end{aligned} \tag{25}$$

Since $\zeta > 1/\beta_1$ has been selected in (21), it can be concluded that the hitting condition $\sigma(t)\dot{\sigma}(t) < 0$ is satisfied. Thus, the proof is achieved completely. □

Remark 2. The controller in (21) demonstrates a discontinuous control law and the phenomenon of chattering will appear. In order to eliminate the chattering, controller (21) can be modified as

$$u(t) = -\zeta\eta\frac{\sigma}{|\sigma| + \varepsilon}, \quad \zeta > \frac{1}{\beta_1}, \tag{26}$$

where ε is a sufficiently small positive constant. From the works [21, 25], the solution of system (8) with (21) can be made arbitrarily close to solution (8) with (26), if one chooses ε sufficiently small.

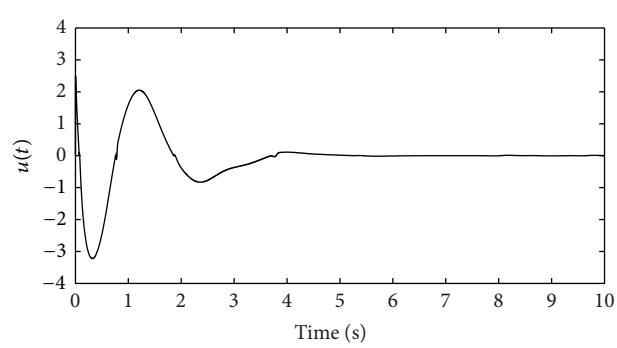

FIGURE 6: The time response of control input $u(t)$.

Remark 3. Obviously, for the case of $q = 1$, the considered system (1) degenerates to an integer-order chaotic system and the design method developed in this paper is also available just by some minor modifications.

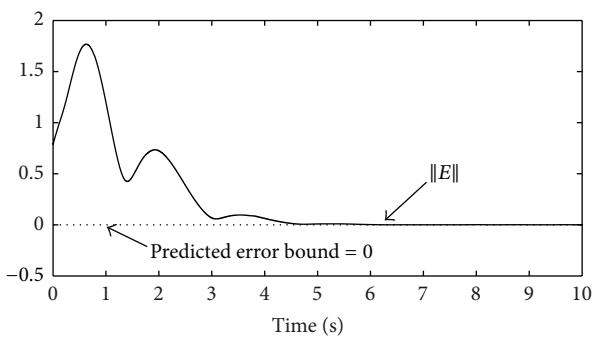

Figure 7: The error bound with matched perturbations.

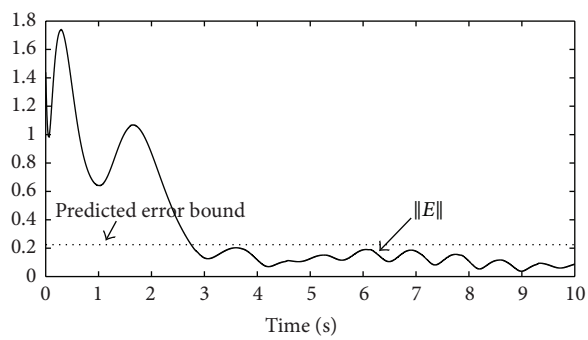

Figure 8: The error bound with unmatched perturbations.

3. An Illustrative Example

In this section, to verify the validity of the proposed synchronization scheme, we numerically examine the synchronization. Here, the drive system and response system accord with (4) and (5), respectively. The input nonlinearity is defined as

$$\phi(u(t)) = [1 + 0.05\sin(u(t))]\, u(t). \tag{27}$$

According to (6), $\beta_1 = 0.95$, $\beta_2 = 1.05$ can be obtained. In numerical simulation, all system's parameters are chosen as $\zeta = 2 > 1/\beta_1$. And the parameter $\varepsilon = 0.01$ in (26) is selected.

Case 1. Consider the case of a nominal system with $d_1(t) = d_2(t) = d_3(t) = d_4(t) = 0$. As mentioned in Section 2, the proposed design procedure can be summarized as follows.

Step 1. According to (9), the switching surface, with $K = [-12.4824 \;\; 6.8483 \;\; -7.8920 \;\; -16.8563]$ satisfying (14), is given by

$$\sigma(t) = I^{1-q} e_2(t) + \int_0^t KE(\tau)\, d\tau. \tag{28}$$

Step 2. According to (26), the sliding mode control law is obtained as follows:

$$u(t) = -2\eta \frac{\sigma}{|\sigma| + 0.01}, \qquad 2 > \frac{1}{\beta_1}, \tag{29}$$

where $\eta = |e_1 - y_s z_s^2 + y_m z_m^2 + KE|$.

In numerical simulations, the simulations are all performed by setting $q = 0.95$ and the initial values of the master and slave systems are given, respectively, as $[x_m(0) \;\; y_m(0) \;\; z_m(0) \;\; w_m(0)]^T = [0.5 \;\; -0.2 \;\; 0.2 \;\; 0.5]^T$ and $[x_s(0) \;\; y_s(0) \;\; z_s(0) \;\; w_s(0)]^T = [0.1 \;\; 0.1 \;\; 0.1 \;\; 0.1]^T$. The simulation results are shown in Figures 3, 4, 5, 6, and 7. Figure 3 shows the corresponding $\sigma(t)$ for the controlled fractional-order hyperchaotic systems under the proposed sliding mode control (29). Figures 4–6 represent, respectively, the state responses, error states' responses, and the control input. From the simulation results, it is shown that the proposed controller (29) can drive the resulting tracking errors $\lim_{t\to\infty}|e_i(t)| = 0$, $i = 1, 2, 3, 4$, which fully coincide with theoretical results in this paper.

Case 2. Consider the case of a non-nominal system with unmatched external perturbations of $d_1(t) = 0.1\sin(4t)$, $d_2(t) = 0.3\sin(4t)$, $d_3(t) = 0$, $d_4(t) = 0$. Under the same simulation conditions as in Case 1, the switching surface with $K = [-12.4824 \;\; 6.8483 \;\; -7.8920 \;\; -16.8563]$ is given by

$$\sigma(t) = I^{1-q} e_2(t) + \int_0^t KE(\tau)\, d\tau \tag{30}$$

and the sliding mode control law is given as in (29).

The time response of the error states, under the proposed sliding mode controller, is shown in Figure 8. It also shows that the error norm $\|E(t)\|$ is bounded in the estimated error bound $\rho = 0.2439$ as predicted.

4. Conclusion

This paper presents a method to design a sliding mode controller for the fractional-order hyperchaotic system subjected to unmatched perturbations and input nonlinearity. A new switching surface of fraction-integer integral (FII) type has been proposed such that the stability of the fractional chaotic system dynamics in the sliding mode is easily ensured. Illustrative examples, including nominal (matched) and nonnominal (unmatched) cases, have been presented to demonstrate the validity of the proposed synchronization scheme.

Conflict of Interests

The authors, Teh-Lu Liao, Jun-Juh Yan, and Jen-Fuh Chang, declare that there is no conflict of interests regarding the publication of this paper.

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