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Research Article

Discontinuous Galerkin Immersed Finite Volume Element Method for Anisotropic Flow Models in Porous Medium

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By choosing the trial function space to the immersed finite element space and the test function space to be piecewise constant function space, we develop a discontinuous Galerkin immersed finite volume element method to solve numerically a kind of anisotropic diffusion models governed by the elliptic interface problems with discontinuous tensor-conductivity. The existence and uniqueness of the discrete scheme are proved, and an optimal-order energy-norm estimate and L^2-norm estimate for the numerical solution are derived.

1. Introduction

Let us consider the following elliptic interface problems in a convex domain $\Omega \subset R^2$:

$$\begin{aligned} -\nabla \cdot (\mathbb{B}\nabla u) &= f, \quad (x, y) \in \Omega, \\ u &= 0, \quad (x, y) \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is separated into two subdomains Ω^+ and Ω^- by interface $\Gamma \in C^2$, see Figure 1 for an illustration, and $f \in L^2(\Omega)$; u satisfies the following homogenous jump conditions on the interface Γ:

$$[u], \left[\mathbb{B}\frac{\partial u}{\partial \mathbf{n}}\right] = 0. \tag{2}$$

Equation (1) describes many real diffusion processes in fluid dynamics and engineering applications, such as the miscible displacement with discontinues conductivity due to complex strata or multiphase flux. It is significant to seek efficiently the numerical solution to the interface problems for better understanding of the mechanism of the flow process and conducting engineering practice.

When $\mathbb{B}(x)$ is a scale function, which corresponds to an isotropic flow case, two classes of numerical methods were developed to approximate (1) in terms of the meshes. One is the fitted finite element or fitted difference method [1–3], which restricts the mesh to be aligned with the smooth interface Γ. Consequently, the fitted methods are costly for more complicated time dependent problems in which the interface moves with time and repeated grid generation is called for. The other one is the immersed interface difference or finite element methods in which the Cartesian grid is naturally used even though it cannot match a curved interface. Although the immersed difference methods [4, 5] were demonstrated to be very effective, convergence analysis of related finite difference methods is extremely difficult and is still open. On the other hand, the immersed finite element method (IFE) was developed, which allows the interface to go through the interior of the element; see the references [6–9] and the references therein. Numerical experiments demonstrated an optimal order of the errors. Once again, it is not easy to analyze this method. Further, to preserve the conservative characteristics of the interface model (1), [10] developed an immersed finite volume element (IFVE) method by combining the finite volume element method [11–16] and the immersed finite element method.

In realistic diffusion processes, the interface problem (1) displays much often its anisotropic type. That is, the conductivity $\mathbb{B}(x)$ becomes a tenser-formed function. The goals of this paper are as follows: (1) to develop a discontinuous Galerkin-immersed interface-finite volume element

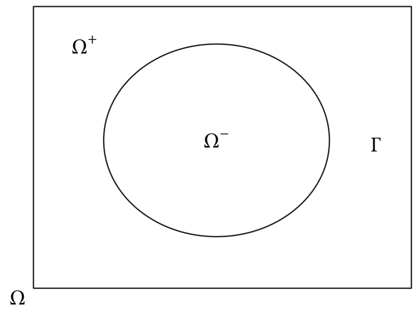

Figure 1

(DGIFVE) method for the second-order elliptic problems with tensor-formed conductivity $\mathbb{B}(\mathbf{x})$ defined by

$$\mathbb{B}(\mathbf{x}) = \mathbb{B}^l(\mathbf{x}) = \begin{pmatrix} m^l & s^l \\ s^l & n^l \end{pmatrix}, \tag{3}$$

where $\mathbf{x} \in \Omega^l$, $m^l > 0$, $n^l > 0$, $m^l n^l > (s^l)^2$, $l = +, -$; by doing so, we can use the ability of the penalty term in discontinuous Galerkin method to control the integrals on an element boundary, in order to prove the solvability of the scheme and derive easily an optimal-order error analysis, and we can use the conservation characteristics of the finite volume element method to construct a conservation-preserved numerical method; (2) to prove the existence and uniqueness of the proposed discontinuous Galerkin-immersed interface-finite volume element procedure based on the nonconforming interface finite element space for anisotropic flow model [17]; (3) to establish its optimal-order energy-norm estimate and L^2-norm estimate.

This paper is organized as follows. In the next section, we will introduce the trial function space and its approximation properties on primal triangulation. In Section 3, we will formulate the DGIFVE procedure. In Section 4, we will introduce some important lemmas. In Section 5, we will prove the existence and uniqueness of the solution of the discrete scheme. In Section 6, we will derive the convergence analysis.

Throughout this paper, the symbol C will be used as a generic positive constant independent of h and may have different values at different places.

2. The Construction of the Trial Function Space

In this section, we recall the definitions of IFE spaces discussed in [7]. Let $\mathcal{T}_h = \{K\}$ be a regular triangulation of Ω with the diameter size h. We can separate the triangles on a partition into two classes:

(1) interface element: the interface Γ passes through the interior of K;

(2) noninterface element: the interface does not intersect with this triangle, or it intersects with this triangle but does not separate its interior into two nontrivial subsets. Let $\mathcal{T}_h^n$ be the collection of all noninterface elements and let $\mathcal{T}_h^m$ be the collection of all interface elements. We will use $A_i = (x_i, y_i)$, $i = 1, 2, 3$ to denote the vertices of T, and we will use $\overline{DE}$ to denote the line segment connecting the intersection of the interface and the edges of a triangle K. This line segment $\overline{DE}$ divides T into two parts K^+ and K^- with $K = K^+ \cup K^- \cup \overline{DE}$ (see Figure 2).

For the analysis, we introduce the spaces

$$\begin{aligned} \widetilde{H}^2(\Omega) &= \left\{u \in H_0^1(\Omega) : u \in H^2(\Omega^s),\ s = +, -\right\}, \\ \widetilde{H}^2(K) &= \left\{u \in H^1(K) : u \in H^2(K \cap \Omega^s),\ s = +, -\right\} \end{aligned} \tag{4}$$

equipped with the norms

$$\begin{aligned} \|u\|_{\widetilde{H}^2(\Omega)}^2 &= \|u\|_{H^2(\Omega^+)}^2 + \|u\|_{H^2(\Omega^-)}^2, \\ \|u\|_{\widetilde{H}^2(K)}^2 &= \|u\|_{H^2(K\cap\Omega^+)}^2 + \|u\|_{H^2(K\cap\Omega^-)}^2, \end{aligned} \tag{5}$$

where $H^m = W_2^m$ $(m = 1, 2)$ is the standard Sobolev spaces. In order to define the bilinear formulation, we introduce the broken Sobolev space $H^1(\mathcal{T}_h) = \{u \in L^2(\Omega) : \forall K \in \mathcal{T}_h, u|_K \in H^1(K)$ and $\forall e \subset \partial K \cap \partial\Omega \neq \emptyset, u|_e = 0\}$.

For a noninterface element K, we use the standard linear shape functions on K whose degrees of freedom are functional values on the vertices of K, and we use $\overline{S}_h(K)$ to denote the linear spaces spanned by the three nodal basis functions on K as follows:

$$\overline{S}_h(K) = \operatorname{span}\{\phi_i \in P_1(K) : i = 1, 2, 3\}, \tag{6}$$

where $P_1(K)$ is the linear space on K. For this space, we have the following estimate of the interpolant:

$$\left\|u - \Pi_h u\right\|_{L^2(K)} + h\left\|u - \Pi_h u\right\|_{H^1(K)} \leqslant Ch^2\|u\|_{H^2(K)}, \tag{7}$$

where $\Pi_h : H^2(K) \to \overline{S}_h(K)$ is the interpolation operator. We use $\overline{S}_h(\Omega)$ to denote the space of the conforming piecewise linear polynomials on the domain $\mathcal{T}_h^n$.

For an interface element K whose geometric configuration is given in Figure 3 in which $A_1 = (0,0)^T$, $A_2 = (h_1, 0)^T$, $A_3 = (0, h_2)^T$, the interface points $D = (bh_1, 0)^T$ and $E = (0, ah_2)^T$, where $0 < a \leqslant 1$ and $0 < b \leqslant 1$. Let ϕ_i, $i = 1, 2, 3$, denote the usual Lagrange nodal basis function associated with the vertex A_i, $i = 1, 2, 3$, respectively. Here we assume that the ratio $r = h_1/h_2$ is bounded below and above by some constants.

By $\widehat{\phi}(A_i) = V_i$, $i = 1, 2, 3$, we can construct the basis function $\widehat{\phi}(\mathbf{x})$ on an interface element K as follows:

$$\widehat{\phi}(\mathbf{x}) = \begin{cases} \widehat{\phi}^-(\mathbf{x}) = V_1\phi_1 + C_1\phi_2 + C_2\phi_2, & \mathbf{x} = (x, y) \in K^-, \\ \widehat{\phi}^+(\mathbf{x}) = C_3\phi_1 + V_2\phi_2 + V_3\phi_2, & \mathbf{x} = (x, y) \in K^+. \end{cases} \tag{8}$$

Satisfy

$$\begin{aligned} &\widehat{\phi}^+(D) = \widehat{\phi}^-(D), \qquad \widehat{\phi}^+(E) = \widehat{\phi}^-(E), \\ &\mathbb{B}^+\frac{\partial\widehat{\phi}^+}{\partial\mathbf{n}_{\overline{DE}}} = \mathbb{B}^-\frac{\partial\widehat{\phi}^-}{\partial\mathbf{n}_{\overline{DE}}}, \end{aligned} \tag{9}$$

where $\mathbf{n}_{\overline{DE}}$ is the unit normal vector on the linesegment $\overline{DE}$.

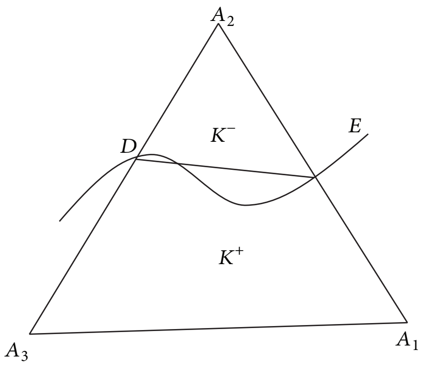

Figure 2

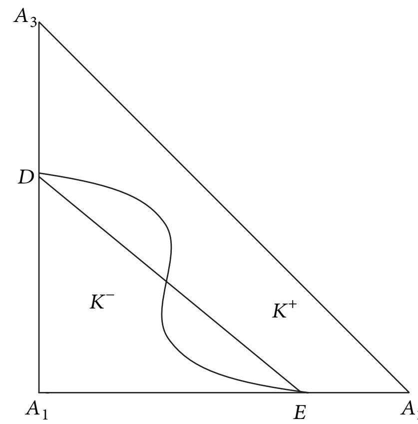

Figure 3

By [17, 18], we have the following conclusions.

Lemma 1. *When* $s^l \geqslant 0$, $l = +, -$, *the piecewise linear function* $\widehat{\phi}(x, y)$ *defined by* (8) *is uniquely decided by three conditions in* (9).

Remark 2. By [17], the condition $s^l \geq 0$, $l = +, -$ is necessary in Lemma 1. For some specially selected entries of $\mathbb{B}^l$ and the intersection points of the interface with the edges on K, $\widehat{\phi}(\mathbf{x})$ satisfying (9) is uniquely undetermined by $\widehat{\phi}(A_i) = V_i$, $i = 1, 2, 3$.

Based on the above results, the finite element space $\widehat{S}_h(K)$ on a typical interface element $K \in \mathcal{T}_h^m$ is defined by

$$\widehat{S}_h(K) = \left\{\widehat{\phi} : \widehat{\phi} \text{ is piecewise linear and satisfies (9)}\right\}. \tag{10}$$

We call $\widehat{S}_h(K)$ the immersed interface element space. For any $u \in \widetilde{H}^2(\Omega)$ and $K \in \mathcal{T}_h^m$, we define $(\Pi_h u)|_K = \Pi_h(u|_K) \in \widehat{S}_h(K)$ by

$$\Pi_h u(A_i) = u(A_i), \quad i = 1, 2, 3, \tag{11}$$

and we call $\Pi_h u$ the interpolant of u in $\widehat{S}_h(K)$. Similar to [7], we have an estimate of the interpolant given in the following theorem.

Theorem 3. *For* $\forall K \in \mathcal{T}_h^m$, *there exists a constant* $C > 0$ *such that the interpolation operator* $\Pi_h : \widetilde{H}^2(K) \to \widehat{S}_h(K)$ *satisfies*

$$\begin{aligned}\|u - \Pi_h u\|_{L^2(K)} + h\|u - \Pi_h u\|_{H^1(K)} \leqslant Ch^2\|u\|_{\widetilde{H}^2(K)}, \\ \forall u \in \widetilde{H}^2(K).\end{aligned} \tag{12}$$

Finally, we define trial function space $S_h(\Omega)$ as the collection of functions such that

$$\begin{aligned}\phi|_K &\in \overline{S}_h(K), \quad K \text{ is a noninterface element}, \\ \phi|_K &\in \widehat{S}_h(K), \quad K \text{ is an interface element}.\end{aligned} \tag{13}$$

The space $S_h(\Omega) \subset L^2(\Omega)$ is a subspace of $H^1(\mathcal{T}_h)$. We also use the space $S_{0h}(\Omega) = \{v_h \in S_h(\Omega), v_h|_{\partial\Omega} = 0\}$.

3. DGIFVE Procedure

In this section, we will construct a dual grid $\mathcal{T}_h^*$ based on $\mathcal{T}_h$. Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. For a given triangle $K \in \mathcal{T}_h$, we divide $K \in \mathcal{T}_h$ into three triangles by connecting the barycenter Q and the three corners of the triangle as shown in Figure 4. Let $\mathcal{T}_h^*$ consist of all these triangles T.

For the $\mathcal{T}_h^*$, we define the test function space as follows:

$$S_h^*(\Omega) = \{\phi(\mathbf{x}) \mid \phi(\mathbf{x})|_T = \text{constant}, \forall T \in \mathcal{T}_h^*\}. \tag{14}$$

Analogous to the operator Π_h, we introduce the interpolation operator $\gamma_h : H_h = S_h(\Omega) + \widetilde{H}^2(\Omega) \cap H_0^1(\Omega) \longrightarrow S_h^*(\Omega)$ defined by, for $\forall v \in H_h$,

$$\gamma_h v|_T = \frac{1}{|e|}\int_e v(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{x} \in T. \tag{15}$$

Let e be an interior edge shared by two elements K_1 and K_2 in $\mathcal{T}_h$. Define the unite normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ on e pointing exterior to K_1 and K_2, respectively. For scalar v function and vector function $\mathbf{q}$, we define their average $\{\cdot\}$ and jump $[\cdot]$ on e, as follows (see [19]):

$$\begin{aligned}\{v\}|_e &= \frac{1}{2}\left(v|_{\partial K_1} + v|_{\partial K_2}\right), \qquad [v]|_e = v|_{\partial K_1} \cdot \mathbf{n}_1 + v|_{\partial K_2} \cdot \mathbf{n}_2, \\ \{\mathbf{q}\}|_e &= \frac{1}{2}\left(\mathbf{q}|_{\partial K_1} + \mathbf{q}|_{\partial K_2}\right), \qquad [\mathbf{q}]|_e = \mathbf{q}|_{\partial K_1} \cdot \mathbf{n}_1 + \mathbf{q}|_{\partial K_2} \cdot \mathbf{n}_2.\end{aligned} \tag{16}$$

If e is an edge on the boundary of Ω, we define

$$\{v\}|_e = v, \qquad [\mathbf{q}]|_e = \mathbf{q} \cdot \mathbf{n}. \tag{17}$$

Let ε_h denote the union of the boundaries of the triangle K of $\mathcal{T}_h$ and let $\varepsilon_h^0 := \varepsilon_h \setminus \partial\Omega$, ε_h^* be the union of the boundaries cutting by the Γ. A straightforward computation gives

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} v\mathbf{q} \cdot \mathbf{n}\, ds = \sum_{e \in \varepsilon_h} \int_e [v]\{\mathbf{q}\}\, ds + \sum_{e \in \varepsilon_h^0} \int_e \{v\}[\mathbf{q}]\, ds. \tag{18}$$

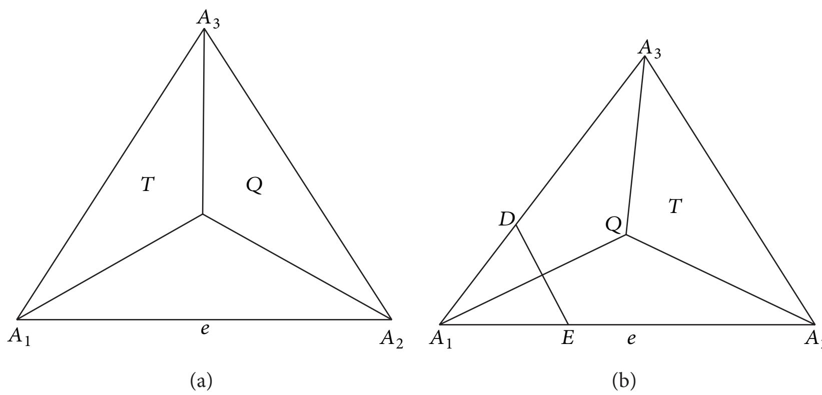

Figure 4: (a) K is a noninterface element and (b) K is an interface element.

We multiply (1) by $v_h \in S_h^*(\Omega)$; using $[\mathbb{B}\nabla u]|_\Gamma = 0$ and Green's formula, we have

$$-\sum_{T\in\mathcal{T}_h^*}\int_{\partial T}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds = (f, v_h), \tag{19}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂T. Let $T_j \in \mathcal{T}_h^*$ $(j = 1, 2, 3)$ be three triangles in $K \in \mathcal{T}_h$. Then, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^*}\int_{\partial T}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds &= \sum_{K\in\mathcal{T}_h}\sum_{j=1}\int_{\partial T_j}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds\\ &= \sum_{K\in\mathcal{T}_h}\sum_{j=1}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds.\end{aligned} \tag{20}$$

Using (18) and the fact that $[\mathbb{B}\nabla u] = 0$, (20) becomes

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^*}\int_{\partial T}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds &= \sum_{K\in\mathcal{T}_h}\sum_{j=1}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds + \sum_{e\in\varepsilon_h}\int_e\{\mathbb{B}\nabla u\}[v_h]\,ds.\end{aligned} \tag{21}$$

By (19) and (21), we can get

$$\begin{aligned}&-\sum_{K\in\mathcal{T}_h}\sum_{j=1}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u\cdot\mathbf{n}v_h ds\\ &\quad-\sum_{e\in\varepsilon_h}\int_e\{\mathbb{B}\nabla u\}[v_h]\,ds = (f, v_h).\end{aligned} \tag{22}$$

By the definition of γ_h, the discontinuous Galerkin immersed finite volume element formulation is equivalent to finding $u_h \in S_{0h}(\Omega)$ such that

$$a_h^*(u_h, \gamma_h\omega_h) = (f, \gamma_h\omega_h), \quad \omega_h \in S_{0h}(\Omega), \tag{23}$$

where

$$\begin{aligned}a_h^*(u_h, \gamma_h\omega_h) &= -\sum_{K\in\mathcal{T}_h}\sum_{j=1}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds\\ &\quad-\sum_{e\in\varepsilon_h}\int_e\{\mathbb{B}\nabla u_h\}[\gamma_h\omega_h]\,ds\\ &\quad+\sum_{e\in\varepsilon_h}\frac{\sigma_0}{h_e}\int_e[u_h][\omega_h]\,ds\end{aligned} \tag{24}$$

is the bilinear formulation defined on $S_{0h}(\Omega) \times S_{0h}(\Omega)$, and in addition to penalty term $\sum_{e\in\varepsilon_h}(\sigma_0/h_e)\int_e[u_h][\omega_h]ds$, the penalty parameter $\sigma_0 > 0$. Since $[\gamma_h u]_e = 0$, it is easy to see that u satisfies the solution of (1) as follows:

$$a_h^*(u, \gamma_h\omega_h) = f(u, \gamma_h\omega_h). \tag{25}$$

Let

$$A(u_h, \omega_h) = -\sum_{K\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds. \tag{26}$$

If $K \in \mathcal{T}_h^n$, we have by $\nabla\cdot(\mathbb{B}\nabla u_h) = 0$, $u_h \in S_{0h}(\Omega)$

$$\begin{aligned}0 &= \int_K\nabla\cdot(\mathbb{B}\nabla u_h)\gamma_h\omega_h dx\\ &=\sum_{j=1}^{3}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds\\ &\quad+\sum_{j=1}^{3}\int_{A_jA_{j+1}}\mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds.\end{aligned} \tag{27}$$

Then,

$$-\sum_{j=1}^{3}\int_{A_{j+1}QA_j}\mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds = \sum_{j=1}^{3}\int_{A_jA_{j+1}}\mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds. \tag{28}$$

Similarly,

$$0 = -\int_K \nabla\cdot(\mathbb{B}\nabla u_h)\,\omega_h dx = -\sum_{j=1}^{3}\int_{A_jA_{j+1}} \mathbb{B}\nabla u_h\cdot\mathbf{n}\omega_h ds + \int_K \mathbb{B}\nabla u_h\cdot\nabla\omega_h dx. \quad (29)$$

We find that

$$\int_{\partial K} \mathbb{B}\nabla u_h\cdot\mathbf{n}\,(\omega_h - \gamma_h\omega_h)\,ds = 0, \quad (30)$$

due to the fact that $\mathbb{B}\nabla u_h$ is a constant vector on each edge and the definition of γ_h. Thus, we can get

$$\int_K \mathbb{B}\nabla u_h\nabla\omega_h dx = -\sum_{j=1}^{3}\int_{A_{j+1}QA_j} \mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds \quad (31)$$

following from (28) and (29). For $K\in\mathscr{T}_h^m$ (see Figure 4(b)), it follows from the same arguments above and the $[B\nabla u_h]|_{\overline{DE}} = 0$ that

$$\begin{aligned}\int_K \mathbb{B}\nabla u_h\nabla\omega_h dx = &-\sum_{j=1}^{3}\int_{A_{j+1}QA_j} \mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds \\ &+\Big\{\int_{\overline{A_1D}} \mathbb{B}^+\nabla u_h^+\cdot\mathbf{n}\,(\gamma_h\omega_h-\omega_h^+)\,ds \\ &+\int_{\overline{DA_2}} \mathbb{B}^-\nabla u_h^-\cdot\mathbf{n}\,(\gamma_h\omega_h-\omega_h^-)\,ds \\ &+\int_{\overline{A_2E}} \mathbb{B}^-\nabla u_h^-\cdot\mathbf{n}\,(\gamma_h\omega_h-\omega_h^-)\,ds \\ &+\int_{\overline{EA_3}} \mathbb{B}^+\nabla u_h^+\cdot\mathbf{n}\,(\gamma_h\omega_h-\omega_h^+)\,ds\Big\}.\end{aligned} \quad (32)$$

Summarizing the results above, we have

$$\begin{aligned}A(u_h,\omega_h) &= -\sum_K\sum_{j=1}^{3}\int_{A_{j+1}QA_j} \mathbb{B}\nabla u_h\cdot\mathbf{n}\gamma_h\omega_h ds \\ &= \sum_{K\in\mathscr{T}_h}\int_K \mathbb{B}\nabla u_h\nabla\omega_h dx \\ &\quad -\sum_{K\in\mathscr{T}_h^m}\sum_{e\in\varepsilon_h^*}\int_e \mathbb{B}\nabla u_h\cdot\mathbf{n}\,(\omega_h-\gamma_h\omega_h)\,ds \\ &= \sum_{K\in\mathscr{T}_h}\int_K \mathbb{B}\nabla u_h\nabla\omega_h dx \\ &\quad -\sum_{e\in\varepsilon_h^*}\int_e [\mathbb{B}\nabla u_h]\,\{\omega_h-\gamma_h\omega_h\}\,ds \\ &\quad -\sum_{e\in\varepsilon_h^*}\int_e \{\mathbb{B}\nabla u_h\}\,[\omega_h-\gamma_h\omega_h]\,ds.\end{aligned} \quad (33)$$

Thus, (24) can be written by

$$a_h^*(u_h,\gamma_h\omega_h) = A(u_h,\omega_h) - \sum_{e\in\varepsilon_h}\int_e \{\mathbb{B}\nabla u_h\}\,[\gamma_h\omega_h]\,ds + \sum_{e\in\varepsilon_h}\frac{\sigma_0}{|e|^{\alpha_0}}\int_e [u_h]\,[\omega_h]\,ds. \quad (34)$$

4. Some Lemmas

We define a norm $|||\cdot|||$ for H_h as follows:

$$|||v|||^2 = |v|_{1,h}^2 + \sum_e \frac{1}{h_e}\int_e [v]^2 ds + \sum_{e\in\varepsilon_h^*} h_e\int_e [\beta\nabla v]^2 ds. \quad (35)$$

In order to prove the existence and uniqueness of the solution to (24) and conduct its convergence analysis, we need the following lemmas.

Lemma 4. *The operator γ_h in* (15) *has the following properties:*

$$\int_e (\omega-\gamma_h\omega)\,ds = 0, \quad \forall\omega\in H_h,\ \forall e\in\varepsilon_h; \quad (36)$$

$$[\omega]|_e = 0 \Longrightarrow [\gamma_h\omega]|_e = 0, \quad \forall\omega\in H_h; \quad (37)$$

$$\|\gamma_h\omega-\omega\|_{0,K} \leqslant Ch_K|\omega|_{1,K}, \quad \forall K\in\mathscr{T}_h,\ \omega\in H_h. \quad (38)$$

Proof. Obviously, (36) and (37) follow from the definition of γ in (15). We only prove (38) below.

Let K be a noninterface element; we have the conclusion (38) by [12]. Therefore, we focus (38) on interface element K (Figure 4(b)). For $\forall\omega\in S_h(\Omega)$, we have the following form:

$$\omega = \begin{cases}\omega^+ = a_0 + b_0x + c_0y, & \mathbf{x}=(x,y)\in K^+,\\ \omega^- = a_1 + b_1x + c_1y, & \mathbf{x}=(x,y)\in K^-.\end{cases} \quad (39)$$

The jump conditions on $\overline{DE}$ lead to (see [7])

$$\nabla\omega^+ = \begin{pmatrix}\mathbf{n}_y^2+\rho\mathbf{n}_x^2 & (\rho-1)\,\mathbf{n}_x\mathbf{n}_y\\ (\rho-1)\,\mathbf{n}_x\mathbf{n}_y & \mathbf{n}_x^2+\rho\mathbf{n}_y^2\end{pmatrix}\nabla\omega^-, \quad (40)$$

or

$$\nabla\omega^+ = N_{\overline{DE}}^-\nabla\omega^-, \quad (41)$$

where $\mathbf{n}_{\overline{DE}} = (\mathbf{n}_x,\mathbf{n}_y)^T$ and $\rho = (\beta^-/\beta^+)$. We know that

$$\|\omega-\gamma_h\omega\|_{0,K}^2 = \sum_{j=1}^{3}\int_{T_j}(\omega-\gamma_h\omega)^2 d\mathbf{x}, \quad (42)$$

where

$$\int_{T_i}(\omega-\gamma_h\omega)^2 d\mathbf{x} = \int_{T_i^+}(\omega^+-\gamma_h\omega)^2 d\mathbf{x} + \int_{T_i^-}(\omega^--\gamma_h\omega)^2 d\mathbf{x}, \quad i=1,2. \quad (43)$$

Since ω is continuous on $\overline{A_1A_2}$, there exists a point ξ such that

$$\gamma_h\omega|_{\overline{A_1A_2}} = \omega(\xi). \tag{44}$$

We suppose that ξ fall on $\overline{A_1D}$; then, we have

$$\begin{aligned}\int_{T_1}(\omega-\gamma_h\omega)^2 d\mathbf{x} &= \int_{T_1^+}(\omega^+-\omega^+(\xi))^2 d\mathbf{x} + \int_{T_1^-}(\omega^- - \omega^+(\xi))^2 d\mathbf{x}\\ &= \int_{T_1^+}(\omega^+-\omega^+(\xi))^2 d\mathbf{x}\\ &\quad + \int_{T_1^-}(\omega^- - \omega^-(D) + \omega_h^+(D) - \omega^+(\xi))^2 d\mathbf{x},\end{aligned} \tag{45}$$

where we used $\omega^-(D) = \omega^+(D)$. Because $\omega^+(\mathbf{x})$ and $\omega^-(\mathbf{x})$ are linear polynomial, we have

$$\begin{aligned}\omega^+(\mathbf{x}) &= \omega^+(\xi) + \nabla\omega^+(\mathbf{x}-\xi),\\ \omega^-(\mathbf{x}) &= \omega^-(D) + \nabla\omega^-(\mathbf{x}-D).\end{aligned} \tag{46}$$

From these expansions of ω and (41), we have

$$\begin{aligned}\int_{T_1}(\omega-\gamma_h\omega)^2 d\mathbf{x} &= \int_{T_1^+}(\nabla\omega^+(\mathbf{x}-\xi))^2 d\mathbf{x}\\ &\quad + \int_{T_1^-}\left(\nabla\omega^-(\mathbf{x}-D) + N_{\overline{DE}}^-\nabla\omega^-(D-\xi)\right)^2 d\mathbf{x}.\end{aligned} \tag{47}$$

Then,

$$\int_{T_1}(\omega-\gamma_h\omega)^2 d\mathbf{x} \leqslant Ch^2|\omega|_{1,T_1}^2. \tag{48}$$

If ξ is on $\overrightarrow{DA_2}$, similarly, we also have (48). Analogously, we can have the following inequality:

$$\int_{T_i}(\omega-\gamma_h\omega)^2 d\mathbf{x} \leqslant Ch^2|\omega|_{1,T_i}^2, \quad i=2,3. \tag{49}$$

This completes the proof of (38) by (48) and (49). □

Lemma 5. *For any $\omega \in H_h$ and $e \in \varepsilon_h$, one has*

$$\left\|[\gamma_h\omega]\right\|_{L^2(e)} \leqslant \|[\omega]\|_{L^2(e)}; \tag{50}$$

$$\left\|[\omega-\gamma_h\omega]\right\|_{L^2(e)} \leqslant 2\|[\omega]\|_{L^2(e)}. \tag{51}$$

Proof. If e is the common side of $T_1, T_2 \in \mathcal{T}_h^*$, and $T_1 \subset K_1$, $T_2 \subset K_2$, by the definition of γ_h, we have

$$\begin{aligned}\left\|[\gamma_h\omega]\right\|_{L^2(e)}^2 &= \left\|\gamma_h\omega|_{\partial T_1}\cdot\mathbf{n}_1 + \gamma_h\omega|_{\partial T_2}\cdot\mathbf{n}_2\right\|_{L^2(e)}^2\\ &= \left\|h_e^{-1}\left(\int_e \omega|_{T_1}ds\cdot\mathbf{n}_1 + \int_e \omega|_{T_2}ds\cdot\mathbf{n}_2\right)\right\|_{L^2(e)}^2\\ &= h_e^{-2}\left\|\int_e[\omega]ds\right\|_{L^2(e)}^2\\ &= h_e^{-1}\left(\int_e[\omega]\,ds\right)^2.\end{aligned} \tag{52}$$

Using the Hölder inequality, we can get

$$\left\|[\gamma_h\omega]\right\|_{L^2(e)}^2 \leqslant \int_e[\omega]^2 ds = \|[\omega]\|_{L^2(e)}^2. \tag{53}$$

If $e \in \partial\Omega$, there exists a $T \in \mathcal{T}_h^*$, such that $e \in \partial T$ and $T \subset K$; we have

$$\begin{aligned}\left\|[\gamma_h\omega]\right\|_{L^2(e)}^2 &= \left\|\gamma_h\omega\cdot\mathbf{n}\right\|_{L^2(e)}^2\\ &= \left\|h_e^{-1}\int_e\omega|_T ds\cdot\mathbf{n}\right\|_{L^2(e)}^2\\ &= h_e^{-1}\left(\int_e[\omega]\,ds\right)^2\\ &\leqslant \|[\omega]\|_{L^2(e)}^2.\end{aligned} \tag{54}$$

Thus, (50) is valid.

For (51), we have

$$\begin{aligned}\left\|[\omega-\gamma_h\omega]\right\|_{L^2(e)} &= \left\|[\omega]-[\gamma_h\omega]\right\|_{L^2(e)}\\ &\leqslant 2\|[\omega]\|_{L^2(e)}\end{aligned} \tag{55}$$

following from (50). □

Lemma 6 (see [20]). *Let $\mathcal{T}$ be a regular triangulation; then, there exists a constant $C > 0$ independent of h_K such that, for $w \in H^1(K)$ and $K \in \mathcal{T}_h$, the following inequality holds:*

$$\int_{\partial K}|w|^2 ds \leqslant C\left\{h_K^{-1}\|w\|_{0,K}^2 + h_K|w|_{1,K}^2\right\}. \tag{56}$$

5. Existence, Uniqueness, and Convergence of DIFVE Solution

In this section, we will prove the existence and uniqueness of the solution to (24) and conduct its convergence analysis in the broken $|||\cdot|||$ norm.

Lemma 7. *There is a constant C independent of h such that for σ_0 large enough and ϵ small enough*

$$a_h^*(u_h, \gamma_h u_h) \geqslant C|||u_h|||^2. \tag{57}$$

Proof. By Cauchy-Schwarz's inequality, we have

$$\sum_{e\in\varepsilon_h^*}\int_e [\mathbb{B}\nabla u_h\cdot\mathbf{n}]\,\{u_h-\gamma_h u_h\}\,ds \leqslant \left(\sum_{e\in\varepsilon_h^*}\|[\mathbb{B}\nabla u_h\cdot\mathbf{n}]\|_{L^2(e)}^2\right)^{1/2}\left(\sum_{e\in\varepsilon_h^*}\|\{u_h-\gamma_h u_h\}\|_{L^2(e)}^2\right)^{1/2}. \tag{58}$$

Using the trace inequality (56), we have

$$\begin{aligned}\|\{u_h-\gamma_h u_h\}\|_{L^2(e)}^2 &\leqslant \frac{1}{2}\left(\left\|u_h-\gamma_h u_h|_{K_1^e}\right\|_{L^2(e)}^2+\left\|u_h-\gamma_h u_h|_{K_2^e}\right\|_{L^2(e)}^2\right)\\ &\leqslant C\left(h_{K_1^e}|u_h|_{1,K_1}^2+h_{K_2^e}|u_h|_{1,K_2}^2\right)\\ &\leqslant Ch\left(|u_h|_{1,K_1}^2+|u_h|_{1,K_2}^2\right),\end{aligned} \tag{59}$$

where edge e is shared by the elements K_1^e and K_2^e. Therefore,

$$\left(\sum_{e\in\varepsilon_h^*}\|\{u_h-\gamma_h u_h\}\|_{L^2(e)}^2\right)^{1/2} \leqslant Ch^{1/2}\left(\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,k}^2\right)^{1/2}. \tag{60}$$

By Young's inequality, we have, for $\varepsilon > 0$,

$$\begin{aligned}\sum_{e\in\varepsilon_h^*}\int_e [\mathbb{B}\nabla u_h\cdot\mathbf{n}]\,\{u_h-\gamma_h u_h\}\,ds &\leqslant Ch^{1/2}\left(\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,k}^2\right)^{1/2}\left(\sum_{e\in\varepsilon_h^*}\|[\mathbb{B}\nabla u_h\cdot\mathbf{n}]\|_{L^2(e)}^2\right)^{1/2}\\ &\leqslant C_0\sum_{e\in\varepsilon_h^*}h\|[\mathbb{B}\nabla u_h\cdot\mathbf{n}]\|_{L^2(e)}^2+\frac{\varepsilon}{3}\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,k}^2.\end{aligned} \tag{61}$$

Similarly, we obtain

$$\begin{aligned}\sum_{e\in\varepsilon_h^*}\int_e \{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\,[u_h-\gamma_h u_h]\,ds &\leqslant \left(\sum_{e\in\varepsilon_h^*}\|\{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\|_{L^2(e)}^2\right)^{1/2}\\ &\quad\times\left(\sum_{e\in\varepsilon_h^*}\|[u_h-\gamma_h u_h]\|_{L^2(e)}^2\right)^{1/2}.\end{aligned} \tag{62}$$

On the one hand, we have

$$\|\{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\|_{L^2(e)}^2 \leqslant Ch^{-1}\left(|u_h|_{1,K_1}^2+|u_h|_{1,K_2}^2\right), \tag{63}$$

and thus

$$\left(\sum_{e\in\varepsilon_h^*}\|\{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\|_{L^2(e)}^2\right)^{1/2} \leqslant Ch^{-1/2}\left(\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,k}^2\right)^{1/2}. \tag{64}$$

On the other hand, we get

$$\left(\sum_{e\in\varepsilon_h^*}\|[u_h-\gamma_h u_h]\|_{L^2(e)}^2\right)^{1/2} \leqslant 2\left(\sum_{e\in\varepsilon_h^*}\|[u_h]\|_{L^2(e)}^2\right)^{1/2} \tag{65}$$

following from (51). Therefore,

$$\begin{aligned}\sum_{e\in\varepsilon_h^*}\int_e \{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\,[u_h-\gamma_h u_h]\,ds &\leqslant Ch^{-1/2}\left(\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,K}^2\right)^{1/2}\left(\sum_{e\in\varepsilon_h^*}\|[u_h]\|_{L^2(e)}^2\right)^{1/2}\\ &\leqslant Ch^{(\alpha_0-1)/2}\left(\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,K}^2\right)^{1/2}\\ &\quad\times\left(\sum_{e\in\varepsilon_h^*}\frac{1}{|e|^{\alpha_0}}\|[u_h]\|_{L^2(e)}^2\right)^{1/2},\end{aligned} \tag{66}$$

where $\alpha_0 > 1$. By ε-inequality, we have

$$\begin{aligned}\sum_{e\in\varepsilon_h^*}\int_e \{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\,[u_h-\gamma_h u_h]\,ds &\leqslant \frac{\epsilon}{3}\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,K}^2+C_1\sum_{e\in\varepsilon_h^*}\frac{1}{|e|^{\alpha_0}}\|[u_h]\|_{L^2(e)}^2.\end{aligned} \tag{67}$$

Similarly, we can get

$$\begin{aligned}\sum_{e\in\varepsilon_h}\int_e \{\mathbb{B}\nabla u_h\cdot\mathbf{n}\}\,[\gamma_h u_h]\,ds &\leqslant \frac{\epsilon}{3}\sum_{K\in\mathcal{T}_h}|u_h|_{1,K}^2+C_2\sum_{e\in\varepsilon_h}\frac{1}{|e|^{\alpha_0}}\|[u_h]\|_{L^2(e)}^2.\end{aligned} \tag{68}$$

Combining (61), (67), and (68), we obtain

$$\begin{aligned}a_h^*(u_h,\gamma_h u_h) &\geqslant \left(C-\frac{\epsilon}{3}\right)\sum_{K\in\mathcal{T}_h^n}|u_h|_{1,K}^2+(C-\epsilon)\sum_{K\in\mathcal{T}_h^m}|u_h|_{1,K}^2\\ &\quad+\sum_{e\in\varepsilon_h^*}\frac{\sigma_0-C_1-C_2}{|e|^{\alpha_0}}\|[u_h]\|_{L^2(e)}^2\\ &\quad+\sum_{e\in\varepsilon_h\setminus\varepsilon_h^*}\frac{\sigma_0-C_2}{|e|^{\alpha_0}}\|[u_h]\|_{L^2(e)}^2\\ &\quad+\sum_{e\in\varepsilon_h^*}C_0h\|[\mathbb{B}\nabla u_h\cdot\mathbf{n}]\|_{L^2(e)}^2.\end{aligned} \tag{69}$$

Choosing σ_0 large enough and ϵ small enough, we have

$$a_h^*(u_h,\gamma_h u_h) \geqslant C|||u_h|||^2. \tag{70}$$

□

Lemma 8. *For $u, \omega \in H_h$, one has*

$$a_h^*(u, \gamma_h \omega) \leqslant C + \left(|||u||| + \left(\sum_{K \in \mathcal{T}_h} h^2 |u|^2_{\widetilde{H}^2(K)} \right)^{1/2} \right) |||\omega||| . \quad (71)$$

If $u_h, \omega_h \in S_h(\Omega)$, then

$$a_h^*(u_h, \gamma_h \omega_h) \leqslant C\, |||u_h|||\, |||\omega_h||| . \quad (72)$$

Proof. By (38) and the trace inequality (56), we have, for any $e^s \in \varepsilon_h^*$, $s = +, -$,

$$\begin{aligned}
&\int_{e^s} \mathbb{B}\nabla u \cdot \mathbf{n} (\omega - \gamma_h \omega)\, ds \\
&\leqslant C\big(h^{-1} |u|^2_{1,K^s} + h |u|^2_{2,K^s}\big)^{1/2} \big(h^{-1} |\omega - \gamma_h \omega|^2_{L^2(K^s)} + h |\omega - \gamma_h \omega|^2_{1,K^s}\big)^{1/2} \\
&\leqslant C\big(|u|^2_{1,K^s} + h^2 |u|^2_{2,K^s}\big)^{1/2} |\omega|_{1,K^s} \\
&\leqslant C\big(|u|^2_{1,K} + h^2 |u|^2_{\widetilde{H}^2(K)}\big)^{1/2} |\omega|_{1,K}.
\end{aligned} \quad (73)$$

By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&|A(u, \omega)| \\
&\leqslant \left| \sum_{K \in \mathcal{T}_h} \int_K \mathbb{B}\nabla u \nabla \omega dx \right| + \left| \sum_{K \in \mathcal{T}_h^m} \sum_{e \in \varepsilon_h^*} \int_e B\nabla u \cdot \mathbf{n} (\omega - \gamma_h \omega)\, ds \right| \\
&\leqslant C \left(|u|_{1,h} + \left(\sum_{K \in \mathcal{T}_h} \big(|u|^2_{1,K} + h^2 |u|^2_{\widetilde{H}^2(K)}\big) \right)^{1/2} \right) |\omega|_{1,h}.
\end{aligned} \quad (74)$$

The definition of $a_h(u, \gamma_h \omega)$ and the inequality above imply that

$$\begin{aligned}
|a_h(u, \gamma_h \omega)| &\leqslant C \Big\{ |u|_{1,h} |\omega|_{1,h} + \left(\sum_{K \in \mathcal{T}_h} \big(|u|^2_{1,K} + h^2 |u|^2_{\widetilde{H}^2(K)}\big) \right)^{1/2} |\omega|_{1,h} \\
&\quad + \left(\sum_{K \in \mathcal{T}_h} \big(|u|^2_{1,K} + h^2 |u|^2_{\widetilde{H}^2(K)}\big) \right)^{1/2} \times \left(\sum_{e \in \varepsilon_h} \|[\omega]\|^2_{L^2(e)} \right)^{1/2} \\
&\quad + \left(\sum_{e \in \varepsilon_h} \|[u]\|^2_{L^2(e)} \right)^{1/2} \left(\sum_{e \in \varepsilon_h} \|[\omega]\|^2_{L^2(e)} \right)^{1/2} \Big\} \\
&\leqslant C \left(|||u||| + \left(\sum_{K \in \mathcal{T}_h} h^2 |u|^2_{\widetilde{H}^2(K)} \right)^{1/2} \right) |||\omega||| .
\end{aligned} \quad (75)$$

This completes the proof of (71). For (72), we can get the following from (71) and $|u_h|_{\widetilde{H}^2(K)} = 0, \forall u_h \in S_h(\Omega)$. □

Lemmas 7 and 8 guarantee the existence and uniqueness of the discontinuous immersed finite volume element solution to (23) when choosing σ_0 large enough.

6. Error Estimates in the Energy Norm

We will derive an optimal-order error estimate in the norm $|||\cdot|||$ defined in (35) and a first order error estimate in L^2-norm. We start with the following lemmas.

Lemma 9. *Let u be the solutions of* (1)*; one has the conclusion*

$$|||u - \Pi_h u||| \leqslant Ch \|u\|_{\widetilde{H}^2(\Omega)}. \quad (76)$$

Proof. By the definition of norm $|||\cdot|||$, we have

$$\begin{aligned}
&|||u - \Pi_h u|||^2 \\
&= |u - \Pi_h u|^2_{1,h} + \sum_{e \in \varepsilon_h} \frac{1}{|e|^{\alpha_0}} \|[u - \Pi_h u]\|^2_{L^2(e)} + \sum_{e \in \varepsilon_h^*} h \|[\nabla(u - \Pi_h u) \cdot \mathbf{n}]\|^2_{L^2(e)}.
\end{aligned} \quad (77)$$

Using trace inequality and (12), we have

$$\begin{aligned}
&\|[u - \Pi_h u]\|^2_{L^2(e)} \\
&\leqslant C \left(\|u - \Pi_h u|_{K_1^e}\|^2_{L^2(e)} + \|u - \Pi_h u|_{K_2^e}\|^2_{L^2(e)} \right) \\
&\leqslant C \big(h^{-1} \|u - \Pi_h u\|^2_{L^2(K_1)} + h |u - \Pi_h u|^2_{1,K_1} + h^{-1} \|u - \Pi_h u\|^2_{L^2(K_2)} + h |u - \Pi_h u|^2_{1,K_2} \big) \\
&\leqslant Ch^2 \left(\|u\|^2_{\widetilde{H}^2(K_1)} + \|u\|^2_{\widetilde{H}^2(K_2)} \right),
\end{aligned} \quad (78)$$

where e is shared by the elements K_1 and K_2. Thus, we obtain

$$\sum_{e \in \varepsilon_h} \|[u - \Pi_h u]\|^2_{L^2(e)} \leqslant Ch^2 \|u\|^2_{\widetilde{H}^2(\Omega)}. \quad (79)$$

Analogously, we can get

$$\sum_{e\in\varepsilon_h^*} h\left\|[\nabla(u-\Pi_h u)\cdot \mathbf{n}]\right\|_{L^2(e)}^2 \leqslant Ch^2\|u\|_{\widetilde{H}^2(\Omega)}^2. \tag{80}$$

By (12), (79), and (80), we obtain (76). □

Lemma 10 (see [21]). *There exists a constant C independent of h such that*

$$\|\omega\| \leqslant C\,|||\omega|||, \quad \forall\omega\in S_h(\Omega). \tag{81}$$

Theorem 11. *Let $u_h \in S_{0h}(\Omega)$ and $u\in \widetilde{H}^2(\Omega)\cap H_0^1(\Omega)$ be the solutions of* (23) *and* (1)*, respectively; then, there exists a constant C independent of h such that*

$$|||u-u_h||| \leqslant Ch\|u\|_{\widetilde{H}^2(\Omega)}, \tag{82}$$

$$\|u-u_h\| \leqslant Ch\|u\|_{\widetilde{H}^2(\Omega)}. \tag{83}$$

Proof. Subtracting (25) from (23) gives

$$a_h^*\left(u-u_h,\gamma_h\omega_h\right)=0, \quad \forall\omega_h\in S_{0h}(\Omega). \tag{84}$$

Using (71), (76), and (84), we have

$$\begin{aligned}
&|||u_h-\Pi_h u|||^2 \\
&\quad\leqslant Ca_h^*\left(u_h-\Pi_h u_h,\gamma_h\left(u_h-\Pi_h u_h\right)\right) \\
&\quad= Ca_h^*\left(u-\Pi_h u,\gamma_h\left(u_h-\Pi_h u_h\right)\right) \\
&\quad\leqslant C\left(|||u-\Pi_h u|||+\left(\sum_{K\in\mathcal{T}_h} h^2|u-\Pi_h u|_{\widetilde{H}^2(K)}^2\right)^{1/2}\right) \\
&\qquad\times |||u_h-\Pi_h u||| \\
&\quad\leqslant Ch^2\|u\|_{\widetilde{H}^2(\Omega)}^2.
\end{aligned} \tag{85}$$

Thus, we can get

$$|||u-u_h||| \leqslant Ch\|u\|_{\widetilde{H}^2(\Omega)} \tag{86}$$

following from triangle inequality, (76), and (85).

For (83), we can get by (12), (81) and triangle inequality. We have completed the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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