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Research Article

Bell Polynomials Approach Applied to (2 + 1)-Dimensional Variable-Coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada Equation

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The bilinear form, bilinear Bäcklund transformation, and Lax pair of a (2 + 1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation are derived through Bell polynomials. The integrable constraint conditions on variable coefficients can be naturally obtained in the procedure of applying the Bell polynomials approach. Moreover, the N-soliton solutions of the equation are constructed with the help of the Hirota bilinear method. Finally, the infinite conservation laws of this equation are obtained by decoupling binary Bell polynomials. All conserved densities and fluxes are illustrated with explicit recursion formulae.

1. Introduction

It is well known that investigation of integrable properties of nonlinear evolution equations (NEEs) can be considered as a pretest and the first step of its exact solvability. The integrability features of soliton equations can be characterized by Hirota bilinear form, Lax pair, infinite symmetries, infinite conservation laws, Painlevé test, Hamiltonian structure, Bäcklund transformation (BT), and so on. The bilinear form of a soliton equation can not only be used to produce many of the known families of multisoliton solutions, but also be employed to derive the bilinear BT, Lax pair, and infinite sets of conserved quantities [1–6]. However, it relies on a particular skill and tedious calculation. In the early 1930s, the classical Bell polynomials were introduced by Bell which are specified by a generating function and exhibiting some important properties [7]. Recently, Lambert and coworkers have proposed a relatively convenient procedure based on Bell polynomials which enables us to obtain bilinear forms, bilinear BTs, Lax pairs, and Darboux covariant Lax pairs for NEEs [8–11]. It is shown that Bell polynomials play an important role in the characterization of bilinearizable equations and a deep relation between the integrability of an NEE and the Bell polynomials. Furthermore, Fan [12], Fan and Chow [13], and Wang and Chen [14, 15] developed the approach to construct infinite conservation laws by decoupling binary-Bell-polynomial-type BT into a Riccati type equation and a divergence type equation. Afterwards, Fan [16] and Fan and Hon [17] extended this method to supersymmetric equations. On the basis of their work, we apply the bell polynomials approach to the high-dimensional variable-coefficient NEEs.

Many physical and mechanical situations are governed by variable-coefficient NEEs, which might be more realistic than the constant coefficient ones in modeling a variety of complex nonlinear phenomena in physical and engineering fields [18–20].

The (2 + 1)-dimensional analogue of the Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation is in the form of

$$\begin{aligned} 36u_t = & -u_{5x} - 15(uu_{2x})_x - 45u^2u_x + 5u_{2x,y} \\ & + 15uu_y + 15u_x\partial_x^{-1}u_y + 5\partial_x^{-1}u_{2y}, \end{aligned} \tag{1}$$

with $\partial_x^{-1} = \int \cdot dx$. Equation (1) is first proposed by Konopelchenko and Dubrovsky [21] and then considered by many authors in various aspects such as its quasiperiodic solutions [22], algebraic-geometric solution [23], N-soliton solutions [24], nonlocal symmetry [25], and symmetry reductions [26]. Based on (1), we will consider a $(2+1)$-dimensional variable-coefficient CDGKS equation as

$$\begin{aligned} & u_t + a_1 u_{5x} + a_2 u_x u_{2x} + a_3 u u_{3x} \\ & \quad + a_4 u^2 u_x + a_5 u_{2x,y} + a_6 \partial_x^{-1} u_{2y} \\ & \quad + a_7 u_x \partial_x^{-1} u_y + a_8 u u_y + a_9 u = 0, \end{aligned} \tag{2}$$

where $a_i = a_i(t)$, $i = 1, \ldots, 9$, are analytic functions with respect to t. The aim of this paper is applying the Bell polynomials approach to systematically investigate the integrability of (2), which includes bilinear form, bilinear BT, Lax pair, and infinite conservation laws.

The layout of this paper is as follows. Basic concepts and identities about Bell polynomials will be briefly introduced in Section 2. In Section 3, by virtue of Bell polynomials and the Hirota bilinear method, the bilinear form and N-soliton solutions of (2) are obtained. In Sections 4 and 5, with the aid of Bell polynomials, the bilinear BT, Lax pair, and infinite conservation laws of (2) are systematically presented, respectively. Section 6 will be our conclusions.

2. Bell Polynomials

The Bell polynomials [7, 9, 10] used here are defined as

$$Y_{nx}(f) = Y_n(\{f_{rx}(1 \le r \le n)\}) = e^{-f}\partial_x^n e^f, \qquad Y_{0x} \equiv 1, \tag{3}$$

where $f(x)$ is a C^∞ function and $f_{rx} = \partial_x^r f$; according to formula (3), the first three are

$$\begin{aligned} & Y_x(f) = f_x, \qquad Y_{2x}(f) = f_{2x} + f_x^2, \\ & Y_{3x}(f) = f_{3x} + 3 f_x f_{2x} + f_x^3. \end{aligned} \tag{4}$$

Based on one-dimensional Bell polynomials, the multidimensional Bell polynomials are expressed as

$$\begin{aligned} Y_{n_1x_1,\ldots,n_lx_l}(f) &= Y_{n_1,\ldots,n_l}\left(\left\{f_{r_1x_1,\ldots,r_lx_l}(1 \le r_i \le n_i, 0 \le i \le l)\right\}\right) \\ &= e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l} e^f, \end{aligned} \tag{5}$$

with $f = f(x_1, \ldots, x_l)$ being a C^∞ function and $f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l} f$; the associated two-dimensional Bell polynomials can be written as

$$\begin{aligned} Y_{mx,nt}(f) &= Y_{m,n}(\{f_{rx,st}(1 \le r \le m, 1 \le s \le n)\}) \\ &= e^{-f}\partial_x^m \partial_t^n e^f. \end{aligned} \tag{6}$$

The most important multidimensional binary Bell polynomials, namely, $\mathscr{Y}$-polynomials, can be defined as

$$\begin{aligned} \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v, w) &= Y_{n_1x_1,\ldots,n_lx_l}(f) \\ &= Y_{n_1,\ldots,n_l}\left(\left\{f_{r_1x_1,\ldots,r_lx_l}\right\}\right) \end{aligned} \tag{7}$$

for

$$f_{r_1x_1,\ldots,r_lx_l} = \begin{cases} v_{r_1x_1,\ldots,r_lx_l} & \sum_{i=0}^{l} r_i \text{ is odd} \\ w_{r_1x_1,\ldots,r_lx_l} & \sum_{i=0}^{l} r_i \text{ is even} \end{cases} \tag{8}$$

with the first few lowest order binary Bell polynomials being

$$\begin{aligned} & \mathscr{Y}_x(v) = v_x, \qquad \mathscr{Y}_{2x}(v, w) = w_{2x} + v_x^2, \\ & \mathscr{Y}_{x,t}(v, w) = w_{x,t} + v_x v_t, \\ & \mathscr{Y}_{2x,t}(v, w) = v_{2x,t} + w_{2x} v_t + 2 w_{xy} v_x + v_x^2 v_y, \ldots. \end{aligned} \tag{9}$$

The $\mathscr{Y}$-polynomials can be linked to the standard Hirota expressions through the identity [10]

$$\begin{aligned} & \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}\left(v = \ln\left(\frac{F}{G}\right), w = \ln(FG)\right) \\ & \quad = (FG)^{-1} D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} F \cdot G, \end{aligned} \tag{10}$$

in which $\sum_{i=1}^{l} n_i \ge 1$ and the operators $D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}$ are classical Hirota bilinear operators defined by [1]

$$\begin{aligned} & D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} F \cdot G \\ & \quad = \left(\partial_{x_1} - \partial_{x_1'}\right)^{n_1}\cdots\left(\partial_{x_l} - \partial_{x_l'}\right)^{n_l} \\ & \quad \times F(x_1, \ldots, x_l)\, G(x_1', \ldots, x_l')\big|_{x_1'=x_1,\ldots,x_l'=x_l}. \end{aligned} \tag{11}$$

Introducing a new field $q = w - v$, in the particular case $F = G$ one has

$$\begin{aligned} G^{-2} D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} G \cdot G &= \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = w - v) \\ &= \begin{cases} 0 & \sum_{i=0}^{l} n_i \text{ is odd,} \\ \mathscr{P}_{n_1x_1,\ldots,n_lx_l}(q) & \sum_{i=0}^{l} n_i \text{ is even,} \end{cases} \end{aligned} \tag{12}$$

in which the even-order $\mathscr{Y}$-polynomials is called $\mathscr{P}$-polynomials; that is,

$$\mathscr{P}_{n_1x_1,\ldots,n_lx_l}(q) = \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = w - v), \tag{13}$$

with

$$\begin{aligned} & \mathscr{P}_{2x}(q) = q_{2x}, \qquad \mathscr{P}_{x,t}(q) = q_{x,t}, \\ & \mathscr{P}_{4x}(q) = q_{4x} + 3q_{2x}^2, \\ & \mathscr{P}_{6x}(q) = q_{6x} + 15 q_{2x} q_{4x} + 15 q_{2x}^3, \ldots. \end{aligned} \tag{14}$$

Moreover, the binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)$ can be written as the combination of $\mathscr{P}$-polynomials and Y-polynomials:

$$\begin{aligned}
&(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\\
&\quad=\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)\Big|_{v=\ln(F/G),w=\ln(FG)}\\
&\quad=\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,v+q)\Big|_{v=\ln(F/G),q=2\ln G}\\
&\quad=\sum_{p_1=0}^{n_1}\cdots\sum_{p_l=0}^{n_l}\binom{n_1}{p_1}\cdots\binom{n_l}{p_l}\mathscr{P}_{n_1x_1,\ldots,n_lx_l}(q)\\
&\qquad\times Y_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}(v).
\end{aligned}\tag{15}$$

Under the Hopf-Cole transformation $v=\ln\psi$, the Y-polynomials can be linearized into the form

$$Y_{n_1x_1,\ldots,n_lx_l}(v)\Big|_{v=\ln\psi}=\frac{\psi_{n_1x_1,\ldots,n_lx_l}}{\psi},\tag{16}$$

which provides a straightforward way for the related Lax systems of NEEs.

3. Bilinear Form and N-Soliton Solutions for (2)

Firstly, introduce a dimensionless potential field q by setting

$$u=cq_{2x},\tag{17}$$

with $c=c(t)$ to be determined. Substituting (17) into (2), integration with respect to x yields the following potential version of (2):

$$\begin{aligned}
&\left(\frac{c_t}{c}+a_9\right)q_x+q_{x,t}-\frac{1}{6}c\,(a_7-a_8)\,\partial_x^{-1}\partial_y\left(q_{4x}+3q_{2x}^2\right)\\
&\quad+a_1q_{6x}+\frac{1}{2}c\,(a_2-a_3)\,q_{3x}^2+ca_3q_{2x}q_{4x}+\frac{1}{3}c^2a_4q_{2x}^3\\
&\quad+\left[a_5+\frac{1}{6}c\,(a_7-a_8)\right]q_{3x,y}+a_6q_{2y}+ca_7q_{2x}q_{x,y}=0;
\end{aligned}\tag{18}$$

on account of the dimension of u ($\dim u=-2$), we find that setting $c=c_0e^{-\int a_9dt}$, where c_0 is an arbitrary constant. In order to write (18) in local bilinear form, here are two cases which are considered to eliminate the effect of the integration ∂_x^{-1}. The bilinear form and N-soliton solutions for each case will be discussed by selecting appropriate constraints on variable coefficients a_i, $i=1,\ldots,9$.

3.1. Case 1. Let $a_7=a_8$; (18) becomes

$$\begin{aligned}
&q_{x,t}+a_1q_{6x}+\frac{1}{2}c\,(a_2-a_3)\,q_{3x}^2+ca_3q_{2x}q_{4x}+\frac{1}{3}c^2a_4q_{2x}^3\\
&\quad+a_5q_{3x,y}+a_6q_{2y}+ca_7q_{2x}q_{x,y}=0.
\end{aligned}\tag{19}$$

This equation can be viewed as a homogeneous $\mathscr{P}$-condition [8] of weight 6 (the weight of each term being defined as minus its dimension, a weight 3 to y). That means (19) can be written as a linear combination of $\mathscr{P}$-polynomials of weight 6:

$$\mathscr{P}_{x,t}(q)+a_1\mathscr{P}_{6x}(q)+a_5\mathscr{P}_{3x,y}(q)+a_6\mathscr{P}_{2y}(q)=0;\tag{20}$$

under the following constraint condition:

$$\begin{aligned}
&ca_3-15a_1=0,\qquad ca_7-3a_5=0,\\
&\frac{1}{2}c\,(a_2-a_3)=0,\qquad \frac{1}{3}c^2a_4-15a_1=0,
\end{aligned}\tag{21}$$

namely,

$$\begin{aligned}
&a_2=a_3=\frac{15a_1}{c_0}e^{\int a_9dt},\qquad a_4=\frac{45a_1}{c_0^2}e^{2\int a_9dt},\\
&a_7=a_8=\frac{3a_5}{c_0}e^{\int a_9dt}.
\end{aligned}\tag{22}$$

According to the property (12), via the following transformation:

$$q=2\ln G\Longleftrightarrow u=cq_{2x}=2c_0e^{-\int a_9dt}(\ln G)_{2x},\tag{23}$$

$\mathscr{P}$-polynomials expression (20) produces the bilinear form of (2) as follows:

$$\left(D_xD_t+a_1D_x^6+a_5D_x^3D_y+a_6D_y^2\right)G\cdot G=0.\tag{24}$$

Starting from this bilinear equation, the one-soliton solution of (2) can be easily obtained by regular perturbation method

$$u=2c_0e^{-\int a_9dt}\left[\ln\left(1+e^{\eta_1}\right)\right]_{2x},\tag{25}$$

with

$$\begin{aligned}
&\eta_1=k_1x+l_1y+\omega_1(t)+\xi_1,\\
&\omega_1(t)=-\int\frac{k_1^6a_1+k_1^3l_1a_5+l_1^2a_6}{k_1}dt.
\end{aligned}\tag{26}$$

However, the multisoliton solutions cannot be derived by means of bilinear equation (24). For the sake of obtaining multisoliton solutions of (2), we take

$$a_5=5c_1a_1,\qquad a_6=-5c_1^2a_1,\tag{27}$$

where c_1 is an arbitrary constant; the bilinear equation can be expressed as

$$\left(D_xD_t+a_1D_x^6+5c_1a_1D_x^3D_y-5c_1^2a_1D_y^2\right)G\cdot G=0,\tag{28}$$

with the conditions (22) and (27); that is,

$$\begin{aligned}
&a_2=a_3=\frac{15a_1}{c_0}e^{\int a_9dt},\qquad a_4=\frac{45a_1}{c_0^2}e^{2\int a_9dt},\\
&a_5=5c_1a_1,\qquad a_6=-5c_1^2a_1,\qquad a_7=a_8=\frac{15c_1a_1}{c_0}e^{\int a_9dt}.
\end{aligned}\tag{29}$$

Based on the bilinear equation (28), the N-soliton solutions for (2) can be constructed as

$$u = 2c_0 e^{-\int a_9 dt}\left[\ln\left(\sum_{\mu=0,1} e^{\sum_{j=1}^{N}\mu_j\eta_j+\sum_{1\le i<j}^{N}\mu_i\mu_j A_{ij}}\right)\right]_{2x}, \quad (30)$$

where

$$\begin{aligned}
\eta_j &= k_j x + l_j y + \omega_j(t) + \xi_j,\\
\omega_j(t) &= -\frac{k_j^6 + 5c_1 k_j^3 l_j - 5c_1^2 l_j^2}{k_j}\int a_1 dt,\\
e^{A_{ij}} &= \Big\{(k_i-k_j)\left[c_1 k_i k_j^2 l_i(2k_i-k_j) + c_1 k_i^2 k_j l_j (k_i-2k_j)\right.\\
&\qquad \left. + k_i^2k_j^2\left(k_i^2-k_ik_j+k_j^2\right)(k_i-k_j)\right]\\
&\quad + c_1^2\left(k_i l_j - k_j l_i\right)^2\Big\}\\
&\quad \times\Big\{(k_i+k_j)\left[c_1k_ik_j^2l_i(2k_i+k_j)\right.\\
&\qquad + c_1k_i^2k_jl_j(k_i+2k_j)\\
&\qquad + k_i^2k_j^2\left(k_i^2+k_ik_j+k_j^2\right)\\
&\qquad \left.\times (k_i+k_j)\right] + c_1^2\left(k_il_j-k_jl_i\right)^2\Big\}^{-1},
\end{aligned} \quad (31)$$

with k_j, l_j, and ξ_j $(j = 1, 2, \ldots, N)$ being arbitrary constants; $\sum_{\mu=0,1}$ indicates a summation over all possible combinations of $\mu_j = 0, 1$ $(j = 1, 2, \ldots, N)$. For $N = 1$, the one-soliton solution for (2) can be written as follows:

$$\begin{aligned}
u &= \frac{1}{2}c_0 k_1^2 e^{-\int a_9 dt}\\
&\quad\times \operatorname{sech}^2\left[\frac{1}{2}\left(k_1x + l_1y - \frac{k_1^6 + 5c_1k_1^3l_1 - 5c_1^2l_1^2}{k_1}\right.\right.\\
&\qquad \left.\left.\times \int a_1 dt + \xi_1\right)\right].
\end{aligned} \quad (32)$$

For $N = 2$, we can obtain the two-soliton solution for (2) as

$$u = 2c_0 e^{-\int a_9 dt}\left[\ln\left(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}\right)\right]_{2x}. \quad (33)$$

Based on solutions (32) and (33), we present some figures to describe the propagations and collisions of the solitary waves. Figure 1 shows the propagation of one-soliton solution via solution (32) when $t = -2$, $t = -1$, and $t = 2$, which maintains its shape except for the phase shift, and the propagation direction can be changed. Figures 2 and 3 illustrate the oblique collision between the two solitons, which keep their original shapes invariant except for phase shifts as mentioned above. It is obvious that the large-amplitude soliton moves faster than the small one. Different from Figure 2, Figure 3 displays that both solitons change their directions during the collision.

3.2. Case 2. As another case, we introduce an auxiliary variable s and a subsidiary condition

$$q_{4x} + 3q_{2x}^2 + q_{x,s} = 0, \quad (34)$$

in virtue of which, similarly, (18) can be written as a linear combination of $\mathscr{P}$-polynomials of weight 6 (a weight 3 to s):

$$\begin{aligned}
&\mathscr{P}_{x,t}(q) + \beta\mathscr{P}_{6x}(q) + \gamma\mathscr{P}_{3x,y}(q)\\
&\quad + a_6\mathscr{P}_{2y}(q) + \frac{1}{6}c(a_7-a_8)\mathscr{P}_{y,s}(q)\\
&\quad + \delta\mathscr{P}_{3x,s}(q) + \alpha\mathscr{P}_{s,s}(q) = 0,
\end{aligned} \quad (35)$$

with the following constraint condition:

$$\begin{gathered}
ca_7 - 3\gamma = 0, \qquad a_5 - \gamma + \frac{1}{6}c(a_7 - a_8) = 0,\\
ca_3 - 15\beta + 9\delta - 12\alpha = 0, \qquad a_1 - \beta + \delta - \alpha = 0,\\
\frac{1}{3}c^2a_4 - 15\beta + 9\delta - 12\alpha = 0,\\
\frac{1}{2}c(a_2 - a_3) + 6\delta - 3\alpha = 0.
\end{gathered} \quad (36)$$

Solving for (36) yields

$$\begin{aligned}
\gamma &= \frac{1}{3}c_0e^{-\int a_9dt}a_7,\\
\beta &= -\frac{3}{2}a_1 + \frac{1}{6}c_0e^{-\int a_9dt}a_3 - \frac{1}{2}\alpha,\\
\delta &= -\frac{5}{2}a_1 + \frac{1}{6}c_0e^{-\int a_9dt}a_3 + \frac{1}{2}\alpha,\\
a_2 &= -a_3 + \frac{30a_1}{c_0}e^{\int a_9dt},\\
a_4 &= \frac{3a_3}{c_0}e^{\int a_9dt},\\
a_5 &= \frac{1}{6}c_0e^{-\int a_9dt}(a_7 + a_8).
\end{aligned} \quad (37)$$

Thus, the $\mathscr{P}$-polynomials expression of (2) and (34) reads

$$\begin{aligned}
&\mathscr{P}_{4x}(q) + \mathscr{P}_{x,s}(q) = 0,\\
&\mathscr{P}_{x,t}(q) + \left(-\frac{3}{2}a_1 + \frac{1}{6}c_0e^{-\int a_9dt}a_3 - \frac{1}{2}\alpha\right)\mathscr{P}_{6x}(q)\\
&\quad + \frac{1}{3}c_0e^{-\int a_9dt}a_7\mathscr{P}_{3x,y}(q)\\
&\quad + a_6\mathscr{P}_{2y}(q) + \frac{1}{6}c_0e^{-\int a_9dt}(a_7 - a_8)\mathscr{P}_{y,s}(q)\\
&\quad + \left(-\frac{5}{2}a_1 + \frac{1}{6}c_0e^{-\int a_9dt}a_3 + \frac{1}{2}\alpha\right)\mathscr{P}_{3x,s}(q)\\
&\quad + \alpha\mathscr{P}_{s,s}(q) = 0,
\end{aligned} \quad (38)$$

in which $\alpha = \alpha(t)$ is an arbitrary function.

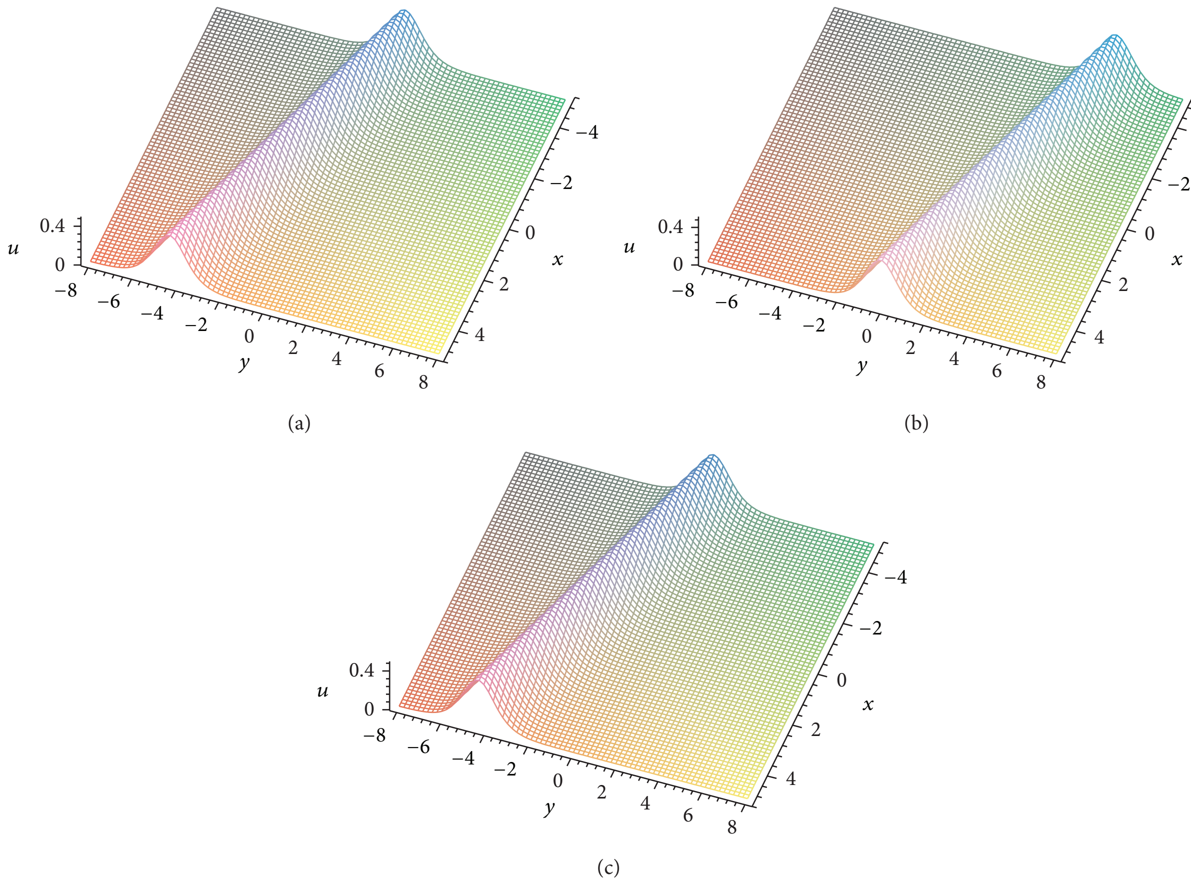


Figure 1: One-soliton solution via solution (32) with $a_9 = 0.01$, $k_1 = 1$, $l_1 = 2$, $c_0 = 1$, $c_1 = 1$, $a_1 = \sin(t)$, and $\xi_1 = 0$. (a) $t = -2$; (b) $t = -1$; (c) $t = 2$.

System (38) produces the bilinear form of (2) as follows:

$$\begin{gathered}\left(D_x^4 + D_x D_s\right) G \cdot G = 0, \\ \left[D_x D_t + \left(-\frac{3}{2}a_1 + \frac{1}{6}c_0 e^{-\int a_9 dt} a_3 - \frac{1}{2}\alpha\right) D_x^6 \right. \\ + \frac{1}{3}c_0 e^{-\int a_9 dt} a_7 D_x^3 D_y \\ + a_6 D_y^2 + \frac{1}{6}c_0 e^{-\int a_9 dt}\left(a_7 - a_8\right) D_y D_s \\ + \left(-\frac{5}{2}a_1 + \frac{1}{6}c_0 e^{-\int a_9 dt} a_3 + \frac{1}{2}\alpha\right) \\ \left. \times D_x^3 D_s + \alpha D_s^2\right] G \cdot G = 0,\end{gathered} \tag{39}$$

by property (12) and transformation (23). From the bilinear equation (39), we can only get the one-soliton solution which is the same as the above formulae (25) and (26). Therefore, (2) under the constraint conditions (37) is not integrable since its multisoliton solutions cannot be obtained.

4. Bilinear BT and Lax Pair for (2)

In order to search for the bilinear BT and Lax pair of (2), under the integrable constraint condition (29) in case 1, we have

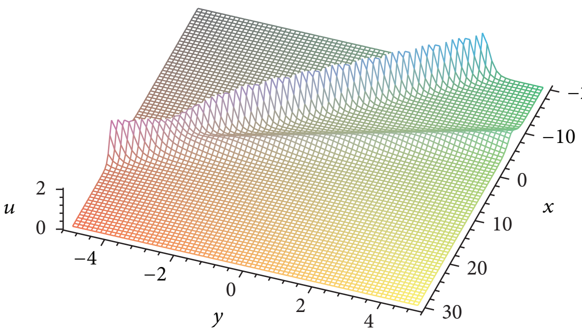
$$\begin{aligned} E(q) = {} & q_{x,t} + a_1\left(q_{6x} + 15 q_{2x} q_{4x} + 15 q_{2x}^3\right) \\ & + 5 c_1 a_1 \left(q_{3x,y} + 3 q_{2x} q_{x,y}\right) - 5 c_1^2 a_1 q_{2y} = 0. \end{aligned} \tag{40}$$

Let

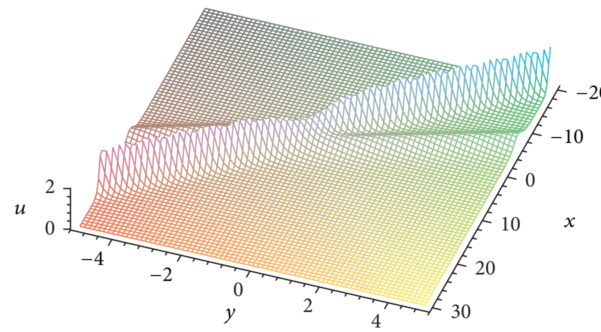
$$q = 2\ln G, \qquad q' = 2\ln F \tag{41}$$

be two solutions of (40), respectively. On introducing two new variables

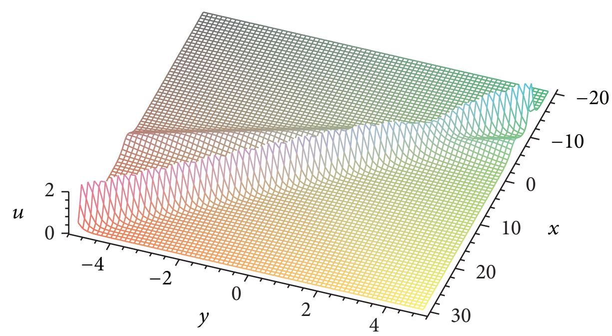
$$\begin{aligned} v &= \frac{q' - q}{2} = \ln\left(\frac{F}{G}\right), \\ w &= \frac{q' + q}{2} = \ln(FG), \end{aligned} \tag{42}$$



(a)

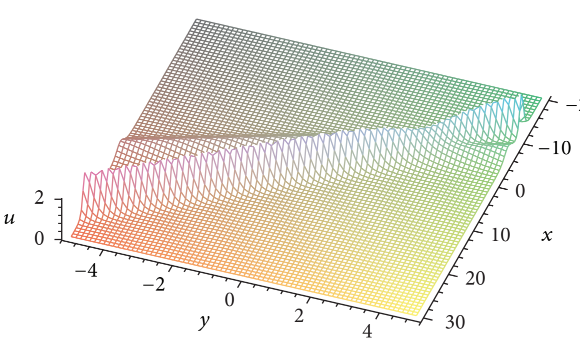


(b)

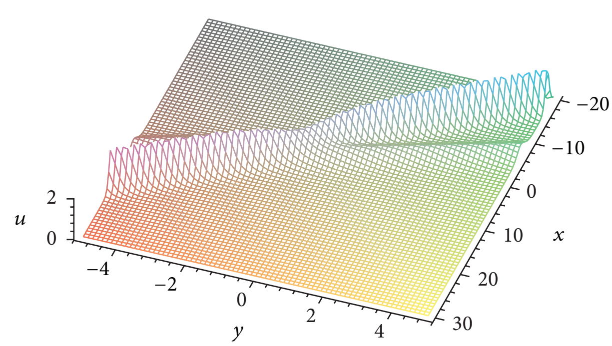


(c)

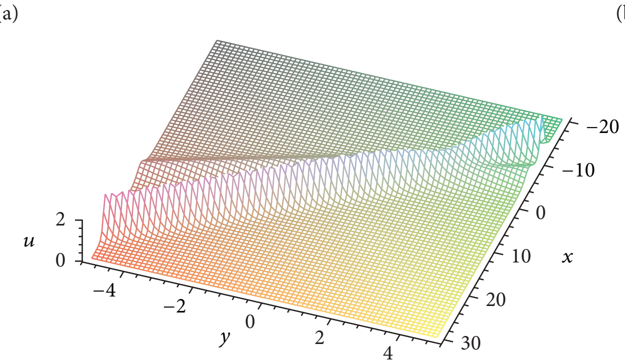
FIGURE 2: Two-soliton solution via solution (33) with $a_9 = 0.01$, $k_1 = 1$, $k_2 = 2$, $l_1 = 2$, $l_2 = 8$, $c_0 = 1$, $c_1 = 0.02$, $a_1 = 0.2$, and $\xi_1 = \xi_2 = 0$. (a) $t = -2$; (b) $t = 0$; (c) $t = 2$.



(a)



(b)



(c)

FIGURE 3: Two-soliton solution via solution (33) with $a_9 = 0.01$, $k_1 = 1$, $k_2 = 2$, $l_1 = 2$, $l_2 = 7$, $c_0 = 1$, $c_1 = 0.02$, $a_1 = t$, and $\xi_1 = \xi_2 = 0$. (a) $t = -0.8$; (b) $t = 0$; (c) $t = 0.8$.

one has the corresponding two-field condition

$$\begin{aligned} E(q') - E(q) &= E(w+v) - E(w-v) \\ &= 2\left[v_{x,t} + 15a_1 v_{2x}^3 \right. \\ &\quad + \left(15a_1 w_{4x} + 45a_1 w_{2x}^2 + 15c_1 a_1 w_{xy}\right) v_{2x} \\ &\quad - 5c_1^2 a_1 v_{2y} + a_1 v_{6x} + 15a_1 w_{2x} v_{4x} \\ &\quad \left. + 5c_1 a_1 v_{3x,y} + 15c_1 a_1 w_{2x} v_{x,y}\right] \\ &= 2\partial_x \left[\mathscr{Y}_t(v) + a_1 \mathscr{Y}_{5x}(v,w) + 5c_1 a_1 \mathscr{Y}_{2x,y}(v,w)\right] \\ &\quad + 2R(v,w) = 0, \end{aligned} \tag{43}$$

with

$$\begin{aligned} R(v,w) = -5a_1 &\left(v_x^4 v_{2x} + 2w_{3x} v_x^3 + 6w_{2x} v_x^2 v_{2x} \right. \\ &+ c_1 v_{x,y} v_x^2 + 2v_{4x} v_x^2 + 2c_1 v_{2x} v_x v_y \\ &+ 2c_1 w_{2xy} v_x + 4v_{3x} v_x v_{2x} + 6w_{2x} v_x w_{3x} \\ &+ w_{5x} v_x + c_1 w_{3x} v_y - 3v_{2x}^3 - 6w_{2x}^2 v_{2x} \\ &- c_1 v_{2x} w_{xy} - 2w_{4x} v_{2x} - 2c_1 w_{2x} v_{xy} \\ &\left. + c_1^2 v_{2y} - v_{4x} w_{2x} + 2v_{3x} w_{3x}\right). \end{aligned} \tag{44}$$

The simplest possible choice is a homogeneous $\mathscr{Y}$-constraint[8] of weight 2; it can only be of form

$$\mathscr{Y}_{2x}(v,w) + a\mathscr{Y}_y(v) = \lambda. \tag{45}$$

It is easy to find that eliminating w_{2x} (and its derivatives) by means of form (45) does not enable one to express the remainder $R(v,w)$ as the x-derivative of a linear combination of $\mathscr{Y}$-polynomials. However, a homogeneous $\mathscr{Y}$-constraint of weight 3

$$\begin{gathered} \mathscr{Y}_{3x}(v,w) + c_1 \mathscr{Y}_y(v) = \lambda, \\ \lambda = \text{arbitrary parameter of weight } 3, \end{gathered} \tag{46}$$

can be used to express $R(v,w)$ as follows:

$$\begin{aligned} R(v,w) &= -\frac{5}{2} a_1 \partial_x \left[\mathscr{Y}_{5x}(v,w) - c_1 \mathscr{Y}_{2x,y}(v,w) \right. \\ &\quad \left. + 3\lambda \mathscr{Y}_{2x}(v,w)\right]. \end{aligned} \tag{47}$$

Thus, the two-field condition (43) becomes

$$\begin{aligned} \partial_x &\left[\mathscr{Y}_t(v) - \frac{3}{2} a_1 \mathscr{Y}_{5x}(v,w) + \frac{15}{2} c_1 a_1 \mathscr{Y}_{2x,y}(v,w) \right. \\ &\left. - \frac{15}{2} a_1 \lambda \mathscr{Y}_{2x}(v,w)\right] = 0 \quad (\text{weight } 6), \end{aligned} \tag{48}$$

where we prefer the equation in the conserved form, which is useful to construct conservation laws later. It is seen that the two-field condition (43) can be decoupled into a pair of parameter-dependent $\mathscr{Y}$-constraints (of weight 3 and weight 5):

$$\begin{gathered} \mathscr{Y}_{3x}(v,w) + c_1 \mathscr{Y}_y(v) - \lambda = 0, \\ \mathscr{Y}_t(v) - \frac{3}{2} a_1 \mathscr{Y}_{5x}(v,w) + \frac{15}{2} c_1 a_1 \mathscr{Y}_{2x,y}(v,w) \\ - \frac{15}{2} a_1 \lambda \mathscr{Y}_{2x}(v,w) = 0. \end{gathered} \tag{49}$$

In view of (10), the bilinear BT for (2) is obtained:

$$\begin{gathered} \left(D_x^3 + c_1 D_y - \lambda\right) F \cdot G = 0, \\ \left(D_t - \frac{3}{2} a_1 D_x^5 + \frac{15}{2} c_1 a_1 D_x^2 D_y - \frac{15}{2} a_1 \lambda D_x^2\right) F \cdot G = 0. \end{gathered} \tag{50}$$

By application of formulae (15) and (16), the system (50) is linearized to be the Lax pair of (2) as

$$\begin{gathered} \psi_{3x} + 3q_{2x}\psi_x + c_1 \psi_y = \lambda \psi, \\ \psi_t - 9a_1 \psi_{5x} - 45a_1 q_{2x} \psi_{3x} - 45 q_{3x} \psi_{2x} \\ - \left(30a_1 q_{4x} + 45a_1 q_{2x}^2 - 15c_1 a_1 q_{x,y}\right) \psi_x = 0. \end{gathered} \tag{51}$$

Starting from this Lax pair with $a_1 = -1$, $a_9 = 0$, $c_0 = 3$, and $c_1 = 1$, the Darboux transformation and nonlocal symmetry of the equation can be established [25]. Checking that the compatibility condition of system (51) is just the potential of (40).

5. Infinite Conservation Laws for (2)

In what follows, we present the infinite conservation laws by recursion formulae for (2). The conservation laws actually have been hinted in the binary-Bell-polynomial-type BT (46) and (48), which can be rewritten in the conserved form

$$\begin{aligned} &v_{3x} + 3v_x w_{2x} + v_x^3 + c_1 v_y = \lambda, \\ &\partial_t(v_x) + \partial_x \left[-\frac{3}{2} a_1 \left(v_{5x} + 5w_{4x} v_x + 10v_{3x} w_{2x} \right.\right. \\ &\qquad + 10 v_{3x} v_x^2 + 15 w_{2x}^2 v_x \\ &\qquad \left. + 10 w_{2x} v_x^3 + v_x^5\right) \\ &\qquad - \frac{15}{2} a_1 \lambda \left(w_{2x} + v_x^2\right) \\ &\qquad \left. + \frac{15}{2} c_1 a_1 \left(w_{2x} v_y + 2w_{x,y} v_x + v_x^2 v_y\right)\right] \\ &\quad + \partial_y \left(\frac{15}{2} c_1 a_1 v_{3x}\right) = 0, \end{aligned} \tag{52}$$

by using the relation

$$\begin{gathered} \partial_t(v_x) = \partial_x(v_t) = v_{x,t}, \\ \partial_y(v_x) = \partial_x(v_y) = v_{x,y}. \end{gathered} \tag{53}$$

By introducing a new potential function

$$\eta = \frac{q'_x - q_x}{2}, \tag{54}$$

in this way, there are

$$v_x = \eta, \qquad w_x = q_x + \eta. \tag{55}$$

Substituting (55) into system (52), we obtain

$$\eta_{2x} + 3\eta\left(q_{2x} + \eta_x\right) + \eta^3 + c_1\partial_x^{-1}\eta_y = \lambda = \varepsilon^3, \tag{56}$$

$$\begin{aligned}
\eta_t + \partial_x \Big[-\frac{3}{2}a_1 \big(& \eta_{4x} + 5q_{4x}\eta + 5\eta_{3x}\eta + 10q_{2x}\eta_{2x} \\
& + 10\eta_x\eta_{2x} + 10\eta^2\eta_{2x} + 15q_{2x}^2\eta \\
& + 30q_{2x}\eta_x\eta + 15\eta_x^2\eta + 10q_{2x}\eta^3 \\
& + 10\eta_x\eta^3 + \eta^5 \big) \\
& - \frac{15}{2}a_1\varepsilon^3\left(q_{2x} + \eta_x + \eta^2\right) \\
& + \frac{15}{2}c_1a_1\big(q_{2x}\partial_x^{-1}\eta_y + \eta_x\partial_x^{-1}\eta_y + 2q_{x,y}\eta \\
& + 2\eta_y\eta + \eta^2\partial_x^{-1}\eta_y\big)\frac{3}{2}\Big] \\
& + \partial_y\left(\frac{15}{2}c_1a_1\eta_{2x}\right) = 0.
\end{aligned} \tag{57}$$

It may be noticed that (56) is not a Riccati-type equation. Similar to [27], inserting expansion

$$\eta = \varepsilon + \sum_{n=1}^{\infty} I_n\left(q, q_x, q_y, \ldots\right)\varepsilon^{-n} \tag{58}$$

into (56) would lead to

$$\begin{aligned}
&\sum_{n=1}^{\infty} I_{n,2x}\varepsilon^{-n} + 3\left(\varepsilon + \sum_{n=1}^{\infty} I_n\varepsilon^{-n}\right)\left(q_{2x} + \sum_{n=1}^{\infty} I_{n,x}\varepsilon^{-n}\right) \\
&\quad + 3\varepsilon^2\sum_{n=1}^{\infty} I_n\varepsilon^{-n} + 3\varepsilon\left(\sum_{n=1}^{\infty} I_n\varepsilon^{-n}\right)^2 \\
&\quad + \left(\sum_{n=1}^{\infty} I_n\varepsilon^{-n}\right)^3 + c_1\sum_{n=1}^{\infty}\partial_x^{-1}I_{n,y}\varepsilon^{-n} = 0;
\end{aligned} \tag{59}$$

collecting the coefficients for the power of ε, we explicity obtain the recursion relations for the conserved densities I'_ns:

$$\begin{aligned}
I_1 &= -q_{2x}, \\
I_2 &= q_{3x}, \\
I_3 &= -\frac{1}{3}\left(2q_{4x} - c_1q_{x,y}\right), \\
I_4 &= \frac{1}{3}\left(q_{5x} - 2c_1q_{2x,y}\right), \\
&\vdots, \\
I_{n+1} &= -\frac{1}{3}\Bigg(I_{n-1,2x} + 3I_{n,x} + 3q_{2x}I_{n-1} \\
&\quad + 3\sum_{k=1}^{n-2} I_kI_{n-1-k,x} + 3\sum_{k=1}^{n-1} I_kI_{n-k} \\
&\quad + \sum_{i+j+k=n-1} I_iI_jI_k + c_1\partial_x^{-1}I_{n-1,y}\Bigg), \\
&\quad (n \geq 4).
\end{aligned} \tag{60}$$

Applying (58) to divergence-type equation (57) and comparing the power of ε provide us with an infinite sequence of conservation laws:

$$I_{n,t} + F_{n,x} + G_{n,y} = 0, \quad (n = 1, 2, \ldots), \tag{61}$$

where the first fluxes F'_ns are given explicitly by

$$\begin{aligned}
F_1 &= -q_{6x}a_1 + \frac{5}{2}c_1a_1q_{3x,y} - 15a_1q_{2x}^3 \\
&\quad - 15c_1a_1q_{2x}q_{x,y} + 5c_1^2a_1q_{2y} - 15a_1q_{2x}q_{4x}, \\
&\vdots, \\
F_n &= -\frac{3}{2}a_1\Bigg[I_{n,4x} + 5q_{4x}I_n + 5\sum_{k=1}^{n-1} I_{k,3x}I_{n-k} \\
&\quad + 5I_{n+1,3x} + 10q_{2x}I_{n,2x} \\
&\quad + 10\sum_{k=1}^{n-1} I_{k,x}I_{n-k,2x} + 10I_{n+2,2x} \\
&\quad + 20\sum_{k=1}^{n} I_kI_{n+1-k,2x} + 10\sum_{i+j+k=n} I_iI_jI_{k,2x} \\
&\quad + 15q_{2x}^2I_n + 30q_{2x}\left(\sum_{k=1}^{n-1} I_{k,x}I_{n-k} + I_{n+1,x}\right) \\
&\quad + 15\sum_{i+j+k=n} I_{i,x}I_{j,x}I_k + 15\sum_{k=1}^{n} I_{k,x}I_{n+1-k,x} \\
&\quad + 10q_{2x}\left(\sum_{i+j+k=n} I_iI_jI_k + 3\sum_{k=1}^{n} I_kI_{n+1-k} + 3I_{n+2}\right) \\
&\quad + 10\sum_{i+j+k+l=n} I_{i,x}I_jI_kI_l + 30\sum_{i+j+k=n+1} I_{i,x}I_jI_k \\
&\quad + 30\sum_{k=1}^{n+1} I_{k,x}I_{n+2-k} + 10I_{n+3,x}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i+j+k+l+m=n} I_i I_j I_k I_l I_m + 5 \sum_{i+j+k+l=n+1} I_i I_j I_k I_l \\
&\left. + 10 \sum_{i+j+k=n+2} I_i I_j I_k + 10 \sum_{k=1}^{n+2} I_k I_{n+3-k} + 5 I_{n+4} \right] \\
&- \frac{15}{2} a_1 \left(I_{n+3,x} + \sum_{k=1}^{n+2} I_k I_{n+3-k} + 2 I_{n+4} \right) \\
&+ \frac{15}{2} c_1 a_1 \left(q_{2x} \partial_x^{-1} I_{n,y} + \sum_{k=1}^{n-1} \partial_x^{-1} I_{k,y} I_{n-k,x} \right. \\
&+ 2 q_{x,y} I_n + 2 I_{n+1,y} + 2 \sum_{k=1}^{n-1} I_{n-k,y} I_k \\
&+ \sum_{i+j+k=n} I_i I_j \partial_x^{-1} I_{k,y} \\
&\left. + 2 \sum_{k=1}^{n} \partial_x^{-1} I_{k,y} I_{n+1-k} + \partial_x^{-1} I_{n+2,y} \right),
\end{aligned} \tag{62}
$$

and the second flues G_n's are

$$
\begin{aligned}
G_1 &= -\frac{15}{2} c_1 a_1 q_{4x}, \\
&\vdots, \\
G_n &= \frac{15}{2} c_1 a_1 I_{n,2x}, \quad n = 2, 3, \ldots.
\end{aligned} \tag{63}
$$

With the recursion formulae (60), (62), and (63) presented above, the infinite conservation laws for (2) can be constructed. In particular, the first conservation law is

$$
\begin{aligned}
&q_{2x,t} + a_1 q_{7x} + 15 a_1 q_{3x} q_{4x} + 15 a_1 q_{2x} q_{5x} \\
&+ 45 a_1 q_{2x}^2 q_{3x} - 5 c_1^2 a_1 q_{x,2y} \\
&+ 15 c_1 a_1 q_{3x} q_{x,y} + 15 c_1 a_1 q_{2x} q_{2x} q_{2x,y} = 0,
\end{aligned} \tag{64}
$$

or equivalently

$$
\begin{aligned}
&u_t + a_1 u_{5x} + \frac{15 a_1}{c_0} e^{\int a_9 dt} u_x u_{2x} \\
&+ \frac{15 a_1}{c_0} e^{\int a_9 dt} u u_{3x} + \frac{45 a_1}{c_0^2} e^{2 \int a_9 dt} u^2 u_x \\
&+ 5 c_1 a_1 u_{2x,y} - 5 c_1^2 a_1 \partial_x^{-1} u_{2y} \\
&+ \frac{15 c_1 a_1}{c_0} e^{\int a_9 dt} u_x \partial_x^{-1} u_y + \frac{15 c_1 a_1}{c_0} e^{\int a_9 dt} u u_y \\
&+ a_9 u = 0,
\end{aligned} \tag{65}
$$

which is exactly (2) under the constraint conditions (29).

6. Conclusion

In this paper, a (2 + 1)-dimensional variable-coefficient CDGKS equation has been investigated by the Bell polynomials approach. For case 1, the CDGKS equation is completely integrable in the sense that it admits bilinear BT, Lax pair, and infinite conservation laws which are derived in a direct and systematic way. By means of the bilinear equation, the N-soliton solutions for the variable-coefficient CDGKS equation are presented. Different parameters and functions are selected to obtain some soliton solutions and also analyze their graphics in Figures 1–3. However, for case 2, the variable-coefficient CDGKS equation under the constraint conditions (37) is not integrable since its multisoliton solutions cannot be obtained. In addition, the integrable constraint conditions on variable coefficients of the equation can be naturally found in the procedure of applying the Bell polynomials approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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