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Research Article

Shape-Preserving and Convergence Properties for the q-Szász-Mirakjan Operators for Fixed $q \in (0, 1)$

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We introduce a q-generalization of Szász-Mirakjan operators $S_{n,q}$ and discuss their properties for fixed $q \in (0, 1)$. We show that the q-Szász-Mirakjan operators $S_{n,q}$ have good shape-preserving properties. For example, $S_{n,q}$ are variation-diminishing, and preserve monotonicity, convexity, and concave modulus of continuity. For fixed $q \in (0, 1)$, we prove that the sequence $\{S_{n,q}(f)\}$ converges to $B_{\infty,q}(f)$ uniformly on $[0, 1]$ for each $f \in C[0, 1/(1-q)]$, where $B_{\infty,q}$ is the limit q-Bernstein operator. We obtain the estimates for the rate of convergence for $\{S_{n,q}(f)\}$ by the modulus of continuity of f, and the estimates are sharp in the sense of order for Lipschitz continuous functions.

1. Introduction

Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] := [k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases} \tag{1}$$
$$[k]! := \begin{cases} [k]\,[k-1]\cdots[1], & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!}. \tag{2}$$

We give the following two q-analogues of exponential function e^x:

$$e_q(x) := \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{((1-q)\,x; q)_\infty}, \quad |x| < \frac{1}{1-q} \quad \text{for } q < 1;$$
$$E_q(x) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{[k]!} = (-(1-q)\,x; q)_\infty, \quad x \in \mathbb{R} \text{ for } q < 1, \tag{3}$$

where $(x; q)_\infty := \prod_{k=1}^{\infty}(1 - xq^{k-1})$. Clearly, we have

$$e_q(x)\,E_q(-x) = 1, \qquad \lim_{q\to 1-} e_q(x) = \lim_{q\to 1-} E_q(x) = e^x. \tag{4}$$

In [1], Phillips proposed the q-Bernstein polynomials: for each positive integer n and $f \in C[0, 1]$, the q-Bernstein polynomial of f is

$$B_{n,q}(f)(x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x). \tag{5}$$

Note that for $q = 1$, $B_{n,q}(f)$ is the classical Bernstein polynomial. In [2], Il'inskiia and Ostrovska proved that, for each $f \in C[0, 1]$ and $q \in (0, 1)$, the sequence $\{B_{n,q}(f)(x)\}$

converges to $B_{\infty,q}(f)(x)$ as $n \to \infty$ uniformly on $x \in [0,1]$, where

$$B_{\infty,q}(f)(x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1-q^k\right) p_{\infty k}(q;x), & 0 \le x < 1, \\ f(1), & x = 1. \end{cases} \tag{6}$$

The operators $B_{\infty,q}$ are called the limit q-Bernstein operators. They also arise as the limit for a sequence of q-Meyer-König Zeller operators (see [3]). For results about properties of $B_{\infty,q}(f,x)$ we refer to [2, 4, 5].

In [6], Aral introduced the following q-Szász-Mirakjan operator: for each positive integer n and $f \in C[0,\infty)$,

$$S_{n,q}^{b}(f)(x) := E_q\left(-[n]\frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]\,b_n}{[n]}\right)\frac{([n]\,x)^k}{[k]!(b_n)^k}, \tag{7}$$

where $0 \le x < \alpha_q(n)$, $\alpha_q(n) := b_n/(1-q^n)$, and $b = \{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. In this paper, we introduce the following q-Szász-Mirakjan operator: for each positive integer n and $f \in C[0, 1/(1-q^n)]$,

$$S_{n,q}(f)(x) := \begin{cases} E_q(-[n]\,x)\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right)\frac{([n]\,x)^k}{[k]!}, & x \in \left[0, \frac{1}{1-q^n}\right), \\ f(1), & x = \frac{1}{1-q^n}. \end{cases} \tag{8}$$

Obviously, the operators $S_{n,q}$ are equal to the operators $S_{n,q}^{b}$ with $b = \{b_n\}$, $b_n = 1$. When $q = 1$, the q-Szász-Mirakjan operators $S_{n,q}$ reduce to the classical Szász-Mirakjan operators.

In recent years, generalizations of linear operators connected with q-Calculus have been investigated intensively. The pioneer work has been made by Lupas [7] and Phillips [1] who proposed generalizations of Bernstein polynomials based on the q-integers. There are also other important q-operators, for example, the two-parametric generalization of q-Bernstein polynomials [8], the q-Bernstein-Durrmeyer operator [9], q-Meyer-König Zeller operators [10], q-Bleimann, Butzer and Hahn operators [11], and q-Szász-Mirakjan operators [6, 12–15]. Among these generalizations, q-Bernstein polynomials proposed by Phillips attracted the most attention and were studied widely by a number of authors (see [1, 2, 5, 16–24]).

In this paper, we will discuss convergence and shape-preserving properties of the q-Szász-Mirakjan operators $S_{n,q}$ for fixed $q \in (0,1)$. We will show that the operators $S_{n,q}$ share good shape-preserving properties such as the variation-diminishing properties, and for each $f \in C[0, 1/(1-q)]$ the sequence $\{S_{n,q}(f)(x)\}$ converges to the function $B_{\infty,q}(f)(x)$ uniformly on $[0,1]$, where $B_{\infty,q}$ are the limit q-Bernstein operators defined by (6). We also investigate the rate of convergence of the q-Szász-Mirakjan operators $S_{n,q}$ for fixed $q \in (0,1)$. Our results demonstrate that in general convergence properties of the q-Szász-Mirakjan operators $S_{n,q}$ are essentially different from those for the classical Szász-Mirakjan operators; however, they are very similar to those for the q-Bernstein polynomials. Notice that different q-generalizations of Szász-Mirakjan operators were introduced and studied by Aral and Gupta [6, 12], by Radu [13], and by Mahmudov [14, 15]. However, our q-Szász-Mirakjan operators have better convergence properties than the other q-generalizations of Szász-Mirakjan operators for fixed $q \in (0,1)$.

The paper is organized as follows. In Section 2, we recall some properties of the q-Szász-Mirakjan operators $S_{n,q}$ and discuss their shape-preserving properties. In Section 3 we investigate the convergence of $S_{n,q}(f)$ for fixed $q \in (0,1)$ and obtain the rate of convergence of $S_{n,q}(f)$ by the modulus of continuity of f, and the estimates are sharp in the sense of order for Lipschitz continuous functions.

2. Shape-Preserving Properties of $S_{n,q}$ for $0 < q < 1$

In the sequel we always assume that $q \in (0,1)$. First we show that the q-Szász-Mirakjan operators $S_{n,q}$ are the positive linear operators on $C[0, 1/(1-q^n)]$. Clearly, it suffices to prove that, for $f \in C[0, 1/(1-q^n)]$,

$$\lim_{x\to(1/(1-q^n))-} S_{n,q}(f)(x) = f\left(\frac{1}{1-q^n}\right). \tag{9}$$

Indeed, for arbitrary $\varepsilon > 0$, there exist a constant $M > 0$ and a $\delta > 0$ such that $|f(x)| \le M$ for all $x \in [0, 1/(1-q^n)]$, and $|f(x) - f(1/(1-q^n))| \le \varepsilon$ for $x \in (1/(1-q^n) - \delta, 1/(1-q^n))$. We choose A to be the minimum positive integer greater than $\log_q((1-q^n)\delta$. Then, for any $k > A$,

$$\begin{gathered} \left|\frac{[k]}{[n]} - \frac{1}{1-q^n}\right| = \frac{q^k}{1-q^n} < \delta, \\ \left|f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right| \le \varepsilon. \end{gathered} \tag{10}$$

It follows from the Euler identity that

$$\begin{gathered} E_q(-[n]\,x)\sum_{k=0}^{\infty}\frac{([n]\,x)^k}{[k]!} = 1 \quad \text{for } x \in \left[0, \frac{1}{1-q^n}\right), \\ E_q(-[n]\,x) = \left(\left(1-q^n\right)x; q\right)_\infty \\ = \prod_{s=0}^{\infty}\left(1 - q^s\left(1-q^n\right)x\right) \longrightarrow 0+, \\ \text{as } x \longrightarrow \frac{1}{1-q^n} - . \end{gathered} \tag{11}$$

This implies that, for $x \in [0, 1/(1-q^n))$,

$$\begin{aligned}
&\left|S_{n,q}(f)(x) - f\left(\frac{1}{1-q^n}\right)\right| \\
&= \left|E_q(-[n]x)\sum_{k=0}^{\infty}\left(f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right)\frac{([n]x)^k}{[k]!}\right| \\
&\le E_q(-[n]x)\left(\sum_{k=0}^{A}\left|f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right|\frac{([n]x)^k}{[k]!}\right. \\
&\qquad\left.+\sum_{k=A+1}^{\infty}\left|f\left(\frac{[k]}{[n]}\right) - f\left(\frac{1}{1-q^n}\right)\right|\frac{([n]x)^k}{[k]!}\right) \\
&\le 2ME_q(-[n]x)\sum_{k=0}^{A}\frac{1}{(1-q)^k[k]!} \\
&\quad + \varepsilon E_q(-[n]x)\sum_{k=A+1}^{\infty}\frac{([n]x)^k}{[k]!} \\
&\le BE_q(-[n]x) + \varepsilon,
\end{aligned} \tag{12}$$

where $B := 2M\sum_{k=0}^{A}(1/(1-q)^k[k]!)$ is a constant independent of x and $E_q(-[n]x) \to 0$ as $x \to (1/(1-q^n))-$. This proves (9).

The q-Szász-Mirakjan operators $S_{n,q}$ possess the end-point interpolation property:

$$S_{n,q}(f)(0) = f(0), \qquad S_{n,q}(f)\left(\frac{1}{1-q^n}\right) = f\left(\frac{1}{1-q^n}\right), \quad n \in \mathbb{N}. \tag{13}$$

They leave invariant linear functions:

$$S_{n,q}(at+b)(x) = ax + b \tag{14}$$

and are degree-preserving on polynomials; that is, if T is a polynomial of degree m, then $S_{n,q}(T)$ is a polynomial of degree m (see [6, Lemma 1] or [25, Theorem 1]).

The following representation of the q-Szász-Mirakjan operators $S_{n,q}$, called the q-difference form, was obtained in [6, Corollary 4]:

$$S_{n,q}(f)(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} f\left(\left[\frac{[0]}{[n]}; \frac{[1]}{[n]}; \dots; \frac{[k]}{[n]}\right]\right)x^k, \tag{15}$$

where $f([x_0; x_1; \dots; x_k])$ denotes the usual divided difference; that is,

$$\begin{aligned}
&f([x_0]) = f(x_0); \qquad f([x_0; x_1]) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \dots, \\
&f([x_0; x_1; \dots; x_k]) = \frac{f([x_1; \dots; x_k]) - f([x_0; \dots; x_{k-1}])}{x_k - x_0}.
\end{aligned} \tag{16}$$

Aral and Gupta discussed the shape-preserving properties of the q-Szász-Mirakjan operators in [12, Corollary 3.2]. We say a function f on an interval I is *i-convex*, $i \ge 1$, if $f \in C(I)$ and all ith forward differences

$$\begin{aligned}
&\Delta_h^i f(t) := \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}f(t+kh), \\
&0 \le h \le \frac{1}{i}, \quad t, t+kh \in I
\end{aligned} \tag{17}$$

are nonnegative. Obviously, a 1-*convex* function is nondecreasing and a 2-*convex* function is convex. Aral and Gupta obtained that, for an i-convex function on $[0,\infty)$, there exists $\hat{q} \in (0,1)$ such that $S_{n,q}(f)$ is also i-convex on $[0, 1/(1-q^n))$ for $q \in (\hat{q}, 1)$.

In this section we also study the shape-preserving properties of the operators $S_{n,q}$. We use a completely different method from the one in [12], and our results hold for all $q \in (0,1)$. In order to state the results, we introduce some notations.

For any real sequence a, finite or infinite, we denote by $S^-(a)$ the number of strict sign changes in a. For $f \in C(I)$, where I is an interval, we define $S^-(f)$ to be the number of sign changes of f; that is,

$$S^-(f) = \sup S^-(f(x_0), \dots, f(x_m)), \tag{18}$$

where the supremum is taken over all increasing sequences $x_0 < \cdots < x_m$ and $x_0, x_m \in I$ for all positive integers m.

Let L be a positive linear operator on $C(I)$. We say that L is variation-diminishing if, for all functions $f \in C(I)$, we have

$$S^-(L_n f) \le S^-(f). \tag{19}$$

A function $\omega(t)$ on $[0, A]$, $A > 0$ is called a modulus of continuity if $\omega(t)$ is continuous, nondecreasing, and semiadditive and $\omega(0) = 0$. We denote by H^ω the class of continuous functions f on $[0, A]$ satisfying the inequality $\omega(f,t) \le \omega(t)$, where $\omega(f,t) = \max_{|x_1-x_2|\le t}|f(x_2) - f(x_1)|$ is the modulus of continuity of $f(x)$. Note that if $f(x)$ is a concave modulus of continuity, then $x^{-1}f(x)$ is nonincreasing on $(0, A]$. Also, if $f(x)$ is a nondecreasing function such that $f(0) = 0$ and $x^{-1}f(x)$ is nonincreasing on $(0, A]$, then $f(x)$ is a modulus of continuity.

Our main results of this section can be formulated as follows.

Theorem 1. *(i) The operators $S_{n,q}$ are variation-diminishing on $[0, 1/(1-q^n)]$.*

(ii) If a function f is i-convex on $[0, 1/(1-q^n)]$, then the functions $S_{n,q}(f)$ are also i-convex on $[0, 1/(1-q^n)]$. Specially, if a function f is nondecreasing (nonincreasing) on $[0, 1/(1-q^n)]$, then $S_{n,q}(f)$ are also nondecreasing (nonincreasing) on $[0, 1/(1-q^n)]$ and if f is convex (concave) on $[0, 1/(1-q^n)]$, then so are $S_{n,q}(f)$.

(iii) If a function f is convex on $[0, 1/(1-q^n)]$, then $S_{n,q}(f)(x) \ge f(x)$, $x \in [0, 1/(1-q^n)]$.

(iv) If $\omega(t)$ is a modulus of continuity, then $f \in H^{\omega}$ implies that, for each $n \geq 1$, $S_{n,q}(f) \in H^{2\omega}$; if $\omega(t)$ is concave, then, for each $n \geq 1$, $S_{n,q}(f) \in H^{\omega}$.

(v) If $\omega(t)$ is a concave modulus of continuity, then, for each $n \geq 1$, $S_{n,q}(\omega)$ is also a concave modulus of continuity and $S_{n,q}(\omega)(t) \leq \omega(t)$.

(vi) If $f(x)$ is a nonnegative function such that $x^{-1}f(x)$ is nonincreasing on $(0, 1/(1-q^n))$], then, for each $n \geq 1$, $x^{-1}S_{n,q}(f)(x)$ is nonincreasing also.

Proof. (i) Let I be an interval, $I \subset [0,\infty)$. We assume that, for a real sequence $a = \{a_k\}_{k=0}^{\infty}$, the power series $\sum_{k=0}^{\infty} a_k x^k$ converges to the function g on I. By means of the well-known Descartes' rule of sign it is easy to prove that

$$S^{-}(g) = S^{-}\left(\sum_{k=0}^{\infty} a_k x^k\right) \leq S^{-}(a). \tag{20}$$

Obviously, if $h(x) > 0$ for any $x \in I$ and $b_k > 0$ for $k \geq 0$, then

$$S^{-}(f) = S^{-}(f \cdot h), \qquad S^{-}\left(\{a_k b_k\}_{k=0}^{\infty}\right) = S^{-}\left(\{a_k\}_{k=0}^{\infty}\right). \tag{21}$$

It follows that

$$\begin{aligned} S^{-}\left(S_{n,q}(f)\right) &= S^{-}\left(\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right)\frac{[n]^k}{[k]!}x^k\right) \\ &\leq S^{-}\left(\left\{f\left(\frac{[k]}{[n]}\right)\right\}_{k=0}^{\infty}\right) \leq S^{-}(f), \end{aligned} \tag{22}$$

which implies that $S_{n,q}$ are variation-diminishing.

(ii) The operators $S_{n,q}$ possess the end-point interpolation property and are degree-preserving on polynomials and variation-diminishing. Then, (ii) follows from [26, Lemma 15].

(iii) It follows from [27, p. 281] that if a positive operator L on $C[0, A]$ reproduces linear functions, then $L(f, x) \geq f(x)$ for any convex function f and for any $x \in [0, A]$. Since $S_{n,q}$ are the positive linear operators and reproduce linear functions, we obtain (iii).

(iv) From [26, Corollary 8], we know that if a positive linear operator L on $C[0, A]$ $(A > 0)$ is variation-diminishing and reproduces linear functions, then, for all $f \in C[0, A]$ and $t \in (0, A]$,

$$\omega(Lf, t) \leq \tilde{\omega}(f, t). \tag{23}$$

Thus, if $f \in H^{\omega}$, then

$$\omega\left(S_{n,q}(f), t\right) \leq \tilde{\omega}(f, t) \leq \tilde{\omega}(t), \tag{24}$$

where $\tilde{\omega}(t)$ and $\tilde{\omega}(f, t)$ denote the least concave majorant of $\omega(t)$ and $\omega(f, t)$, respectively. It is well known that for each modulus of continuity ω there exists a concave modulus of continuity $\tilde{\omega}$ such that $\omega(t) \leq \tilde{\omega}(t) \leq 2\omega(t)$ for $t \in [0, A]$. Thence, $S_{n,q}(f) \in H^{2\omega}$ and furthermore $S_{n,q}(f) \in H^{\omega}$ if ω is concave, which means (iv) holds.

(v) From (i) we know that, for a concave modulus of continuity ω and each $n \geq 1$, the function $S_{n,q}(\omega)$ is nondecreasing and concave on $(0, A]$, where $A = 1/(1-q^n)$. We also have $S_{n,q}(\omega)(0) = 0$. This means that $S_{n,q}(\omega)$ is a concave modulus of continuity. The inequality $S_{n,q}(\omega)(t) \leq \omega(t)$ follows directly from (iii).

(vi) Since, for any constant c,

$$\begin{aligned} &S^{-}\left(\frac{S_{n,q}(f)(x)}{x} - c\right) \\ &\quad = S^{-}\left(S_{n,q}(f)(x) - cx\right) = S^{-}\left(S_{n,q}(f(t) - ct)(x)\right) \\ &\quad \leq S^{-}(f(x) - cx) = S^{-}\left(\frac{f(x)}{x} - c\right) \leq 1 \end{aligned} \tag{25}$$

we get that $S_{n,q}(f)(x)/x$ is nondecreasing or nonincreasing on $(0, A]$, where $A = 1/(1-q^n)$. For any $t \in (0, A)$, $f(t)/t \geq f(A)/A$, we have $f(t) \geq f(A)t/A$, and thus $S_{n,q}(f)(x) \geq S_{n,q}(f(A)t/A)(x) = f(A)x/A$. Hence,

$$\frac{S_{n,q}(f)(x)}{x} \geq \frac{f(A)}{A} = \frac{S_{n,q}(f)(A)}{A}, \tag{26}$$

which implies that $S_{n,q}(f)(x)/x$ is nonincreasing on $[0, A]$.

Theorem 1 is proved. □

3. The Rate of Convergence for the q-Szász-Mirakjan Operators $S_{n,q}$ for Fixed $q \in (0, 1)$

The approximation properties of the sequence $\{S_{n,q_n}^{b}(f)\}$ in weighted spaces as $\lim_{n\to\infty} q_n = 1-$ were investigated in [6, Theorem 2] and [25, Theorem 6]. The obtained results are similar to the ones of the classical Szász-Mirakjan operators. However, there are few results about convergence properties of $S_{n,q}$ for fixed $q \in (0, 1)$. This section is devoted to discussing the convergence properties of the q-Szász-Mirakjan operators $S_{n,q}$ for fixed $q \in (0, 1)$.

We set

$$s_{n,k}(q; x) = E_q(-[n]x)\frac{([n]x)^k}{[k]!} = \frac{([n]x)^k}{[k]!}((1-q^n)x; q)_{\infty}, \tag{27}$$

$$p_{\infty,k}(q; x) = \frac{x^k}{(1-q)^k [k]!}(x; q)_{\infty}. \tag{28}$$

Formerly, for $f \in C[0, 1/(1-q)]$ and each $k \geq 0$, $\{f([k]/[n])\}$ converges to $f(1-q^k)$, $\{s_{n,k}(q; x)\}$ converges to $p_{\infty,k}(q; x)$, and

$$\begin{aligned} S_{n,q}(f)(x) &= \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) s_{n,k}(q; x) \\ &\longrightarrow \sum_{k=0}^{\infty} f\left(1-q^k\right) p_{\infty,k}(q; x) \\ &= B_{\infty,q}(f)(x), \end{aligned} \tag{29}$$

as $n \to \infty$. Indeed, the above conclusion holds. We have the following stronger results.

Theorem 2. *Let $f \in C[0, 1/(1-q)]$. Then, we have*

$$\sup_{x\in[0,1]} \left|S_{n,q}(f)(x) - B_{\infty,q}(f)(x)\right| \le C_q \omega(f, q^n), \tag{30}$$

where $C_q = 4 + q/(1-q) + (1/(1-q))e^{q^2/(1-q)^2}$. This estimate is sharp in the following sense of order: for each α, $0 < \alpha \le 1$, there exists a function $f_\alpha(x)$ which belongs to the Lipschitz class $\mathrm{Lip}\,\alpha := \{f \in C[0,1] \mid \omega(f; t) \le t^\alpha\}$ *such that*

$$\sup_{x\in[0,1]} \left|S_{n,q}(f_\alpha)(x) - B_{\infty,q}(f_\alpha)(x)\right| \ge Cq^{n\alpha}, \tag{31}$$

where C is a positive constant independent of n.

Remark 3. It follows from (30) that, for $f \in C[0, 1/(1-q)]$, $\lim_{n\to\infty} S_{n,q}(f)(x) = B_{\infty,q}(f)(x)$ uniformly on $x \in [0,1]$ as $n \to \infty$. Since $B_{\infty,q}(f)(x) = f(x)$, $x \in [0,1]$, if and only if f is linear on $[0,1]$ (see [2, Theorem 6]), we get that the sequence $S_{n,q}(f)(x)$ converges to f uniformly on $[0,1]$ if and only if f is linear on $[0,1]$.

Remark 4. It should be emphasized that the proof of Theorem 2 requires estimation techniques involving the infinite product. Also, it is a little more difficult than the one used for q-Bernstein polynomials (see [23]), since $S_{n,q}(f)(1) \ne f(1) = B_{\infty,q}(f)(1)$.

Proof. Since the operators $S_{n,q}$ and $B_{\infty,q}$ reproduce linear functions, we get that, for $x \in [0,1)$,

$$\sum_{k=0}^{\infty} s_{n,k}(q;x) = 1, \qquad \sum_{k=0}^{\infty} \frac{[k]}{[n]} s_{n,k}(q;x) = x, \qquad x \in \left[0, \frac{1}{1-q^n}\right), \tag{32}$$

$$\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1, \qquad \sum_{k=0}^{\infty} \left(1-q^k\right) p_{\infty,k}(q;x) = x, \qquad x \in [0,1), \tag{33}$$

where $s_{n,k}(q;x)$ and $p_{\infty,k}(q;x)$ are defined by (27) and (28), respectively. By means of (32) and (33), direct calculations give that

$$\sum_{k=0}^{n} q^k s_{n,k}(q;x) = 1 - x + q^n x, \qquad \sum_{k=0}^{\infty} q^k p_{\infty,k}(q;x) = 1 - x. \tag{34}$$

For $x = 0$, we have

$$\left|S_{n,q}(f)(0) - B_{\infty,q}(f)(0)\right| = |f(0) - f(0)| = 0. \tag{35}$$

For $x = 1$, it follows that

$$\begin{aligned}
&\left|S_{n,q}(f)(1) - B_{\infty,q}(f)(1)\right| \\
&= \left|\sum_{k=0}^{\infty} \left(f\left(\frac{[k]}{[n]}\right) - f(1)\right) s_{n,k}(q;1)\right| \\
&\le \sum_{k=0}^{\infty} \left(\left|f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right)\right| \right. \\
&\qquad \left. + \left|f(1) - f\left(1-q^k\right)\right|\right) s_{n,k}(q;1) \\
&\le \sum_{k=0}^{\infty} \left(\omega\left(f, \frac{[k]q^n}{[n]}\right) + \omega\left(f, q^k\right)\right) s_{n,k}(q;1) \\
&\le \omega(f, q^n) \sum_{k=0}^{\infty} \left(2 + \frac{[k]}{[n]} + \frac{q^k}{q^n}\right) s_{n,k}(q;1) \\
&= 4\omega(f, q^n),
\end{aligned} \tag{36}$$

where in the first equality we used (32); in the last inequality we used the inequality $\omega(f, \lambda t) \le (1+\lambda)\omega(f,t)$ for any $\lambda, t > 0$; in the last equality we used (32) and (34).

Now for $x \in (0,1)$, by (32) and (33), we have

$$\begin{aligned}
&\left|S_{n,q}(f,x) - B_{\infty,q}(f,x)\right| \\
&= \left|\sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) s_{n,k}(q;x) - \sum_{k=0}^{\infty} f\left(1-q^k\right) p_{\infty,k}(q;x)\right| \\
&= \left|\sum_{k=0}^{\infty} \left(f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right)\right) s_{n,k}(q;x) \right. \\
&\qquad \left. + \sum_{k=0}^{\infty} \left(f\left(1-q^k\right) - f(1)\right)\left(s_{n,k}(q;x) - p_{\infty,k}(q;x)\right)\right| \\
&\le \sum_{k=0}^{\infty} \left|f\left(\frac{[k]}{[n]}\right) - f\left(1-q^k\right)\right| s_{n,k}(q;x) \\
&\quad + \sum_{k=0}^{\infty} \left|f\left(1-q^k\right) - f(1)\right| \left|s_{n,k}(q;x) - p_{\infty,k}(q;x)\right| \\
&=: J_1 + J_2.
\end{aligned} \tag{37}$$

Since

$$\left|\frac{[k]}{[n]} - \left(1-q^k\right)\right| = \frac{[k]q^n}{[n]}, \qquad \omega(f; \lambda t) \le (1+\lambda)\,\omega(f;t), \quad \lambda, t > 0, \tag{38}$$

we get by (32)

$$\begin{aligned} J_1 &\le \sum_{k=0}^{\infty} \omega\left(f, \frac{[k]\, q^n}{[n]}\right) s_{n,k}(q;x) \\ &\le \omega(f, q^n) \sum_{k=0}^{\infty} \left(1 + \frac{[k]}{[n]}\right) s_{n,k}(q;x) \\ &= (1+x)\,\omega(f,q^n) \le 2\omega(f,q^n). \end{aligned} \tag{39}$$

In order to estimate J_2, we need to estimate $|s_{n,k}(q;x) - p_{\infty,k}(q;x)|$. We have

$$\begin{aligned} &\left|s_{n,k}(q;x) - p_{\infty,k}(q;x)\right| \\ &\quad= \left|\frac{([n]\,x)^k}{[k]!}\prod_{s=0}^{\infty}\left(1-(1-q^n)\,q^s x\right) - \frac{x^k}{(1-q)^k\,[k]!}\prod_{s=0}^{\infty}(1-q^s x)\right| \\ &\quad\le \frac{x^k}{(1-q)^k\,[k]!}\left|\prod_{s=0}^{\infty}\left(1-q^s(1-q^n)\,x\right)(1-q^n)^k - \prod_{s=0}^{\infty}(1-q^s x)(1-q^n)^k\right| \\ &\qquad+ \frac{x^k}{(1-q)^k\,[k]!}\prod_{s=0}^{\infty}(1-q^s x)\left|1-(1-q^n)^k\right| \\ &\quad\le p_{\infty,k}(q;x)\left(\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| + \left|1-(1-q^n)^k\right|\right). \end{aligned} \tag{40}$$

We note that

$$\begin{aligned} q^k\left(1-(1-q^n)^k\right) &= q^{k+n}\left(1+(1-q^n)+\cdots+(1-q^n)^{k-1}\right) \\ &\le k q^{k+n} \le \frac{q^{n+1}}{1-q}. \end{aligned} \tag{41}$$

It follows that

$$\begin{aligned} J_2 &\le \sum_{k=0}^{\infty}\omega\left(f,q^k\right)\left|s_{n,k}(q;x) - p_{\infty,k}(q;x)\right| \\ &\le \omega(f,q^n)\sum_{k=0}^{\infty}\left(1+\frac{q^k}{q^n}\right)\left|s_{n,k}(q;x) - p_{\infty,k}(q;x)\right| \\ &\le \omega(f,q^n)\left(\sum_{k=0}^{\infty}\left(s_{n,k}(q;x)+p_{\infty,k}(q;x)\right) + \sum_{k=0}^{\infty}\frac{q^k}{q^n}\left|s_{n,k}(q;x)-p_{\infty,k}(q;x)\right|\right) \\ &\le \omega(f,q^n)\left(2+\sum_{k=0}^{\infty}\frac{q^k}{q^n}p_{\infty,k}(q;x) \times\left(\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| + \left|1-(1-q^n)^k\right|\right)\right) \\ &\le \omega(f,q^n)\left(2+\sum_{k=0}^{\infty}p_{\infty,k}(q;x) \times\left(q^{k-n}\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| + \frac{q}{1-q}\right)\right) \\ &\le \omega(f,q^n)\left(2+q^{-n}(1-x) \times\left|\prod_{s=0}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| + \frac{q}{1-q}\right) \\ &=: \omega(f,q^n)\left(2+H+\frac{q}{1-q}\right), \end{aligned} \tag{42}$$

where in the fourth inequality we used (32) and (33); in the last inequality we used (34) and (33). We estimate H. We have

$$\begin{aligned} H &= q^{-n}\left|(1-x+q^n x)\prod_{s=1}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right) - (1-x)\right| \\ &= x\prod_{s=1}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right) + q^{-n}(1-x) \times\left|\prod_{s=1}^{\infty}\left(1+\frac{q^{s+n}x}{1-q^s x}\right)-1\right| \\ &=: x e^K + q^{-n}(1-x)\left(e^K - 1\right), \end{aligned} \tag{43}$$

where $K := \sum_{s=1}^{\infty}\ln(1+q^{s+n}x/(1-q^s x))$. Using the inequality $\ln(1+t) \le t$, $t \ge 0$, we get that

$$\begin{aligned} K &\le \sum_{s=1}^{\infty}\frac{q^{s+n}x}{1-q^s x} \le \sum_{s=1}^{\infty}\frac{q^{s+n}}{1-qx} \le \frac{q^{n+1}}{(1-q)(1-qx)} \\ &\le \frac{q^2}{(1-q)^2}. \end{aligned} \tag{44}$$

It follows that

$$\begin{gathered} e^K \le e^{q^2/(1-q)^2}, \\ e^K - 1 = K e^{\xi} \le K e^K \le \frac{q^{n+1}}{(1-q)(1-qx)} e^{q^2/(1-q)^2}, \\ \xi \in [0, K]. \end{gathered} \tag{45}$$

This deduces that, for $x \in (0,1)$,

$$H \le e^{q^2/(1-q)^2} + (1-x)\frac{q}{(1-q)(1-qx)}e^{q^2/(1-q)^2} \le \frac{1}{1-q}e^{q^2/(1-q)^2}, \tag{46}$$

and thence

$$J_2 \le \omega(f, q^n)\left(2 + \frac{q}{1-q} + \frac{1}{1-q}e^{q^2/(1-q)^2}\right). \tag{47}$$

We conclude from (39) and (47) that, for $x \in (0,1)$,

$$\begin{aligned}\left|S_{n,q}(f,x) - B_{\infty,q}(f,x)\right| &\le J_1 + J_2 \\ &\le \omega(f, q^n)\left(4 + \frac{q}{1-q} + \frac{1}{1-q}e^{q^2/(1-q)^2}\right).\end{aligned} \tag{48}$$

Hence, (30) follows from (35), (36), and (48).

At last we show that the estimate (30) is sharp. For each α, $0 < \alpha \le 1$, suppose that $f_\alpha^*(x)$ is a continuous function, which is equal to zero in $[0, 1-q]$ and $[1-q^2, 1]$, equal to $(x-(1-q))^\alpha$ in $[1-q, 1-q+q(1-q)/2]$, and linear in the rest of $[0,1]$. It is easy to see that $\omega(f_\alpha^*, t) \le At^\alpha$. We set $f_\alpha(t) = (1/A)f_\alpha^*(t)$. Then, $f_\alpha \in \operatorname{Lip}\alpha$, and for sufficiently large n, we have

$$\begin{aligned}&\sup_{x\in[0,1]}\left|S_{n,q}(f_\alpha)(x) - B_{\infty,q}(f_\alpha)(x)\right| \\ &\quad = \frac{1}{A}\frac{(1-q)^\alpha q^{n\alpha}}{(1-q^n)^\alpha}\sup_{x\in[0,1]}\left|s_{n,1}(q;x)\right| \\ &\quad \ge \frac{(1-q)^\alpha q^{n\alpha}}{A}\left|s_{n,1}\left(q;\frac{1}{2}\right)\right| \\ &\quad \ge \frac{(1-q)^\alpha}{2A(1-q)}\prod_{s=0}^{\infty}\left(1-\frac{q^s}{2}\right)q^{n\alpha} =: Cq^{n\alpha}.\end{aligned} \tag{49}$$

The proof of Theorem 2 is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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