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Research Article

A Characterization of Completeness via Absolutely Convergent Series and the Weierstrass Test in Asymmetric Normed Semilinear Spaces

N. Shahzad[1] and O. Valero[2]

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia*
[2] *Departamento de Ciencias Matemáticas e Informática, Universidad de las Islas Baleares, Carretera de Valldemossa km. 7.5, 07122 Palma de Mallorca, Spain*

Correspondence should be addressed to N. Shahzad; nshahzad@kau.edu.sa

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Asymmetric normed semilinear spaces are studied. A description of biBanach, left K-sequentially complete, and Smyth complete asymmetric normed semilinear spaces is provided and three appropriate notions of absolute convergence in the asymmetric normed framework are introduced. Some characterizations of completeness are also obtained via absolutely convergent series. Moreover, as an application, a Weierstrass test for the convergence of series is derived.

1. Introduction

Asymmetric normed linear spaces were applied to solve extremal problems arising in a natural way in mathematical programming first by Duffin and Karlovitz in 1968 [1] and then by Krein and Nudelman in 1977 [2]. Since then, the interest in such asymmetric structures has been growing. Asymmetric functional analysis has become a research branch of analysis nowadays. In 2013, Cobzas [3] published a monograph entitled "Functional Analysis in Asymmetric Normed Spaces," which collects a large number of results in the aforesaid research line. On account of [3], a systematic and deep study of asymmetric normed linear spaces and other related structures such as asymmetric normed semilinear spaces has been made by Romaguera and some of his collaborators. Many of the aforesaid results can be found in [4–20].

Inspired by the intense research activity in the field under consideration, our purpose is to study some properties of asymmetric normed semilinear spaces. The main goal of this paper is to delve into the relationship between completeness of asymmetric normed semilinear spaces and the absolute convergence of series in such a way that the classical context of Banach normed linear spaces can be recovered as a particular case. In particular, we introduce three absolute convergence notions which are appropriate to describe biBanach asymmetric normed semilinear spaces, left K-sequentially complete asymmetric normed semilinear spaces, and Smyth complete asymmetric normed semilinear spaces. Moreover, as an application of the developed theory, we derive a criterion, inspired by the celebrated Weierstrass M-test, for the convergence of series of asymmetric normed semilinear space valued bounded mappings.

2. Preliminaries

Throughout, we will denote the set of real numbers, the set of nonnegative real numbers, and the set of positive integer numbers by $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$, respectively.

According to [3], a semilinear space (or cone) Y is a subset of a linear space X such that $x + y \in Y$ and $\lambda \cdot x \in Y$ for all $x, y \in Y$ and $\lambda \in \mathbb{R}^+$. Clearly, every linear space can be considered also as a semilinear space.

Following [3], an asymmetric norm on a linear space X is a function $\|\cdot\| : X \to \mathbb{R}^+$ satisfying the following conditions for all $x, y \in X$ and $\lambda \in \mathbb{R}^+$:

(i) $\|x\| = \|-x\| = 0 \Leftrightarrow x = 0$;

(ii) $\|\lambda \cdot x\| = \lambda\|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

A pair $(X, \|\cdot\|)$ is called an asymmetric normed linear space provided that X is a linear space and $\|\cdot\|$ is an asymmetric norm on X. Observe that if $(X, \|\cdot\|)$ is an asymmetric normed linear space, then the function $\|\cdot\|^s$ defined on X by $\|x\|^s(x) = \max\{\|x\|, \|-x\|\}$ for all $x \in X$ is a norm on X.

On account of [11], an asymmetric normed semilinear space (also called normed cone in [3]) is a pair $(Y, \|\cdot\|)$ where Y is a semilinear space of an asymmetric normed linear space $(X, \|\cdot\|)$. The restriction of $\|\cdot\|$ to Y is also denoted by $\|\cdot\|$.

In our context, by a quasimetric space, we mean a pair (X, d) such that X is a nonempty set and d is a function $d : X \to \mathbb{R}^+$ satisfying the following conditions for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Furthermore, every quasimetric d allows defining a metric d^s on X by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$.

It is well known that, given a quasimetric space (X, d), a topology $\mathcal{T}(d)$ can be induced on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$. We will say that a subset A of X is d-closed whenever A is closed with respect to the topology $\mathcal{T}(d)$. Moreover, if $(x_n)_{n\in\mathbb{N}}$ is a sequence which converges to a point $x \in X$ with respect to the topology $\mathcal{T}(d)$, then we will say that $(x_n)_{n\in\mathbb{N}}$ d-converges to x. Furthermore, a quasimetric space (X, d) is said to be bicomplete provided that the associated metric space (X, d^s) is complete. For a fuller treatment of quasimetric spaces, we refer the reader to [21].

Each asymmetric norm $\|\cdot\|$ on a linear space X induces a quasimetric $d_{\|\cdot\|}$ on X which is defined by $d_{\|\cdot\|}(x, y) = \|y - x\|$ for all $x, y \in X$. According to [11] (see also [3]), an asymmetric normed linear space $(X, \|\cdot\|)$ is biBanach whenever the induced quasimetric space $(X, d_{\|\cdot\|})$ is bicomplete. In addition, if Y is a semilinear space of a linear space X, then $(Y, \|\cdot\|)$ is called a biBanach asymmetric normed semilinear space provided that the quasimetric space $(Y, d_{\|\cdot\|})$ is bicomplete where the restriction of $d_{\|\cdot\|}$ to Y is also denoted by $d_{\|\cdot\|}$.

Notice that, given an asymmetric normed semilinear space $(X, \|\cdot\|)$, $d^s_{\|\cdot\|}(x, y) = d_{\|\cdot\|^s}(x, y)$ for all $x, y \in X$.

In what follows, we will work with asymmetric normed semilinear spaces $(Y, \|\cdot\|)$ in such a way that Y is assumed to be a $d_{\|\cdot\|}$-closed semilinear space of an asymmetric normed linear space $(X, \|\cdot\|)$ (where again the restriction of $\|\cdot\|$ to Y is denoted by $\|\cdot\|$). Of course, in order to work with asymmetric normed linear spaces, we only need to take $Y = X$.

3. The Absolute Convergence of Series in Asymmetric Normed Linear Spaces

3.1. The Notion of Absolute Convergence of Series. Let us recall that a series in a normed linear space $(X, \|\cdot\|)$ is a sequence of the form $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ where $(x_n)_{n\in\mathbb{N}}$ is a sequence in X. However, it is customary to denote the sequence $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ by $\sum_{k=1}^{\infty} x_k$. Of course, when $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ is convergent, we will say that the series $\sum_{k=1}^{\infty} x_k$ is convergent and its limit will be also denoted by $\sum_{k=1}^{\infty} x_k$ [22].

In the classical framework of normed linear spaces, the notion of absolutely convergent series is stated as follows.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a normed linear space $(X, \|\cdot\|)$, then the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent provided that the series of nonnegative real numbers $\sum_{k=1}^{\infty} \|x_k\|$ is convergent.

The preceding notion, among other things, allows characterizing completeness of normed linear spaces [23].

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then, the following assertions are equivalent.*

(1) $(X, \|\cdot\|)$ *is Banach.*

(2) *Every absolutely convergent series is $d_{\|\cdot\|}$-convergent.*

It is clear that the notion of series can be extended to the framework of asymmetric normed semilinear spaces in the following obvious way [3].

A series in an asymmetric normed semilinear space $(X, \|\cdot\|)$ is a sequence of the form $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ where $(x_n)_{n\in\mathbb{N}}$ is a sequence in X. Following the notation used for series in normed linear spaces, a series $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ in an asymmetric normed semilinear space will be also denoted by $\sum_{k=1}^{\infty} x_k$. Of course, when $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ is $d_{\|\cdot\|}$-convergent, we will say that $\sum_{k=1}^{\infty} x_k$ is $d_{\|\cdot\|}$-convergent and its limit will be also denoted by $\sum_{k=1}^{\infty} x_k$.

A natural attempt to extend the notion of absolute convergence to the asymmetric normed semilinear spaces would be as follows.

Definition 2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in asymmetric normed semilinear space $(X, \|\cdot\|)$. Then, we will say that the series $\sum_{k=1}^{n} x_k$ is absolutely convergent provided that the series $\sum_{k=1}^{\infty} \|x_k\|$ is convergent. Moreover, $(X, \|\cdot\|)$ will be said to have the absolute convergence property provided that every absolutely convergent series is $d_{\|\cdot\|}$-convergent.

We must point out that the absolute convergence was introduced by Cobzas in [3] in the framework of asymmetric normed linear spaces (in fact in the context of asymmetric seminormed linear spaces). So, in Definition 2, we extend the absolute convergence to the case of asymmetric normed semilinear spaces.

Clearly, the notion of absolutely convergent series in normed linear spaces is retrieved as a particular case of the preceding one whenever the asymmetric norm in Definition 2 is exactly a norm.

The example below shows that there exists an asymmetric normed semilinear space without the absolute convergence property.

Example 3. Consider the asymmetric normed linear space $(l_0, \|\cdot\|^+_{\sup})$ where l_0 denotes the space of all real sequences with only a finite number of nonzero terms, $\|\cdot\|^+_{\sup}$ is the asymmetric norm defined by

$$\|x\|^+_{\sup} = \sup_{n\in\mathbb{N}} \|x_n\|_{\max} \quad (1)$$

for all $x \in l_0$, and $\|\cdot\|_{\max} : \mathbb{R} \to \mathbb{R}^+$ is the asymmetric norm defined by $\|x\|_{\max} = \max\{x, 0\}$ for all $x \in \mathbb{R}$. Now, take the sequence $(x_n)_{n\in\mathbb{N}}$ in l_0 given by

$$x_n = \left(0, 0, \ldots, 0, \overbrace{\frac{1}{2^n}}^{n}, 0, 0, \ldots \right) \quad (2)$$

for all $n \in \mathbb{N}$. It is clear that the series $\sum_{k=1}^{\infty} \|x_k\|^+_{\sup}$ converges. However, the series $\sum_{k=1}^{\infty} x_k$ does not $d_{\|\cdot\|^+_{\sup}}$-converge.

In Example 5, we will show that there are asymmetric semilinear spaces with the absolute convergence property.

Now, it seems natural to wonder whether the characterization provided by Theorem 1 remains valid when we replace in its statement "Banach normed linear space" with "biBanach asymmetric normed semilinear space." Thus, the desired result could be stated as follows.

"Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space. Then, the following assertions are equivalent:

(1) $(X, \|\cdot\|)$ is biBanach;

(2) $(X, \|\cdot\|)$ has the absolute convergence property."

Nevertheless, Example 21 in Section 4 shows that such a result is not true. Accordingly, we can conclude that the fact that the asymmetric normed semilinear space is biBanach does not guarantee that the absolute convergence property holds. Since biBanach completeness implies that the metric $d_{\|\cdot\|^s}$ is complete, it seems obvious that the characterization of biBanach completeness must require an additional property more restrictive than the absolute convergence property which involves $d_{\|\cdot\|^s}$ and, thus, in some sense, the norm $\|\cdot\|^s$. Inspired by this fact, we propose the following notion which also retrieves as a particular case the classical one.

Definition 4. An asymmetric normed semilinear space $(X, \|\cdot\|)$ will be said to have the strong absolute convergence property provided that every absolutely convergent series is $d_{\|\cdot\|^s}$-convergent.

Clearly asymmetric normed semilinear spaces that hold the strong absolute convergence property form a subclass of those satisfying the absolute convergence property.

The next example shows that there are asymmetric normed semilinear spaces which do not have the strong absolute convergence property.

Example 5. Consider the asymmetric normed linear space $(\mathbb{R}, \|\cdot\|_{\max})$ where $\|\cdot\|_{\max}$ is the asymmetric norm introduced in Example 3. Then, the metric $d_{\|\cdot\|^s_{\max}}$ is exactly the Euclidean metric on $\mathbb{R}$. It follows that the quasimetric space $(\mathbb{R}, d_{\|\cdot\|^s_{\max}})$ is bicomplete and, thus, the asymmetric normed linear space $(\mathbb{R}, \|\cdot\|_{\max})$ is biBanach. Define the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ by $x_n = -1$ for all $n \in \mathbb{N}$. It is clear that $\sum_{k=1}^{\infty} x_k$ is absolutely convergent, since $\|x_n\|_{\max} = 0$ for all $n \in \mathbb{N}$. However, the series $\sum_{k=1}^{\infty} x_k$ is not $d_{\|\cdot\|^s_{\max}}$-convergent. So, $(\mathbb{R}, \|\cdot\|_{\max})$ does not have the strong absolute convergence property.

The following is an example of an asymmetric normed semilinear space with the strong absolute convergence property.

Example 6. Consider the asymmetric normed linear space $(\mathbb{R}, \|\cdot\|_{\max})$ introduced in Example 5. It is clear that the pair $(\mathbb{R}^+, \|\cdot\|_{\max})$ is an asymmetric normed semilinear space. Since $\|x\|^s_{\max} = x$ for all $x \in \mathbb{R}^+$, we have that every absolutely convergent series is $d_{\|\cdot\|^s_{\max}}$-convergent.

In the remainder of this section, we consider a few notions of completeness that arise in a natural way in the asymmetric context. Thus, we focus our efforts on characterizing those asymmetric normed semilinear spaces that enjoy the (strong) absolute convergence property in terms of the aforementioned notions of completeness.

3.2. The Characterization. In order to achieve our goal, let us recall two notions of completeness for quasimetric spaces, the so-called left K-sequential completeness and the Smyth completeness.

A quasimetric space (X, d) is said to be left K-sequentially complete (Smyth complete) provided that every left K-Cauchy sequence is d-convergent (d^s-convergent), where a sequence $(x_n)_{n\in\mathbb{N}}$ is left K-Cauchy whenever for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m \geq n \geq k$.

According to [11, 24], an asymmetric normed semilinear space $(X, \|\cdot\|)$ is left K-sequentially complete (Smyth complete) if the quasimetric space $(X, d_{\|\cdot\|})$ is left K-sequentially complete (Smyth complete).

3.2.1. Absolute Convergence. In this subsection, we provide a description of asymmetric normed semilinear spaces that have the absolute convergence property.

The following result will be crucial for our purpose, whose proof can be found in [25].

Lemma 7. *Let (X, d) be a quasimetric space and let $(x_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence. If there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which d-converges to x, then $(x_n)_{n\in\mathbb{N}}$ d-converges to x.*

From the preceding result, we immediately obtain the following one for asymmetric normed semilinear spaces.

Lemma 8. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space and let $(x_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence. If there exists* *a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which $d_{\|\cdot\|}$-converges to x, then $(x_n)_{n\in\mathbb{N}}$ $d_{\|\cdot\|}$-converges to x.*

Taking into account the preceding lemma, we characterize asymmetric normed semilinear spaces that enjoy the absolute convergence property in the result below. It must be pointed out that the equivalence between assertions (2) and (3) in Theorem 9 is given in [3] for asymmetric normed linear spaces. However, here, we prove the equivalence following a technique that, although related to the one used in [3], allows us to provide a bit more information, in the spirit of [23], about the spaces under study (assertion (1) in the statement of Theorem 9).

Theorem 9. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space. Then, the following assertions are equivalent.*

(1) *$\sum_{k=1}^{\infty} x_k$ is $d_{\|\cdot\|}$-convergent for every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\|x_n\| < 2^{-n}$ for all $n \in \mathbb{N}$.*

(2) *$(X, \|\cdot\|)$ is left K-sequentially complete.*

(3) *$(X, \|\cdot\|)$ has the absolute convergence property.*

Proof. (1) $\Rightarrow$ (2). Assume that $(x_n)_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in $(X, d_{\|\cdot\|})$. Our aim is to show that $(x_n)_{n\in\mathbb{N}}$ has a $d_{\|\cdot\|}$-convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Indeed, we can consider an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$d_{\|\cdot\|}\left(x_{n_k}, x_s\right) < 2^{-k} \tag{3}$$

for all $s \geq n_k$. It follows that

$$d_{\|\cdot\|}\left(x_{n_{k-1}}, x_{n_k}\right) < 2^{-k-1} \tag{4}$$

for all $k \in \mathbb{N}$ with $k > 1$. Hence, we have that

$$x_{n_k} = x_{n_{k-1}} + b_{n_{k,k-1}} \quad \text{with } \left\|b_{n_{k,k-1}}\right\| < 2^{-k-1} \tag{5}$$

for all $k \in \mathbb{N}$ with $k > 1$, where $b_{n_{k,k-1}} = x_{n_k} - x_{n_{k-1}}$. Next, define a sequence $(y_n)_{n\in\mathbb{N}}$ in X as follows:

$$y_1 = x_{n_1}, \qquad y_k = b_{n_{k,k-1}} \quad \forall k \in \mathbb{N} \text{ with } k > 1. \tag{6}$$

It is obvious that $\|y_k\| < 2^{-k}$ for all $k \in \mathbb{N}$. Then, we obtain that $\sum_{m=1}^{\infty} y_m$ is $d_{\|\cdot\|}$-convergent. Since

$$\begin{aligned} x_{n_k} &= x_{n_{k-1}} + b_{n_{k,k-1}} = x_{n_{k-2}} + b_{n_{k-1,k-2}} + b_{n_{k,k-1}} \\ &= x_{n_1} + b_{n_{2,1}} + \cdots + b_{n_{k,k-1}} = y_1 + \cdots + y_k \end{aligned} \tag{7}$$

for all $k \in \mathbb{N}$, we conclude that $(x_{n_k})_{k\in\mathbb{N}}$ is $d_{\|\cdot\|}$-convergent. By Lemma 8, the sequence $(x_n)_{n\in\mathbb{N}}$ is $d_{\|\cdot\|}$-convergent. It follows that the asymmetric normed semilinear space $(X, \|\cdot\|)$ is left K-sequentially complete.

(2) $\Rightarrow$ (3). Consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that the induced series $\sum_{k=1}^{n} x_k$ is absolutely convergent. Then, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} \|x_k\| < \epsilon$ for all $n \geq n_0$. Thus, taking $m, n \geq n_0$ such that $m \geq n \geq n_0$, we have that

$$\begin{aligned} d_{\|\cdot\|}\left(\sum_{k=1}^{n} x_k, \sum_{k=1}^{m} x_k\right) &= \left\|\sum_{k=n+1}^{m} x_k\right\| \leq \sum_{k=n+1}^{m} \|x_k\| \\ &\leq \sum_{k=n+1}^{\infty} \|x_k\| < \epsilon. \end{aligned} \tag{8}$$

It follows that $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ is a left K-Cauchy sequence. Since $(X, \|\cdot\|)$ is left K-sequentially complete, we obtain that $\sum_{k=1}^{\infty} x_k$ is $d_{\|\cdot\|}$-convergent.

(3) $\Rightarrow$ (1). Consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\|x_n\| < 2^{-n}$ for all $n \in \mathbb{N}$. Then, we show that the series $\sum_{k=1}^{n} x_k$ is absolutely convergent. Indeed, $\sum_{k=1}^{n} \|x_k\| \leq \sum_{k=1}^{n} 1/2^k < 1$ for all $n \in \mathbb{N}$. It follows that the series $\sum_{k=1}^{\infty} \|x_k\|$ is convergent. Thus, by hypothesis, we obtain that the series $\sum_{k=1}^{\infty} x_k$ is $d_{\|\cdot\|}$-convergent. This concludes the proof. □

Next, we give a few examples of asymmetric normed semilinear spaces having the absolute convergence property.

Example 10. In [24], the following asymmetric normed semilinear spaces were proved to be left K-sequentially complete and, thus, by Theorem 9 all have the absolute convergence property.

(1) $(l_\infty, \|\cdot\|_{\sup}^{+})$ and $(l_\infty^{+}, \|\cdot\|_{\sup}^{+})$, where l_∞ is the set of real number sequences $(x_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} x_n < \infty$, $l_\infty^{+} = \{x \in l_\infty : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$, and $\|\cdot\|_{\sup}^{+}$ is the asymmetric norm introduced in Example 3.

(2) $(l_p, \|\cdot\|_{+p})$ and $(l_p^{+}, \|\cdot\|_{+p})$, where $1 \leq p < \infty$, l_p is the set of real number sequences $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, $l_p^{+} = \{x \in l_p : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$, and $\|\cdot\|_{+p}$ is the asymmetric norm defined on l_p by

$$\|x\|_{+p} = \left(\sum_{n=1}^{\infty} \|x_n\|_{\max}^{p}\right)^{1/p} \tag{9}$$

for all $x \in l_p$.

(3) $(C([0,1]), \|\cdot\|_{+\infty})$ and $(C^{+}([0,1]), \|\cdot\|_{+\infty})$, where $C([0,1])$ is the set of all functions $f : [0,1] \to \mathbb{R}$ which are continuous from $([0,1], |\cdot|)$ into $(\mathbb{R}, |\cdot|)$, $C^{+}([0,1]) = \{f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1]\}$, and $\|\cdot\|_{+\infty}$ is the asymmetric norm on $C([0,1])$ given by

$$\|f\|_{+\infty} = \sup_{x\in[0,1]} \|f(x)\|_{\max} \tag{10}$$

for all $f \in C[0,1]$.

(4) $(c_0, \|\cdot\|_{\sup}^{+})$ and $(c_0^{+}, \|\cdot\|_{\sup}^{+})$, where c_0 is the set of real number sequences $(x_n)_{n\in\mathbb{N}}$ which converge to 0 and $c_0^{+} = \{x \in c_0 : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}$.

(5) $(\mathbb{R}^m, \|\cdot\|_{\max,m})$ and $(\mathbb{R}^m_+, \|\cdot\|_{\max,m})$, where $m \in \mathbb{N}$, $\mathbb{R}^m = \{(x_1,\ldots,x_m) : x_1,\ldots,x_m \in \mathbb{R}\}$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_1 \geq 0,\ldots x_m \geq 0\}$, and $\|\cdot\|_{\max,m}$ is the asymmetric norm on $\mathbb{R}^m$ defined by

$$\|x\|_{\max,m} = \max_{1\leq i\leq m}\|x_i\|_{\max} \tag{11}$$

for all $x \in \mathbb{R}^m$.

3.2.2. Strong Absolute Convergence. In this subsection, we provide a description of asymmetric normed semilinear spaces that have the strong absolute convergence property.

In the following, the well-known result below plays a central role [3].

Lemma 11. *Let (X, d) be a quasimetric space and let $(x_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence. If there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which d^s-converges to x, then $(x_n)_{n\in\mathbb{N}}$ d^s-converges to x.*

As a direct consequence, we obtain the next result.

Lemma 12. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space and let $(x_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence in $(X, d_{\|\cdot\|})$. If there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which $d^s_{\|\cdot\|}$-converges to x, then $(x_n)_{n\in\mathbb{N}}$ $d^s_{\|\cdot\|}$-converges to x.*

Next, we are able to provide a description of asymmetric normed semilinear spaces having the strong absolute convergence property.

Theorem 13. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space. Then, the following assertions are equivalent.*

(1) *$\sum_{k=1}^{\infty} x_k$ is $d^s_{\|\cdot\|}$-convergent for every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\|x_n\| < 2^{-n}$ for all $n \in \mathbb{N}$.*

(2) *$(X, \|\cdot\|)$ is Smyth complete.*

(3) *$(X, \|\cdot\|)$ has the strong absolute convergence property.*

Proof. (1) $\Rightarrow$ (2). Assume $(x_n)_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in $(X, d_{\|\cdot\|})$. Our aim is to show that $(x_n)_{n\in\mathbb{N}}$ has a $d_{\|\cdot\|^s}$-convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. By the same method as in the proof of (1) $\Rightarrow$ (2) of Theorem 9, we can show the existence of a sequence $(y_n)_{n\in\mathbb{N}}$ such that $x_{n_k} = y_1 + \cdots + y_k$ and $\|y_k\| < 2^{-k}$ for all $k \in \mathbb{N}$. Then, by hypothesis, the series $\sum_{k=1}^{\infty} y_k$ is $d^s_{\|\cdot\|}$-convergent and, hence, as a consequence the sequence $(x_{n_k})_{k\in\mathbb{N}}$ is $d^s_{\|\cdot\|}$-convergent. By Lemma 12, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is $d^s_{\|\cdot\|}$-convergent. So, $(X, \|\cdot\|)$ is Smyth complete.

(2) $\Rightarrow$ (3). Consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that the induced series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent. Then, as in the proof of (2) $\Rightarrow$ (3) of Theorem 9, we have that the sequence $(\sum_{k=1}^{n} x_k)_{n\in\mathbb{N}}$ is left K-Cauchy. Since $(X, \|\cdot\|)$ is Smyth complete, we deduce that $\sum_{k=1}^{\infty} x_k$ is $d_{\|\cdot\|^s}$-convergent.

(3) $\Rightarrow$ (1). Consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\|x_n\| < 2^{-n}$ for all $n \in \mathbb{N}$. Then, according to the proof of (3) $\Rightarrow$ (1) of Theorem 9, we have that the series $\sum_{k=1}^{n} x_k$ is absolutely convergent. Since $(X, \|\cdot\|)$ has the strong absolute convergence property, we conclude that the series $\sum_{k=1}^{\infty} x_k$ is $d_{\|\cdot\|^s}$-convergent. □

Among all asymmetric normed semilinear spaces given in Example 10, the only ones that are Smyth complete are $(l^+_p, \|\cdot\|_{+p})$, $(c^+_0, \|\cdot\|^+_{\sup})$, and $(\mathbb{R}^m_+, \|\cdot\|_{\max,m})$ (see [26]). Hence, by Theorem 13, the aforesaid asymmetric normed semilinear spaces have the strong absolute convergence property.

Since every Cauchy sequence is a left K-Cauchy sequence and every Smyth complete asymmetric normed semilinear space is, at the same time, left K-sequential complete and biBanach, we immediately deduce the next result.

Corollary 14. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space which has the strong absolute convergence property. Then, the following assertions hold:*

(1) *$(X, \|\cdot\|)$ is left K-sequentially complete;*

(2) *$(X, \|\cdot\|)$ is biBanach.*

Remark 15. Observe that Example 10 yields instances of asymmetric normed semilinear spaces that hold the absolute convergence property but not the strong one. Therefore, left K-sequential completeness is not equivalent to the strong absolute convergence property. Moreover, although the strong absolute convergence property has been introduced in Definition 4 with the aim of describing biBanach completeness, next, we show that both the aforesaid notions are not equivalent. Indeed, let $(\mathbb{R}, \|\cdot\|_{\max})$ be the biBanach asymmetric normed linear space given in Example 5. In the aforementioned example, it was shown that, for the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = -1$ for all $n \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} x_k$ is absolutely convergent; however, it is not $d_{\|\cdot\|^s_{\max}}$-convergent. So $(\mathbb{R}, \|\cdot\|_{\max})$ does not have the strong absolute convergence property.

In the light of the preceding remark, it seems clear that we will need a new subclass of absolutely convergent series for providing a characterization of biBanach asymmetric normed semilinear spaces. To this end, we introduce the following notion.

Definition 16. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in an asymmetric normed semilinear space $(X, \|\cdot\|)$, then the series $\sum_{k=1}^{\infty} x_k$ is s-absolutely convergent provided that the series $\sum_{k=1}^{\infty} \|x_k\|^s$ is convergent. Moreover, we will say that the asymmetric normed semilinear space $(X, \|\cdot\|)$ has the s-absolute convergence property whenever every s-absolutely convergent series is $d_{\|\cdot\|^s}$-convergent.

Obviously, every s-absolutely convergent series is absolutely convergent and, besides, all spaces with the strong absolute convergence property have, at the same time, the s-absolute convergence property. However, the converse does not hold. Indeed, $(\mathbb{R}, \|\cdot\|_{\max})$ is an example of asymmetric normed semilinear space which enjoys the s-absolute convergence property but not the strong absolute convergence property.

We end the section yielding the characterization in the case of biBanach asymmetric normed semilinear spaces through the next result whose proof runs as the proof of Theorem 1.

Theorem 17. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space. Then, the following assertions are equivalent.*

(1) *$\sum_{k=1}^{\infty} x_k$ is $d^s_{\|\cdot\|}$-convergent for every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\|x_n\|^s < 2^{-n}$ for all $n \in \mathbb{N}$.*

(2) *$(X, \|\cdot\|)$ is biBanach.*

(3) *$(X, \|\cdot\|)$ has the s-absolute convergence property.*

4. The Weierstrass Test in Asymmetric Normed Semilinear Spaces

The relevance of Theorem 1 is given by its wide number of applications. Among others, it allows to state a criterion, known as Weierstrass M-test, for the uniform convergence of series of bounded mappings [27]. For the convenience of the reader and with the aim of making our exposition self-contained, let us recall the notion of bounded mapping [28].

Let (X, d) be a metric space and let S be a nonempty set. Then, a mapping $f : S \to X$ is said to be bounded if for some $x \in X$ there exists $M_x \in \mathbb{R}^+_0$ such that $d(f(s), x) \le M_x$ for all $s \in S$, where $\mathbb{R}^+_0$ stands for the set of positive real numbers.

Next, consider a normed linear space $(X, \|\cdot\|)$. Then, the set $Nb(S, X) = \{f : S \to X : f \text{ is bounded}\}$ becomes a normed linear space endowed with the norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty = \sup_{s\in S}\|f(s)\|$ for all $f \in Nb(S, X)$. Observe that the boundness of f guarantees that $\|f\|_\infty < \infty$. Furthermore, the normed linear space $(Nb(S, X), \|\cdot\|_\infty)$ is Banach provided that the normed linear space $(X, \|\cdot\|)$ is so.

The following theorem contains the Weierstrass M-test in the normed case.

Theorem 18. *Let $(X, \|\cdot\|)$ be a Banach normed linear space and let S be a nonempty set. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $Nb(S, X)$ and there exists a sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+_0$ such that the series $\sum_{k=1}^{\infty} M_k$ is convergent and $\|f_n\|_\infty \le M_n$ for all $n \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} f_k$ is $d_{\|\cdot\|_\infty}$-convergent.*

Our main goal in this section is to prove a version of Theorem 18 in the realm of asymmetric normed semilinear spaces. To this end, Theorems 9 and 13 will play a crucial role. Of course, the notion of bounded mapping can be formulated in our context in two different ways. Indeed, given a quasimetric space (X, d) and a nonempty set S, a mapping $f : S \to X$ will be said to be bounded from the left if for some $x \in X$ there exists $M_x \in \mathbb{R}^+$ such that $d(f(s), x) \le M_x$ for all $s \in S$. A mapping $f : S \to X$ is bounded from the right if for some $x \in X$ there exists $M_x \in \mathbb{R}^+$ such that $d(x, f(s)) \le M_x$ for all $s \in S$.

It is routine to check that the pair $(ANb_+(S, X), \|\cdot\|_\infty)$ is an asymmetric normed semilinear space whenever $(X, \|\cdot\|)$ is an asymmetric normed semilinear space, where $ANb_+(S, X) = \{f : S \to X : f \text{ is bounded from the left}\}$ and the asymmetric norm $\|\cdot\|_\infty$ is defined as in the normed case. The same occurs with $(ANb_-(S, X), \|\cdot\|_\infty)$ which can be defined dually.

It is clear that the notions of bounded mapping in the quasimetric case allow us to recover the bounded notion for metric spaces. Moreover, $ANb_+(S, X) = ANb_-(S, X) = Nb(S, X)$ when $(X, \|\cdot\|)$ is a normed linear space and, in addition, we impose the constraint "$M_x \in \mathbb{R}^+_0$" in the definition of bounded mapping.

The next example gives mappings which are bounded from both the left and the right.

Example 19. Let $(\mathbb{R}, \|\cdot\|_{\max})$ be the asymmetric normed linear space given in Example 5. Define the mapping $f_0 : \mathbb{R} \to \mathbb{R}$ by

$$f_0(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases} \tag{12}$$

Clearly,

$$d_{\|\cdot\|_{\max}}(f_0(x), 0) = \|-f_0(x)\|_{\max} = 0 \tag{13}$$

for all $x \in \mathbb{R}$. Thus, f_0 is a bounded mapping from the left.

Next, consider the asymmetric normed semilinear space $(\mathbb{R}^+, \|\cdot\|_{\max})$ and define the mapping $f_1 : \mathbb{R}^+ \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 1, \\ 1 & \text{if } x \le 1. \end{cases} \tag{14}$$

Obviously,

$$d_{\|\cdot\|_{\max}}(0, f_1(x)) = \|f_1(x)\|_{\max} \le 1 \tag{15}$$

for all $x \in \mathbb{R}^+$. Hence, we have that f_1 is a bounded mapping from the right.

Of course, the sets $ANb_+(S, X)$ and $ANb_-(S, X)$ are different in general as shown in the preceding example. In fact, $f_0 \notin ANb_-(S, X)$.

In the next result, we discuss the left K-sequential completeness and the Smyth completeness of $ANb_+(S, X)$ and $ANb_-(S, X)$.

Theorem 20. *Let $(X, \|\cdot\|)$ be an asymmetric normed semilinear space and let S be a nonempty set. If $(X, \|\cdot\|)$ is Smyth complete, then $(ANb_+(S, X), \|\cdot\|_\infty)$ and $(ANb_-(S, X), \|\cdot\|_\infty)$ are both left K-sequentially complete.*

Proof. We only prove that $(ANb_+(S, X), \|\cdot\|_\infty)$ is left K-sequentially complete. Similar arguments can be applied to prove that $(ANb_-(S, X), \|\cdot\|_\infty)$ is left K-sequentially complete.

Let $(f_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence in $ANb_+(S, X)$. Then, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_{\|\cdot\|_\infty}(f_n, f_m) < \epsilon$ for all $m \ge n \ge n_0$. Hence, $d_{\|\cdot\|}(f_n(s), f_m(s)) < \epsilon$ for all $s \in S$ and for all $m \ge n \ge n_0$. It follows that, for each $s \in S$, $(f_n(s))_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in $(X, \|\cdot\|)$. Since $(X, \|\cdot\|)$ is Smyth complete, we deduce that there exists $x_s \in X$ such that $(f_n(s))_{n\in\mathbb{N}}$ $d_{\|\cdot\|^s}$-converges to x_s.

Next, define the mapping $f : S \to X$ by $f(s) = x_s$. Then, $f \in ANb_+(S, X)$. Indeed, the fact that f_{n_0} is bounded from the left yields the existence of $x_{f_{n_0}} \in X$ and $M_{x_{f_{n_0}}} \in \mathbb{R}^+$ such that $d_{\|\cdot\|}(f_{n_0}(s), x_{f_{n_0}}) \leq M_{x_{f_{n_0}}}$ for all $s \in S$. Thus, we obtain that

$$\begin{aligned} d_{\|\cdot\|}\left(f(s), x_{f_{n_0}}\right) &\leq d_{\|\cdot\|^s}\left(f(s), f_{n_0}(s)\right) + d_{\|\cdot\|}\left(f_{n_0}(s), x_{f_{n_0}}\right) \\ &\leq \epsilon + M_{x_{f_{n_0}}} \end{aligned} \tag{16}$$

for all $s \in S$. Moreover, the existence of $n_1 \in \mathbb{N}$ is guaranteed such that $n_1 \geq n_0$ and, in addition, $d_{\|\cdot\|}(f(s), f_{n_1}(s)) < \epsilon$ and $d_{\|\cdot\|^s}(f_{n_1}(s), f_n(s)) < \epsilon$ for all $n \geq n_0$. Thus, we deduce that

$$\begin{aligned} d_{\|\cdot\|}\left(f(s), f_n(s)\right) \leq{} & d_{\|\cdot\|}\left(f(s), f_{n_1}(s)\right) \\ & + d_{\|\cdot\|^s}\left(f_{n_1}(s), f_n(s)\right) < 2\epsilon \end{aligned} \tag{17}$$

for all $n \geq n_0$ and for all $s \in S$. So, we have shown that $(f_n)_{n\in\mathbb{N}}$ is $d_{\|\cdot\|_\infty}$-convergent as claimed. □

The next example shows that there are left K-sequentially complete asymmetric normed semilinear spaces whose asymmetric normed semilinear space of bounded mappings is not left K-sequentially complete.

Example 21. Consider the left K-sequentially asymmetric normed linear space $(c_0, \|\cdot\|^+_{\sup})$ introduced in Example 10. Of course, it is routine to check that $(c_0, \|\cdot\|^+_{\sup})$ is not Smyth complete. Let $S = \{1\}$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ given by $f_n(1) = (1/n^2, 1/n^2, \ldots, 1/n^2, 0, 0, \ldots)$. Clearly, $\|f_n\|_\infty = \|f_n(1)\|^+_{\sup} = 1/n^2$ for all $n \in \mathbb{N}$. Hence, f_n belongs to $ANb_+(1, c_0)$. Moreover, $\sum_{k=1}^{\infty} \|f_k\|_\infty$ is convergent. Nevertheless, $\sum_{k=1}^{\infty} f_k$ is not $d_{\|\cdot\|_\infty}$-convergent. Thus, by Theorem 9, we conclude that $(ANb_+(1, c_0), \|\cdot\|_\infty)$ is not left K-sequentially complete. Observe, in addition, that $(ANb_+(1, c_0), \|\cdot\|_\infty)$ is biBanach and, hence, we have an example of a biBanach asymmetric normed linear space which does not have the absolute convergence property as announced in Section 3.1.

The following example provides a Smyth complete asymmetric normed semilinear space whose asymmetric normed semilinear space of bounded mappings is not Smyth complete.

Example 22. Consider the Smyth complete asymmetric normed linear space $(c_0^+, \|\cdot\|^+_{\sup})$ given (see Example 5) and let S be as in Example 21. Take the sequence $(f_n)_{n\in\mathbb{N}}$ given in Example 21. Then, it is clear that f_n belongs to $ANb_+(1, c_0^+)$ for all $n \in \mathbb{N}$ and that the series $\sum_{k=1}^{\infty} \|f_k\|_\infty$ is convergent. However, the series $\sum_{k=1}^{\infty} f_k$ is not $d_{\|\cdot\|^s_\infty}$-convergent. Thus, by Theorem 13, we conclude that $(ANb_+(1, \mathbb{R}), \|\cdot\|_\infty)$ is not Smyth complete.

We end the paper proving the announced Weierstrass test in the asymmetric framework.

Theorem 23. *Let $(X, \|\cdot\|)$ be a Smyth complete asymmetric normed semilinear space and let S be a nonempty set. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $ANb_+(S, X)$ and there exists a sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that the series $\sum_{k=1}^{\infty} M_k$ is convergent and $\|f_n\|_\infty \leq M_n$ for all $n \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} f_k$ is $d_{\|\cdot\|_\infty}$-convergent.*

Proof. Since $\sum_{k=1}^{n} \|f_n\|_\infty \leq \sum_{k=1}^{n} M_n$ and the series $\sum_{k=1}^{\infty} M_k$ is convergent, we have that the series $\sum_{k=1}^{\infty} \|f_k\|_\infty$ is convergent. The Smyth completeness of $(X, \|\cdot\|)$ provides the left K-sequential completeness of $ANb_+(S, X)$ and, by Theorem 9, we deduce that the series $\sum_{k=1}^{\infty} f_k$ is $d_{\|\cdot\|_\infty}$-convergent. □

Using arguments similar to those in the proof of preceding result, one can get the following result.

Theorem 24. *Let $(X, \|\cdot\|)$ be a Smyth complete asymmetric normed semilinear space and let S be a nonempty set. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $ANb_-(S, X)$ and there exists a sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that the series $\sum_{k=1}^{\infty} M_k$ is convergent and $\|f_n\|_\infty \leq M_n$ for all $n \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} f_k$ is $d_{\|\cdot\|_\infty}$-convergent.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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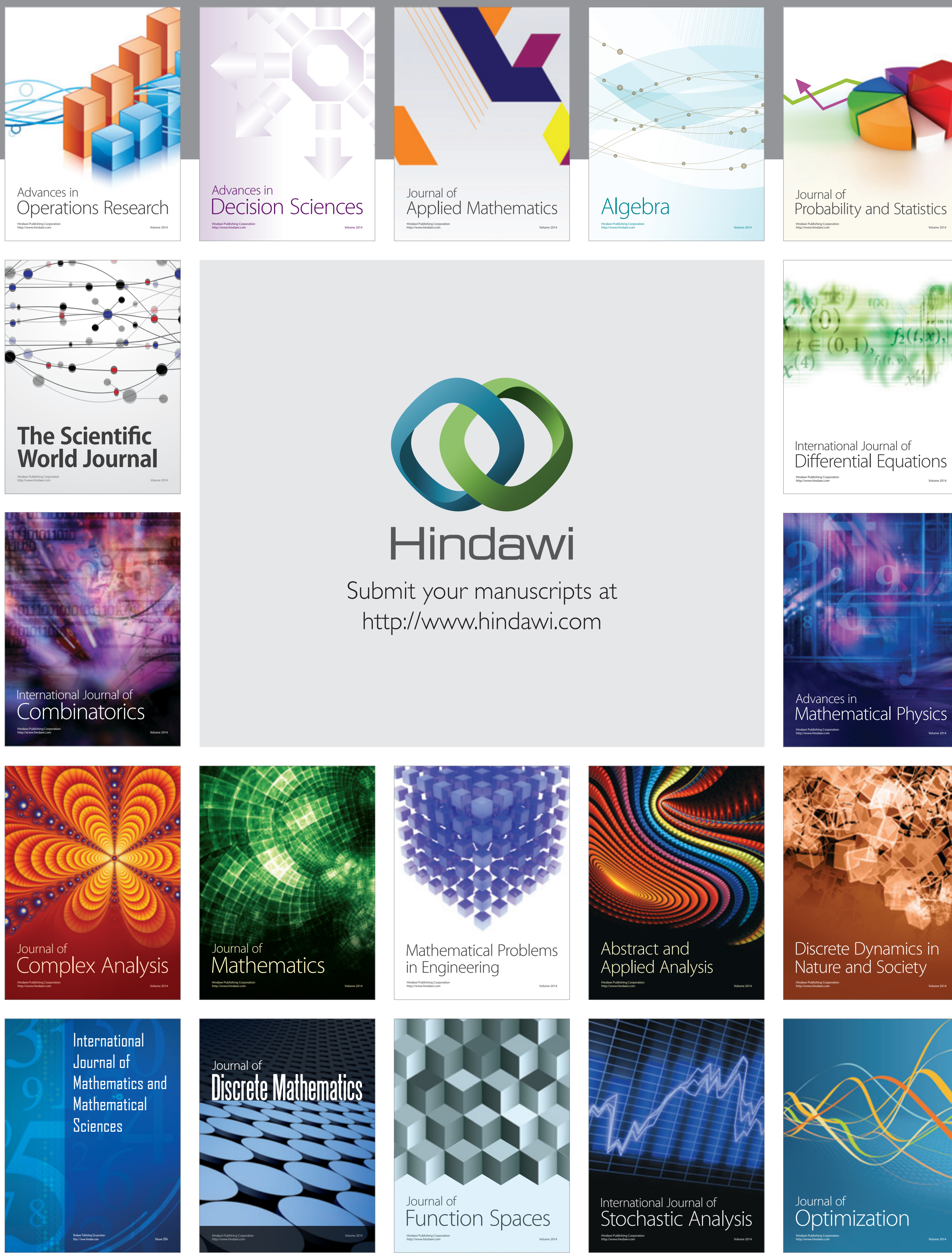

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