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Research Article

An Application of Fixed Point Theory to a Nonlinear Differential Equation

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We introduce a new family of mappings on $[0, +\infty)$ by relaxing the nondecreasing condition on the mappings and by using the properties of this new family we present some fixed point theorems for α-ψ-contractive-type mappings in the setting of complete metric spaces. By applying our obtained results, we also assure the fixed point theorems in partially ordered complete metric spaces and as an application of the main results we provide an existence theorem for a nonlinear differential equation.

1. Introduction and Preliminaries

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic field and this is very active field of research at present. Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models (integral and partial equations, variational inequalities, etc). It can be applied to, for example, variational inequalities, optimization, and approximation theory. The fixed point theory has been continually studied by many researchers (see, e.g., [1–5] and references contained therein). It is well known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach-Caccioppoli theorem which was published in 1922 in [6] and it also appeared in [7]. Later in 1968, Kannan [8] studied a new type of contractive mappings. Since then, there have been many results related to mappings satisfying various types of contractive inequalities; we refer to ([9–12] etc.) and references contained therein.

Recently, Samet et al. [5] introduced a new category of contractive-type mappings known as α-ψ contractive-type mappings. The results obtained by Samet et al. [5] extended and generalized the existing fixed point results in the literature, in particular the Banach contraction principle. Salimi et al. [4] and Karapinar and Samet [3] generalized the α-ψ contractive-type mappings and obtained various fixed point theorems for this generalized class of contractive mappings [3, 4].

Most of papers (see, for instance, [3–5] and references contained therein) have considered the α-ψ contractive-type mapping for a nondecreasing mapping $\psi : [0, +\infty) \to [0, +\infty)$ with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \in (0, +\infty)$. The convergence of $\sum_{n=1}^{\infty} \psi^n(t)$ and nondecreasing condition for ψ are restrictive and it is a fact that such a mapping is differentiable almost everywhere and hence continuous why was one of our aims to write this paper in order to consider a family of mappings $\psi : [0, +\infty) \to [0, +\infty)$ by relaxing nondecreasing condition and the convergence of the series $\sum_{n=1}^{\infty} \psi^n(t)$. This paper is inspired and motivated by research works [4, 5]; we will introduce a new family of mappings on $[0, +\infty)$ and prove the fixed point theorems for mappings using properties of this new family in complete metric spaces. By applying our obtained results, we also assure the fixed point theorems in partially ordered complete metric spaces and give the applications to ordinary differential equations.

In the rest of the paper, we introduce some notations and definitions that will be used in the sequel.

Lemma 1 (see [5]). *Suppose that* $\psi : [0, +\infty) \to [0, +\infty)$. *If* ψ *is nondecreasing, then for each* $t \in (0, +\infty)$, $\lim_{n\to\infty}\psi^n(t) = 0$ *implies that* $\psi(t) < t$.

Remark 2. It is easily seen that if $\psi : [0, +\infty) \to [0, +\infty)$ is nondecreasing and $\psi(t) < t$, for all $t \in (0, +\infty)$, then $\psi(0) = 0$.

Definition 3 (see [5]). Let $T : X \to X$ and let $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if, for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

In 2012, Samet et al. [5] introduced the concept of α-ψ-contractive-type mappings, where $\psi \in \Psi_1$ and

$$\Psi_1 = \Big\{\psi : \psi : [0, +\infty) \longrightarrow [0, +\infty) \text{ is nondecreasing with } \sum_{n=1}^{\infty}\psi^n(t) < \infty, \ \forall t \in (0, +\infty)\Big\}. \tag{1}$$

Definition 4 (see [5]). Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ where $\psi \in \Psi_1$ such that

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y)), \tag{2}$$

for all $x, y \in X$.

In [5], the authors assured the existence of the fixed point theorems for the mentioned mappings satisfying α-admissibility in the complete metric spaces.

Recently, Salimi et al. [4] modified the concept of α-admissibility.

Definition 5 (see [4]). Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$. We say that T is α-admissible with respect to η if, for all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

Remark 6. If we suppose that $\eta(x, y) = 1$, for all $x, y \in X$, Definition 5 is reduced to Definition 3.

Salimi et al. [4] proved the existence of fixed point theorems for generalized α-ψ-contractive-type mappings where $\psi \in \Psi_1$. They also assure the fixed point theorems generalized α-ψ-contractive-type mappings where ψ is a nondecreasing continuous mapping and $\psi(0) = 0$.

In this work, we will introduce a new family of mappings on $[0, +\infty)$ without assuming the nondecreasing condition for ψ and prove the fixed point theorems for α-ψ-contractive-type mappings using properties of this new family in complete metric spaces. We will use our result to obtain fixed point results in partially ordered complete metric spaces and to give an application to nonlinear differential equations.

2. Main Results

We now introduce a new family Ψ_2 of mappings and prove the existence of fixed point results for α-ψ-contractive-type mappings where $\psi \in \Psi_2$.

Denote by Ψ_2 the family of mappings $\psi : [0, +\infty) \to [0, +\infty)$ such that

(i) ψ is an upper semicontinuous mapping from the right;
(ii) $\psi(t) < t$ for all $t \in (0, +\infty)$;
(iii) $\psi(0) = 0$.

Remark 7. By Lemma 1, for each $\psi \in \Psi_1$, we have $\psi(t) < t$ for all $t \in (0, +\infty)$ and by Remark 2 we obtain that $\psi(0) = 0$.

Remark 8. Since every nondecreasing mapping is differentiable almost everywhere (see [13]), we observe that nondecreasing condition is closed to continuity and it is restrictive.

Example 9. The floor function $f(x) = \lfloor x \rfloor$ is upper semicontinuous function from the right and nondecreasing but is not continuous.

Example 10. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a mapping defined by

$$\psi(t) = \begin{cases} 1, & t = 0; \\ 0, & t > 0. \end{cases} \tag{3}$$

We have that ψ is upper semicontinuous from the right and $\psi(t) < t$ for all $t \in (0, +\infty)$. Furthermore, ψ is not nondecreasing.

Example 11. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a mapping defined by

$$\psi(t) = \begin{cases} \dfrac{1}{2}, & t \in \mathbb{N}; \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Thus, ψ is upper semicontinuous from the right, $\psi(t) < t$ for all $t \in (0, +\infty)$ and $\psi(0) = 0$. Moreover, ψ is not nondecreasing.

We now the prove the existence of the fixed point theorem for α-admissible mappings with respect to η where $\psi \in \Psi_2$.

Theorem 12. *Let* (X, d) *be a complete metric space and* $\psi \in \Psi_2$. *Suppose that* $T : X \to X$ *is a mapping satisfying the following conditions:*

(i) *T is α-admissible with respect to η;*
(ii) *if* $x, y \in X$ *and* $\alpha(x, y) \geq \eta(x, y)$, *then* $d(Tx, Ty) \leq \psi(d(x, y))$;
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
(iv) *T is continuous or if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *and then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}$.

Then, T has a fixed point.

Proof. Since $x_0 \in X$, there exists x_1 such that $x_1 = Tx_0$. Therefore, we can construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}. \tag{5}$$

If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then T has a fixed point. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$ and T is α-admissible with respect to η, we obtain that

$$\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq \eta(Tx_0, Tx_1) = \eta(x_1, x_2). \tag{6}$$

By continuing the process as above, we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \quad \forall n \in \mathbb{N}. \tag{7}$$

Applying (ii), we obtain that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \psi(d(x_{n-1}, x_n)), \tag{8}$$

for all $n \in \mathbb{N}$. Since $\psi(t) < t$ for all $t \in (0, +\infty)$, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \tag{9}$$

for all $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence. It follows that there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c. \tag{10}$$

We will prove that $c = 0$. Suppose that $c > 0$. Since ψ is upper semicontinuous from the right using (9), we have

$$\begin{aligned} c &= \limsup_{n\to\infty} d(x_n, x_{n+1}) \\ &\leq \limsup_{n\to\infty} \psi(d(x_{n-1}, x_n)) \leq \psi(c) < c, \end{aligned} \tag{11}$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{12}$$

This implies that for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$d(x_{n_k}, x_{n_k+1}) < \frac{1}{2^k}. \tag{13}$$

We obtain that

$$\sum_{k=1}^{\infty} d(x_{n_k}, x_{n_k+1}) < \infty. \tag{14}$$

Therefore, $\{x_{n_k}\}$ is a Cauchy sequence and so converges to some $x \in X$. By continuity of T, we have

$$\lim_{n\to\infty} x_{n_k+1} = \lim_{n\to\infty} Tx_{n_k} = Tx. \tag{15}$$

This implies that x is a fixed point of T. On the other hand, since

$$\alpha(x_{n_k}, x_{n_k+1}) \geq \eta(x_{n_k}, x_{n_k+1}), \quad \forall k \in \mathbb{N} \tag{16}$$

and $\{x_{n_k}\}$ converges to x, we obtain that

$$\alpha(x_{n_k}, x) \geq \eta(x_{n_k}, x) \quad \forall k \in \mathbb{N}. \tag{17}$$

Using (ii), for each $k \in \mathbb{N}$, we have

$$\begin{aligned} d(Tx, x) &\leq d(Tx, Tx_{n_k}) + d(Tx_{n_k}, x) \\ &\leq \psi(d(x_{n_k}, x)) + d(x_{n_k+1}, x). \end{aligned} \tag{18}$$

Since ψ is upper semicontinuous from the right, we obtain that

$$\limsup_{k\to\infty} \psi(d(x_{n_k}, x)) \leq \psi(0) = 0. \tag{19}$$

By taking the limit as $k \to \infty$, this yields $d(Tx, x) = 0$ and hence $Tx = x$. □

Theorem 13. *Suppose all hypotheses of Theorem 12 hold. Assume that, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z) \geq \eta(y, z)$. Then, T has a unique fixed point.*

Proof. Assume that x and y are two fixed points of T. This implies that there exists $z \in X$ such that

$$\alpha(x, z) \geq \eta(x, z), \qquad \alpha(y, z) \geq \eta(y, z). \tag{20}$$

Since T is α-admissible with respect to η, for each $n \in \mathbb{N}$, we obtain that

$$\alpha(x, T^n z) \geq \eta(x, T^n z), \qquad \alpha(y, T^n z) \geq \eta(y, T^n z). \tag{21}$$

It follows that

$$d(x, T^{n+1} z) = d(Tx, T^{n+1} z) \leq \psi(d(x, T^n z)) < d(x, T^n z). \tag{22}$$

Therefore, $\{d(x, T^n z)\}$ is a nonincreasing sequence and then converges to some $c \in \mathbb{R}$. We will show that $c = 0$. Suppose that $c > 0$. Since ψ is upper semicontinuous from the right, we have

$$c = \limsup_{n\to\infty} d(x, T^{n+1} z) \leq \limsup_{n\to\infty} \psi(d(x, T^n z)) \leq \psi(c) < c, \tag{23}$$

which is a contradiction. It follows that

$$\lim_{n\to\infty} d(x, T^n z) = 0. \tag{24}$$

Similarly, by the same argument, we can prove that

$$\lim_{n\to\infty} d(y, T^n z) = 0. \tag{25}$$

Since the limit of the sequence is unique, we have $x = y$. □

Applying Theorems 12 and 13, we immediately obtain the following result.

Corollary 14. *Let (X, d) be a complete metric space and $\psi \in \Psi_2$. Suppose that $T : X \to X$ is an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous or if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *and then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$;

(iv) *for all* $x, y \in X$, *there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$.

Then, T has a unique fixed point.

Bhaskar and Lakshmikantham [9] introduced the definition of coupled fixed points.

Definition 15 (see [9]). Let $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{26}$$

Remark 16. Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \to X \times X$ by

$$T(x, y) = (F(x, y), F(y, x)) \quad \forall (x, y) \in X \times X. \tag{27}$$

Therefore, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T.

By using the analogous proof appeared in [5], we obtain the coupled fixed point results assuming $\psi \in \Psi_2$.

Theorem 17. *Let* (X, d) *be a complete metric space and* $F : X \times X \to X$ *be a given mapping. Suppose that there exist* $\psi \in \Psi_2$ *and a function* $\alpha : X^2 \times X^2 \to [0, +\infty)$ *such that*

$$\begin{aligned} &\alpha((x, y), (u, v))\, d(F(x, y), F(u, v)) \\ &\quad \leq \frac{1}{2}\psi(d(x, u) + d(y, v)), \end{aligned} \tag{28}$$

for all $(x, y), (u, v) \in X$. *Suppose that,*

(i) *for all* $(x, y), (u, v) \in X \times X$, *one has*

$$\begin{aligned} &\alpha((x, y), (u, v)) \geq 1 \\ &\text{implies } \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1; \end{aligned} \tag{29}$$

(ii) *there exists* $(x_0, y_0) \in X \times X$ *such that*

$$\begin{aligned} &\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1, \\ &\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1; \end{aligned} \tag{30}$$

(iii) *F is continuous.*

Then, F has a coupled fixed point.

Theorem 18. *Let* (X, d) *be a complete metric space and* $F : X \times X \to X$ *be a given mapping. Suppose that there exist* $\psi \in \Psi_2$ *and a function* $\alpha : X^2 \times X^2 \to [0, +\infty)$ *such that*

$$\begin{aligned} &\alpha((x, y), (u, v))\, d(F(x, y), F(u, v)) \\ &\quad \leq \frac{1}{2}\psi(d(x, u) + d(y, v)), \end{aligned} \tag{31}$$

for all $(x, y), (u, v) \in X \times X$. *Suppose that,*

(i) *for all* $(x, y), (u, v) \in X \times X$, *we have*

$$\begin{aligned} &\alpha((x, y), (u, v)) \geq 1 \\ &\quad \Longrightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1; \end{aligned} \tag{32}$$

(ii) *there exists* $(x_0, y_0) \in X \times X$ *such that*

$$\begin{aligned} &\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1, \\ &\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1; \end{aligned} \tag{33}$$

(iii) *if* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\begin{aligned} &\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1, \\ &\alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1, \\ &x_n \longrightarrow x \in X, \quad y_n \longrightarrow y \in X \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{34}$$

then

$$\alpha((x_n, y_n), (x, y)) \geq 1, \quad \alpha((y, x), (y_n, x_n)) \geq 1 \quad \forall n \in \mathbb{N}. \tag{35}$$

Then, F has a coupled fixed point.

Theorem 19. *Suppose that all hypotheses of Theorem 17 (resp., Theorem 18) hold. Assume that, for all* $(x, y), (u, v) \in X \times X$, *there exists* $(z_1, z_2) \in X \times X$ *such that*

$$\begin{aligned} &\alpha((x, y), (z_1, z_2)) \geq 1, \qquad \alpha((z_2, z_1), (y, x)) \geq 1, \\ &\alpha((u, v), (z_1, z_2)) \geq 1, \qquad \alpha((z_2, z_1), (v, u)) \geq 1. \end{aligned} \tag{36}$$

Then, F has a unique coupled fixed point.

3. Consequences

We now prove the fixed point theorems in complete metric spaces and partially ordered complete metric spaces using our obtained results.

Theorem 20 (Banach [6]). *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping satisfying*

$$d(Tx, Ty) \leq k d(x, y), \tag{37}$$

for all $x, y \in X$, *where* $k \in [0, 1)$. *Then, T has a unique fixed point.*

Proof. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be mappings defined by

$$\alpha(x, y) = 1, \quad \eta(x, y) = 1 \quad \forall x, y \in X. \tag{38}$$

It follows that T is α-admissible with respect to η. Suppose that $\psi : [0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = kt$ for all $t \in [0, +\infty)$. This implies that ψ is upper semicontinuous from the right, $\psi(t) < t$ for all $t \in (0, +\infty)$ and $\psi(0) = 0$. Furthermore, we can see that all assumptions in Theorem 13 are now satisfied. This completes the proof. □

Theorem 21 (Ran and Reurings [14]). *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:*

(i) *there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq k(d(x, y))$ for all $x, y \in X$ with $x \preccurlyeq y$;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*

(iii) *T is continuous.*

Then, T has a fixed point.

Proof. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ are mappings defined by

$$\alpha(x, y) = \begin{cases} 1, & x \preccurlyeq y; \\ 0, & \text{otherwise}, \end{cases}$$
$$\eta(x, y) = \begin{cases} \frac{1}{2}, & x \preccurlyeq y; \\ 2, & \text{otherwise}. \end{cases} \tag{39}$$

Let $x, y \in X$ such that $\alpha(x, y) \geq \eta(x, y)$. This implies that $x \preccurlyeq y$. Since T is nondecreasing with respect to $\preccurlyeq$, we obtain that $Tx \preccurlyeq Ty$. Therefore, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. It follows that T is α-admissible with respect to η. Define a mapping $\psi : [0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = kt$ for all $t \in [0, +\infty)$. We can see that $\psi \in \Psi_2$. For each $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$, we obtain that $x \preccurlyeq y$ and this yields

$$d(Tx, Ty) \leq k(d(x, y)) = \psi(d(x, y)). \tag{40}$$

By using (ii), we have $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Hence, all assumptions in Theorem 12 are now satisfied. Thus, we obtain the desired result. □

Theorem 22 (Nieto and Rodríguez-López [12]). *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:*

(i) *there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq k(d(x, y))$ for all $x, y \in X$ with $x \preccurlyeq y$;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;*

(iii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ are mappings defined as in the proof of Theorem 21. Assume that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. This implies that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$. Using (iii), this yield $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$. Therefore, $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$. Hence, all assumptions in Theorem 12 are now satisfied. Thus, we obtain the desired result. □

Theorem 23. *Suppose that all hypotheses of Theorem 21 (resp., Theorem 22) hold. Assume that, for all $x, y \in X$, there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$. Then T has a unique fixed point.*

Proof. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ are mappings defined as in the proof of Theorem 21. Let $x, y \in X$. It follows that there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$. Therefore, $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z) \geq \eta(y, z)$. Hence, all assumptions in Theorem 13 are now satisfied. So, the proof is complete. □

4. Applications to Ordinary Differential Equations

The following ordinary differential equation is taken from Samet et al. [5].

Denote by $C([0, 1])$ the set of all continuous functions defined on $[0, 1]$ and let $d : C([0, 1]) \times C([0, 1]) \to \mathbb{R}$ be defined by

$$d(x, y) = \|x - y\|_\infty = \max_{t \in [0,1]} |x(t) - y(t)|. \tag{41}$$

It is well known that $(C([0, 1]), d)$ is a complete metric space. Let us consider the two-point boundary value problem of the second-order differential equation:

$$-\frac{d^2 x}{dt^2} = f(t, x(t)), \quad t \in [0, 1];$$
$$x(0) = x(1) = 0, \tag{42}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The Green function associated to (42) is defined by

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1; \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{43}$$

Assume that the following conditions hold:

(i) there exists a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that, for all $t \in [0, 1]$, for all $a, b \in \mathbb{R}$ with $\phi(a, b) \geq 0$, we have

$$|f(t, a) - f(t, b)| \leq 8\psi\left(\max_{a,b \in \mathbb{R}, \phi(a,b) \geq 0} |a - b|\right), \tag{44}$$

where $\psi \in \Psi_2$;

(ii) there exists $x_0 \in C([0, 1])$ such that, for all $t \in [0, 1]$, we have

$$\phi\left(x_0(t), \int_0^1 G(t, s) f(s, x_0(s)) \, ds\right) \geq 0; \tag{45}$$

(iii) for all $t \in [0, 1]$, for all $x, y \in C([0, 1])$,

$$\phi(x(t), y(t)) \geq 0$$
$$\text{implies } \phi\left(\int_0^1 G(t, s) f(s, x(s)) \, ds, \right.$$
$$\left. \int_0^1 G(t, s) f(s, y(s)) \, ds\right) \geq 0; \tag{46}$$

(iv) if $\{x_n\}$ is a sequence in $C([0,1])$ such that $x_n \to x \in C([0,1])$ and $\phi(x_n, x_{n+1}) \geq 0$, for all $n \in \mathbb{N}$, then $\phi(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.

We now prove that existence of a solution of the mentioned second-order differential equation. The idea of proving the following theorem is taken from [5] but is slightly different.

Theorem 24. *Under assumptions (i)–(iv),* (42) *has a solution in* $C^2([0,1])$.

Proof. It is well known that $x \in C^2([0,1])$ is a solution of (42) is equivalent to $x \in C([0,1])$ is a solution of the integral equation (see [5])

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds \quad \forall t \in [0,1]. \tag{47}$$

Let $T : C([0,1]) \to C([0,1])$ be a mapping defined by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s))\, ds \quad \forall t \in [0,1]. \tag{48}$$

Suppose that $x, y \in C([0,1])$ such that $\phi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$. By applying (i), we obtain that

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| \int_0^1 G(t,s) \left[f(s, x(s)) - f(s, y(s)) \right] ds \right| \\ &\leq \int_0^1 G(t,s) \left| f(s, x(s)) - f(s, y(s)) \right| ds \\ &\leq 8 \left(\int_0^1 G(t,s)\, ds \right) (\psi(\|x - y\|_\infty)) \\ &\leq 8 \left(\sup_{t\in[0,1]} \int_0^1 G(t,s)\, ds \right) (\psi(\|x - y\|_\infty)). \end{aligned} \tag{49}$$

Since $\int_0^1 G(t,s)ds = -(t^2/2) + (t/2)$, for all $t \in [0,1]$, we have $\sup_{t\in[0,1]} \int_0^1 G(t,s)ds = 1/8$. It follows that

$$\|Tx - Ty\|_\infty \leq \psi(\|x - y\|_\infty), \tag{50}$$

for each $x, y \in C([0,1])$, such that $\phi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$.

Let $\alpha, \eta : C([0,1]) \times C([0,1]) \to [0,\infty)$ be mappings defined by

$$\begin{aligned} \alpha(x,y) &= \begin{cases} 1, & \phi(x(t), y(t)) \geq 0,\ t \in [0,1]; \\ 0, & \text{otherwise}, \end{cases} \\ \eta(x,y) &= \begin{cases} \frac{1}{2}, & \phi(x(t), y(t)) \geq 0,\ t \in [0,1]; \\ 2, & \text{otherwise}. \end{cases} \end{aligned} \tag{51}$$

Let $x, y \in C([0,1])$ such that $\alpha(x,y) \geq \eta(x,y)$. This implies that $\phi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$. Therefore,

$$\|Tx - Ty\|_\infty \leq \psi(\|x - y\|_\infty). \tag{52}$$

Furthermore, if $x, y \in C([0,1])$ such that $\alpha(x,y) \geq \eta(x,y)$, then by using (iii) we have

$$\begin{aligned} &\phi(Tx(t), Ty(t)) \geq 0 \\ &\text{and this yields } \alpha(Tx, Ty) \geq \eta(Tx, Ty). \end{aligned} \tag{53}$$

It follows that T is α-admissible with respect to η. By (ii), there exists $x_0 \in C([0,1])$ such that

$$\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0). \tag{54}$$

Applying Theorem 12, we obtain that T has a fixed point in $C([0,1])$; say x. Hence, x is a solution of (42). □

Corollary 25. *Assume that the following conditions hold:*

(i) $f : [0,1] \times \mathbb{R} \to [0,\infty)$ *is continuous and nondecreasing;*

(ii) *for all* $t \in [0,1]$, *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$, *one has*

$$|f(t,a) - f(t,b)| \leq 8\psi\left(\max_{a,b\in\mathbb{R}, a\leq b} |a - b| \right), \tag{55}$$

where $\psi \in \Psi_2$;

(iii) *there exists* $x_0 \in C([0,1])$ *such that, for all* $t \in [0,1]$, *one has*

$$x_0(t) \leq \int_0^1 G(t,s) f(s, x_0(s))\, ds. \tag{56}$$

Then, (42) *has a unique solution in* $C^2([0,1])$.

Proof. Define a mapping $\phi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi(a,b) = b - a \quad \forall a, b \in \mathbb{R}. \tag{57}$$

By the analogous proof, as in Theorem 24, we obtain that (42) has at least one solution. Since, for each $x, y \in C([0,1])$, there exists a mapping $z = \max\{x, y\}$ such that $\alpha(x,z) \geq \eta(x,z)$ and $\alpha(y,z) \geq \eta(y,z)$. This implies that the solution of (42) is unique by Theorem 13. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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