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Research Article

An Extension of Hypercyclicity for N-Linear Operators

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Grosse-Erdmann and Kim recently introduced the notion of bihypercyclicity for studying the existence of dense orbits under bilinear operators. We propose an alternative notion of orbit for N-linear operators that is inspired by difference equations. Under this new notion, every separable infinite dimensional Fréchet space supports supercyclic N-linear operators, for each $N \geq 2$. Indeed, the nonnormable spaces of entire functions and the countable product of lines support N-linear operators with residual sets of hypercyclic vectors, for $N = 2$.

1. Introduction

The study of linear dynamics has attracted the interest of a number of researchers from different areas over the past decades. Despite the several isolated examples in the literature due to Birkhoff [1], MacLane [2], and Rolewicz [3], it was not until the eighties with the unpublished Ph.D. thesis of Kitai [4] and the papers by Beauzamy [5] and by Gethner and Shapiro [6] when the notion of hypercyclicity started to become popular among mathematicians devoted to operator theory and functional analysis. This interest was fostered by the extension of the definition of chaos in the sense of Devaney to the linear setting by Godefroy and Shapiro [7]. The state of the art on linear dynamics was first described by Grosse-Erdmann in [8] and revisited in [9]; see also [10]. Evidences of the area's maturity are the recent monographs of Bayart and Matheron [11] and of Grosse-Erdmann and Peris-Manguillot [12].

Throughout this paper, X denotes an infinite-dimensional separable Fréchet space over the real or complex scalar field $\mathbb{K}$ and $L(X)$ denorbtes the space of linear and continuous operators on X, endowed with the topology of uniform convergence over bounded sets. We recall that a linear operator $T \in L(X)$ is said to be *hypercyclic* if there exists some $x \in X$ such that the orbit $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, T^3x, \ldots\}$ is dense in X. This notion is equivalent to transitivity in the linear setting by the Birkhoff Transitivity Theorem; see for instance [12, Theorem 1.16].

The notion of hypercyclicity is related to the ones of supercyclicity and cyclicity that appeared in connection with the invariant subspace problem; see for instance [13, 14]. We recall that an operator $T \in L(X)$ is said to be *cyclic* if there is some $x \in X$ such that its orbit has dense linear span (i.e., $\overline{\mathrm{span}\{\mathrm{Orb}(T, x)\}} = X$) and it is said to be *supercyclic* if there exists a vector $x \in X$ such that the set of scalar multiples of the orbit is dense (i.e., $\overline{\{\lambda T^n x : \lambda \in \mathbb{K}, n \in \mathbb{N}_0\}} = X$).

Apart from these, several notions—either adapted from other areas to the linear setting or conceived within linear dynamics themselves—have appeared in recent years to describe the dynamic behaviour of a linear operator: weak mixing [15, 16], frequent hypercyclicity [17], disjoint hypercyclicity [18, 19], distributional chaos [20], the specification property [21], irregular vectors [22], Li Yorke chaos and distributionally irregular vectors [23], and bihypercyclicity [24], among others.

The notion of bihypercyclicity, due to Grosse-Erdmann and Kim, was introduced to extend the notion of hypercyclicity from operators to bilinear operators. Let us recall their

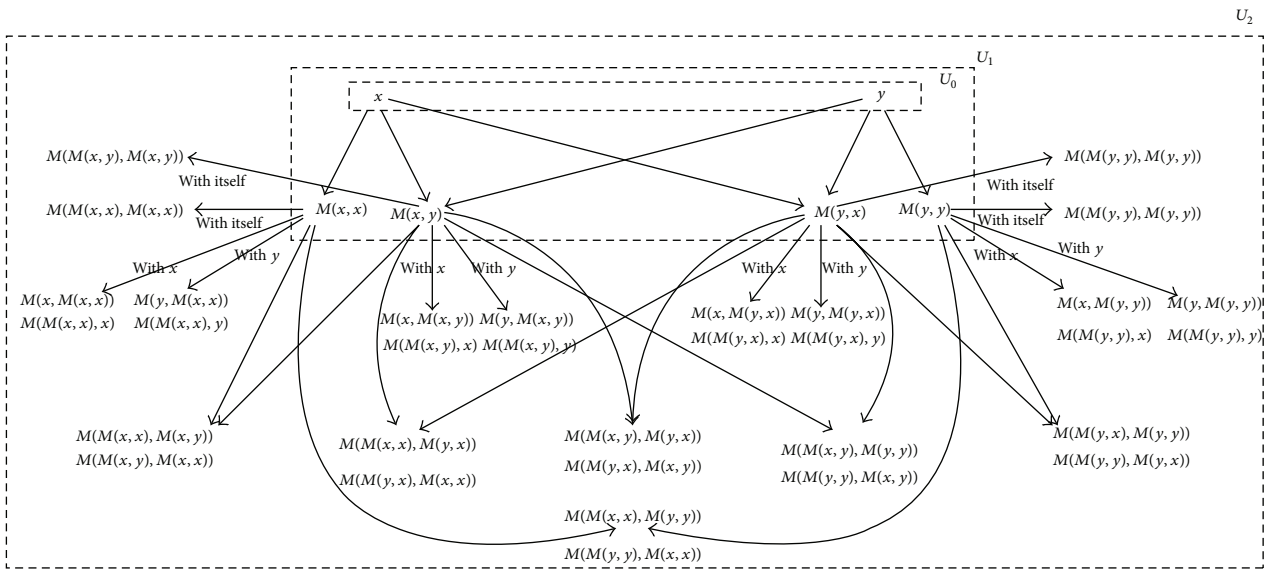

Figure 1: Iterates of a pair (x, y) under a bilinear operator M in the sense of Grosse-Erdmann and Kim [24].

notion of orbit for a given bilinear operator $M : X \times X \to X$. Given $(x, y) \in X \times X$, we define the sequence of sets $\{U_n\}_n$ as follows:

$$U_0 := \{x, y\},$$
$$U_n := U_{n-1} \cup \{M(u, v) : u, v \in U_{n-1}\} \quad \forall n \geq 1. \tag{1}$$

So

$$\begin{aligned} U_0 &= \{x, y\}, \\ U_1 &= U_0 \cup \{M(x,x), M(x,y), M(y,x), M(y,y)\}, \\ U_2 &= U_1 \cup \{M(x, M(x,x)), M(x, M(x,y)), \\ &\qquad M(x, M(y,x)), M(x, M(y,y)), \\ &\qquad M(y, M(x,x)), M(y, M(x,y)), \ldots, \\ &\qquad M(M(x,x),x), M(M(x,y),x), \ldots\}, \end{aligned} \tag{2}$$

and so on. Figure 1 shows an organized scheme for computing the items in U_0, U_1, and U_2. Then, the orbit of (x, y) under M, in the sense of Grosse-Erdmann and Kim, is defined as

$$\mathrm{Orb}_{\text{GE-K}}(M, (x, y)) := \bigcup_{n=0}^{\infty} U_n. \tag{3}$$

The bilinear operator M is said to be *bihypercyclic* if there exists a pair $(x, y) \in X \times X$ whose orbit $\mathrm{Orb}_{\text{GE-K}}(M, (x, y))$ is dense in X. If so, (x, y) is called a *bihypercyclic vector* for M.

Under this definition, the authors in [24] proved several interesting results such as the lack of density of the set of bihypercyclic vectors of $X \times X$ for a bihypercyclic operator [24, Theorem 1], the bihypercyclicity of nonzero scalar multiples of a bihypercyclic operator [24, Theorem 2], and the existence of bihypercyclic operators on finite-dimensional spaces [24, Theorem 10] and on any separable Banach space [24, Theorem 11]. They also provided a general method for constructing bihypercyclic bilinear operators [24, Proposition 4]; if one can find a vector $x \in X$ such that the operator $T(\cdot) := M(x, \cdot)$ is hypercyclic on X, then M is bihypercyclic on $X \times X$. This follows by noting that the orbit of a vector y under T lies inside the orbit of (x, y) under M.

Nevertheless, the notion of iterating an N-linear mapping is not evident when $N \geq 2$ and other interpretations may be worth considering. Our attempt is to consider an orbit of a pair (x, y) under a bilinear operator in such a way that the iterates can be arranged sequentially, and not in a network shape as Figure 1 shows.

In what remains, we consider the following notion of orbit for an N-linear operator, inspired in difference equations. Whereas each state of a discrete dynamical system given by an operator $T \in L(X)$ is determined by one preceding state (i.e., $x_n = Tx_{n-1}$), a state in a system given by an N-linear operator $M \in L(X^N, X)$ relies on N preceding states (i.e., $u_n = M(u_{n-N}, \ldots, u_{n-1})$).

Definition 1 (orbit under an N-linear operator). Let $M : X^N \to X$ be an N-linear operator, where $N \geq 2$. Each N-tuple $(x_1, x_2, \ldots x_N) \in X^N$ determines a unique sequence $\{u_n\}_n \subset X$ satisfyin

$$\begin{aligned} u_i &= x_i \quad \text{for } i = 1, 2, \ldots, N, \\ u_i &= M(u_{i-N}, u_{i-N+1}, \ldots, u_{i-1}) \quad \text{for } i > N. \end{aligned} \tag{4}$$

We say that $\{u_n\}_n$ is the N-linear orbit for M with initial conditions $(x_1, x_2, \ldots, x_N)$ and denote it $\{u_n\}_n = \mathrm{Orb}(M, (x_1, x_2, \ldots, x_N))$.

For the case of a bilinear operator, this definition of orbit is simpler than the one used for bihypercyclicity, thanks to the linear order in computing the "iterates" of the initial conditions, as Figure 2 shows.

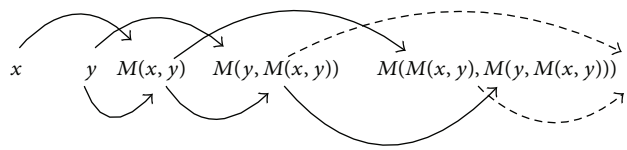

FIGURE 2: Another way of iterating a pair of initial conditions (x, y) under a bilinear operator $M : X \times X \to X$.

With this new type of orbit, it is natural to consider the following definition.

Definition 2. One says that an N-linear operator $M : X^N \to X$ is *hypercyclic* if there exists an N-tuple $x = (x_1, x_2, \ldots, x_N) \in X^N$ whose orbit (in the sense of Definition 1)

$$\{u_n\}_n = \mathrm{Orb}\,(M, x) \tag{5}$$

is dense in X. If $\mathbb{K} \cdot \mathrm{Orb}(M, x) = \{\lambda u_n : \lambda \in \mathbb{K}, n \in \mathbb{N}\}$ is dense, we say that M is *supercyclic*. Such a vector $x \in X^N$ is said to be hypercyclic or supercyclic for M, respectively.

Definition 1 provides the following connection with the theory of universal sequences.

Remark 3. Given a continuous N-linear map $M : X^N \to X$, consider the sequence $\{L_n\}_n$ of continuous maps $L_n : X^N \to X$ inductively defined as follows: for $j = 1, \ldots, N$, $L_j(x_1, \ldots, x_N) = x_j$ is the projection of the jth coordinate of X^N onto X. For $n > N$ and $z \in X^N$, we let

$$L_n(z) = M\left(L_{n-N}(z), \ldots, L_{n-1}(z)\right). \tag{6}$$

Then, the orbit of a vector $x = (x_1, \ldots, x_N) \in X^N$ under M is precisely the "orbit" of x under the action of $\{L_n\}_n$. That is,

$$\{u_n\}_n = \mathrm{Orb}\,(M, x) = \{L_n x\}_n. \tag{7}$$

In particular, x is hypercyclic for M if and only if it is universal for the sequence $\{L_n\}_n$ in $C(X^N, X)$ and a similar observation holds for the supercyclic case.

Given that the set of universal vectors for a given sequence $\{L_n\}_n$ in $C(X^N, X)$ is either residual or not dense [8, Proposition 6], we immediately have the following consequence.

Proposition 4. *Let M be a hypercyclic N-linear operator on a Fréchet space X, where $N \geq 2$. Then the set*

$$HC(M) = \left\{x \in X^N : \mathrm{Orb}\,(M, x) \text{ is dense in } X\right\} \tag{8}$$

of hypercyclic vectors for M is either residual in X^N or not dense in X^N.

On Section 2, we show that every separable infinite-dimensional Fréchet space supports a supercyclic N-linear operator, for any $N \geq 2$ (Theorem 5). On Section 3, we show that the space $\omega = \mathbb{K}^{\mathbb{N}}$, the countably infinite product of lines, supports hypercyclic N-linear operators, for any $N \geq 2$. We also show that in contrast with the set of bihypercyclic vectors being a nondense G_δ set [24, Theorem 1], the set of hypercyclic vectors for an N-linear operator can be residual on X^N. On Section 4, we show that the space $H(\mathbb{C})$ of entire functions—which unlike ω supports continuous norms—does support hypercyclic N-linear operators, for $N = 2$.

2. Existence of Supercyclic N-Linear Operators

Theorem 5. *Every separable infinite-dimensional Fréchet space X supports, for each $N \geq 2$, an N-linear operator having a residual set of supercyclic vectors.*

The proof of Theorem 5 makes use of the following result by Bonet and Peris, which has been a key ingredient to prove the existence of hypercyclic operators on Fréchet spaces different from ω.

Lemma 6 (see [25, Lemma 2]). *Let X be a separable infinite dimensional Fréchet space $X \neq \omega$. There are sequences $\{y_n\}_n \subset X$ and $\{f_n\}_n \subset X'$ such that*

(1) *$\{y_n\}_n$ converges to 0 in X and* $\mathrm{span}\,\{y_n : n \in \mathbb{N}\}$ *is dense in X;*

(2) *$\{f_n\}_n$ is X-equicontinuous in X';*

(3) *$f_m(y_n) = 0$ if $n \neq m$ and $f_n(y_n) \in]0, 1[$ for all $n \in \mathbb{N}$.*

With this notation, Bonet and Peris proved that the operator

$$Sy = y + \sum_{n=1}^{\infty} \frac{1}{2^n} f_{n+1}(y)\, y_n \tag{9}$$

resulted in being hypercyclic on X. Lemma 6 can be compared with the well-known result by Ovsepian and Pełczyńsky on the existence of a fundamental total and bounded pair of biorthogonal sequences on separable Banach spaces [26]. This last result was used by Herzog to show that every infinite-dimensional separable Banach space supports a supercyclic operator [27], which was of the form

$$Ry = \sum_{n=1}^{\infty} \frac{1}{2^n} \tilde{f}_{n+1}(y)\, \tilde{y}_n, \tag{10}$$

where $\{\tilde{y}_n\}_n$ and $\{\tilde{f}_n\}_n$ were a pair of biorthogonal sequences given by the Ovsepian and Pełczyńsky result. This operator R is in fact a *generalized backward shift* operator [7]. In [28], Salas extended to these types of operators previous results, due to Hilden and Wallen, on the supercyclicity of unilateral backward weighted shifts on ℓ_p spaces [14]. The supercyclicity of generalized backward shift operators can be characterized in terms of having R dense range or verifying the supercyclicity criterion [29].

Using the operator R, but in this case using the sequences given by Bonet and Peris lemma instead of the ones by Ovsepian and Pełczyńsky, and taking again the tensor product approach, we can prove that every separable infinite

dimensional Fréchet space X supports an N-supercyclic operator. We point out that the case $X = \omega$ will be established once we show Example 9.

We also use the following lemma, due to Grosse-Erdmann [30].

Lemma 7. *Let $\{L_n\}_n$ be a sequence of continuous mappings $L_n : X \to Y$, $n \in \mathbb{N}$, where X and Y are topological vector spaces. If X is a Baire space and Y is metrizable, then the set of universal vectors for $\{L_n\}_n$ is residual in X if and only if the set $\{(x, L_n x) : x \in X, n \in \mathbb{N}\}$ is dense in $X \times Y$.*

Proof of Theorem 5. Consider the N-linear operator $M = f_1 \otimes \cdots \otimes f_1 \otimes R$, where $N \geq 2$ is fixed. That is, $M : X^N \to X$ is given by

$$M(x_1, x_2, \ldots, x_N) = f_1(x_1) f_1(x_2) \cdots f_1(x_{N-1}) R x_N. \tag{11}$$

Notice that, for any given vectors $a_1, \ldots, a_N$ in X, the orbit

$$\operatorname{Orb}(M, (a_1, \ldots, a_N)) = (u_n)_{n \geq 1} \tag{12}$$

satisfies

$$\begin{gathered}(u_1, \ldots, u_N) = (a_1, \ldots, a_N), \\ u_{N+\ell} = C_\ell R^\ell a_N, \quad \forall \ell \in \mathbb{N},\end{gathered} \tag{13}$$

with the scalar C_ℓ depending only on the scalars in the set

$$A_\ell = \{f_1(a_j)\}_{1 \leq j \leq N-1} \cup \{f_j(a_N)\}_{1 \leq j \leq \ell-1}, \tag{14}$$

where $C_\ell = 0$ if and only if $0 \in A_\ell$. Notice also that for any $y \in \operatorname{span}\{y_1, y_2, \ldots\}$ of the form $y = \sum_{n=1}^{r} a_i y_i$ with $r \in \mathbb{N}$ and $k \in \mathbb{N}$ we have

$$R^k y = \sum_{n=1}^{r-k} a_{n+k} D_{k,n} F_{k,n} y_n, \tag{15}$$

where $D_{k,n} = 2^{-k(2n+k-1)/2}$ and $F_{k,n} = \prod_{i=1}^{k} f_{n+i}(y_{n+i})$ for each $k > r$ and $R^k y = 0$ for each $k \leq r$. Now, let $a_1, \ldots, a_N$, and b be vectors in the linear span of $\{y_1, y_2, \ldots\}$ and let $\varepsilon > 0$ be given. By Remark 3 and Lemma 7, it suffices to show there exist vectors $u_1, \ldots, u_N$ in X, a scalar λ, and a positive integer n so that

$$\begin{gathered}d_{X^N}((u_1, \ldots, u_N), (a_1, \ldots, a_N)) < \epsilon, \\ d(\lambda L_n(u_1, \ldots, u_N), b) < \epsilon,\end{gathered} \tag{16}$$

where d_{X^N} and d are the metrics on X^N and X, respectively, and $\{L_k\}_k$ is the sequence in $C(X^N, X)$ associated with M by relation (6).

Without loss of generality, we may assume that $b = \sum_{n=1}^{r} b_n y_n$ and that $a_j = \sum_{n=1}^{r} a_{j,n} y_n$ with $a_{j,n} \neq 0 \neq b_n$ for each $j = 1, \ldots, N$ and $n = 1, \ldots r$ for some $r \in \mathbb{N}$. Consider the vectors $u_j = a_j$ for $1 \leq j \leq N-1$ and

$$u_N = a_N + \delta \sum_{j=1}^{r} \frac{b_j}{D_{r,j} F_{r,n}} y_{j+r}, \tag{17}$$

where $\delta > 0$ is chosen small enough so that $d(u_N, a_N) < \epsilon$. Then

$$L_{N+r}(u_1, \ldots, u_N) = u_{N+r} = C_r R^r u_N = C_r \delta b, \tag{18}$$

and by (14) the scalar C_r is nonzero, as

$$\begin{gathered}0 \notin A_r = \{f_1(a_1), \ldots, f_1(a_{N-1}), f_1(b), \\ f_2(b), \ldots, f_{r-1}(b)\}.\end{gathered} \tag{19}$$

So (16) is satisfied taking $n = N + r$ and $\lambda = 1/C_r \delta$. □

3. Hypercyclic N-Linear Operators on ω

Let us consider the space ω endowed with the product topology. This can be given either by the metric

$$\begin{gathered}d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x(i) - y(i)|}{1 + |x(i) - y(i)|} \quad \text{for every } x = (x(i))_i, \\ y = (y(i))_i \in \omega,\end{gathered} \tag{20}$$

or by the family of continuous seminorms $\{p_n\}_{n \in \mathbb{N}}$ defined as

$$p_n(x) := \sup\{|x(i)|;\ i \leq n\} \quad \text{for every } x \in \omega. \tag{21}$$

Let $\mathcal{D} := \{g_n\}_n = \{(g_n(1), g_n(2), \ldots)\}_n$ denote a countable dense subset of ω satisfying $g_n(j) = 0$ if and only if $j > n$. Also, B denotes the unweighted backward shift operator defined over a sequence of numbers $x = (x(1), x(2), x(3), \ldots)$ as

$$B(x(1), x(2), x(3), \ldots) = (x(2), x(3), x(4), \ldots). \tag{22}$$

The hypercyclicity phenomenon on ω has been already considered as a particular case of Fréchet spaces, since it is the furthest Fréchet space from having a continuous norm; see, for instance, [25, 31–33]. The main result of this section is the following.

Theorem 8. *For each integer $N \geq 2$, there exists an N-linear operator on ω that supports a dense N-linear orbit.*

We show Theorem 8 by providing two examples of such operators. The first one is constructed as a tensor product like [24, Example 1].

Example 9. Let $N \geq 2$ be fixed and consider the N-linear operator $M = e_1' \otimes \cdots \otimes e_1' \otimes B$ on ω, where e_1' is the first coordinate functional on ω. That is, $M : \omega^N \to \omega$ is given by

$$M(x_1, x_2, \ldots, x_N) = e_1'(x_1) e_1'(x_2) \cdots e_1'(x_{N-1}) B x_N. \tag{23}$$

Then, M is hypercyclic. To see this, notice that for any vector x in ω the orbit

$$\{u_n\}_n = \operatorname{Orb}(M, (e_1, \ldots, e_1, x)) \tag{24}$$

is the following: $u_1 = \cdots = u_{N-1} = e_1$, $u_N = x$, $u_{N+1} = Bx$, and for each $\ell \geq N + 2$ we have

$$u_\ell = C_\ell B^{\ell - N} x, \tag{25}$$

where

$$C_\ell := x(1)^{F_{\ell-N-1}} x(2)^{F_{\ell-N-2}} \cdots x(\ell - N - 1)^{F_1} \tag{26}$$

and where $\{F_n\}_n$ is the Fibonacci sequence of order N and seed

$$\left(1, 2, \ldots, 2^{N-2}, 2^{N-1} - 1\right). \tag{27}$$

That is, $\{F_n\}_n$ is recursively defined by

$$F_n = \begin{cases} 2^{n-1}, & \text{if } n = 1, 2, \ldots, N-1, \\ 2^{n-1} - 1, & \text{if } n = N, \\ F_{n-N} + F_{n-N+1} + \cdots + F_{n-1}, & \text{if } n > N. \end{cases} \tag{28}$$

We now construct $x \in \omega$ so that $\mathrm{Orb}((e_1, \ldots, e_1, x), M)$ is dense in ω, as follows: let $\{n_k\}_k$ be a sequence of integers so that $n_1 := 1$ and $n_{k+1} > (k+N)^2 + n_k$ for all $k \in \mathbb{N}$ and let $\mathscr{D} = \{g_n\}_n$ be a dense sequence in ω so that each g_n satisfies $g_n(j) \neq 0$ if and only if $1 \leq j \leq n$.

We define the first $n_2 - N$ coordinates of x by

$$\left(x(1), x(2), \ldots, x(n_2 - N)\right) = \left(g_1(1), 1, \ldots, 1\right), \tag{29}$$

and for $N + 2 \leq \ell \leq n_2 - N$ we let C_ℓ be defined as in (26).

Next, we define

$$\begin{aligned} &\left(x(n_2 - N + 1), \ldots, x(n_3 - N)\right) \\ &\quad = \left(\frac{g_2(1)}{C_{n_2-N}}, \frac{g_2(2)}{C_{n_2-N}}, 1, \ldots, 1\right), \end{aligned} \tag{30}$$

and for $n_2 - N + 1 \leq \ell \leq n_3 - N$ we again let C_ℓ be defined by (26). Inductively, having defined $x(j)$ and C_j for $n_{k-1} - N < j \leq n_k - N$, we define $x(j)$ and C_j for $n_k - N < j \leq n_{k+1} - N$ as

$$\begin{aligned} &x(j) \\ &= \begin{cases} \dfrac{g_{k+1}(j - n_k + N)}{C_{n_k-N}}, & \text{if } n_k - N < j \leq n_k - N + k + 1, \\ 1, & \text{if } n_k - N + k + 2 \leq j \leq n_{k+1} - N, \end{cases} \end{aligned} \tag{31}$$

and C_j is defined as in (26). So, we have defined x in ω of the form

$$\begin{aligned} x = \Bigg(&g_1(1), 1, \ldots, 1, \frac{g_2(1)}{C_{n_2-N}}, \frac{g_2(2)}{C_{n_2-N}}, 1, \ldots, 1, \frac{g_3(1)}{C_{n_3} - N}, \\ &\frac{g_3(2)}{C_{n_3} - N}, \frac{g_3(3)}{C_{n_3} - N}, 1, \ldots \Bigg). \end{aligned} \tag{32}$$

Finally, to prove the denseness of $\mathrm{Orb}(M, (e_1, \ldots, e_1, x))$ we take an arbitrary y in ω and $\epsilon > 0$. Let $k \in \mathbb{N}$ be large enough so that $\sum_{j \geq k}(1/2^j) < \epsilon/2$ and $d(g_k, y) < \epsilon/2$. Then,

$$d\left(u_{n_k}, y\right) \leq \frac{\epsilon}{2} + d\left(u_{n_k}, g_k\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{33}$$

since by (25) the first k coordinates of $u_{n_k} - g_k$ are all zero.

Remark 10. We note that the set $\mathrm{HC}(M)$ of hypercyclic vectors for the operator M of the previous example is residual in ω^N.

Proof of Remark 10. By Proposition 4, it suffices to verify that $\mathrm{HC}(M)$ is dense in ω^N. The orbit $\{U_n\}_n = \mathrm{Orb}(M, x)$ of a given $x = (x_1, \ldots, x_N)$ in X^N is given by $(U_1, \ldots, U_N) = (x_1, \ldots, x_N)$, $U_{N+1} = x_1(1)x_2(1) \cdots x_{N-1}(1)Bx_N$, and for $\ell \geq 2$

$$U_{N+\ell} = A_\ell Q_\ell B^\ell x_N, \tag{34}$$

where

$$\begin{aligned} A_\ell &= \prod_{j=1}^{N-2} \left(x_j(1)\right)^{F_{j,\ell}}, \\ Q_\ell &= \left(x_{N-1}(1)\right)^{F_{N,\ell}} \prod_{k=1}^{\ell-1} \left(x_N(k)\right)^{F_{N,\ell-k}} \end{aligned} \tag{35}$$

and where each $\{F_{k,\ell}\}_\ell$ is an N-Fibonacci sequence as follows.

For $1 \leq k \leq N-1$,

$$\begin{aligned} F_{k,1} &= 1 \\ F_{k,2} &= 2 \\ F_{k,3} &= 2^2 \\ &\vdots \\ F_{k,k-1} &= 2^{k-2} \\ F_{k,k} &= 2^{k-1} - 1 \\ F_{k,k+1} &= F_{k,1} + F_{k,2} + \cdots + F_{k,k} \\ F_{k,k+2} &= F_{k,1} + F_{k,2} + \cdots + F_{k,k+1} \\ &\vdots \\ F_{k,N} &= F_{k,1} + F_{k,2} + \cdots + F_{k,N-1} \\ F_{k,\ell} &= F_{k,\ell-N} + F_{k,\ell-N+1} + \cdots + F_{k,\ell-1} \quad \text{for } \ell \geq N+1. \end{aligned} \tag{36}$$

For $k = N$,

$$\begin{aligned} F_{N,1} &= 1 \\ F_{N,2} &= 2 \\ F_{N,3} &= 2^2 \\ &\vdots \\ F_{N,N-1} &= 2^{N-2} \\ F_{N,N} &= 2^{N-1} - 1 \\ F_{N,\ell} &= F_{N,\ell-N} + F_{N,\ell-N+1} + \cdots + F_{N,\ell-1} \quad \text{for } \ell \geq N+1. \end{aligned} \tag{37}$$

Let $a_1, \ldots, a_N$ in span$\{e_1, e_2, \ldots\}$ be given and $\epsilon > 0$. We want to find some $x = (x_1, \ldots, x_N)$ in HC(M) with

$$d\left(a_j, x_j\right) < \epsilon \quad \text{for } j = 1, \ldots, N. \tag{38}$$

Let r be large enough so that $\sum_{k \geq r}(1/2^k) < \varepsilon$. Perturbing each a_j if necessary, we may assume without loss of generality that $e_1'(a_j) = a_j(1) \neq 0$ for $j = 1, \ldots, N$ and that $a_N = (a_N(j))_j$ with $a_N(j) \neq 0$ if and only if $1 \leq j \leq r$. Let $\mathscr{D} := \{g_n\}_n = \{(g_n(1), g_n(2), \ldots)\}_n$ denote a countable dense subset of ω satisfying $g_n(j) = 0$ if and only if $j > n$ and let $\{n_k\}_k$ be a sequence of positive integers satisfying $n_{k+1} > (r + N + k)^2 + n_k$. Consider the vectors $x_j = a_j$ for $j = 1, \ldots N - 1$ and

$$x_N = \Bigg(a_N(1), \ldots, a_N(r), \underbrace{1, \ldots, 1}_{n_1 - N - r - 1}, x_N(n_1 - N), g_1(1), \underbrace{1, \ldots, 1}_{n_2 - n_1 - 2}, x_N(n_2 - N), g_2(1), g_2(2), 1, \ldots\Bigg), \tag{39}$$

where the coordinates $x(n_k - N)$ $(k \in \mathbb{N})$ of x_N are to be determined. Notice that $d(a_N, x_N) < \epsilon$ by our selection of r regardless of how the $x(n_k - N)$'s are chosen. Now each scalar $C_\ell = A_\ell Q_\ell$ in (34) depends only on

$$\left\{x_j(1)\right\}_{1 \leq j \leq N} \cup \left\{x_N(i)\right\}_{1 \leq i \leq \ell - 1} \tag{40}$$

and every scalar C_ℓ will be nonzero as long as each of the $x(n_k - N)$'s are nonzero. In particular, by (34) we can define such $\{x_N(n_k - N)\}_k$ so that

$$U_{N+n_k}(j) = g_k(j) \quad \text{for each } j = 1, \ldots, k \text{ and each } k \geq 2. \tag{41}$$

So $x = (x_1, \ldots, x_N)$ is in HC(M) and the conclusion follows. □

We next provide another example of a hypercyclic N-linear operator on ω, avoiding the tensor product technique. In Example 11, each coordinate of an element in X^N is used to conform the iteration under the multilinear operator M. This allows us to provide a much simpler expression of some initial conditions that yield a dense orbit under M. We point out that since roots of different orders are taken, we only consider this example on $\omega = \mathbb{C}^{\mathbb{N}}$.

Example 11. Let $N \geq 2$ be fixed and consider the N-linear operator $M : \omega^N \to \omega$, where $\omega = \mathbb{C}^{\mathbb{N}}$, given by

$$M\left(\left(x_1(i)\right)_i, \ldots, \left(x_N(i)\right)_i\right) := \left(\prod_{\ell=1}^{N} x_\ell(i + N + \ell + 1)\right)_i. \tag{42}$$

Let $\{n_k\}_k$ be a sequence of integers so that $n_1 := 1$ and $n_{k+1} > k^2 + n_k$ for all $k \in \mathbb{N}$ and let $\{G_n\}_n$ denote the Fibonacci sequence of integers recursively defined by

$$G_n := \begin{cases} 1, & \text{if } n = 1, 2, \ldots, N, \\ G_{n-N} + G_{n-N+1} + \cdots + G_{n-1}, & \text{if } n > N. \end{cases} \tag{43}$$

Taking again the set $\mathscr{D}$, we consider the vector $x = (x(n))_n$ in ω given by

$$x(i) = \begin{cases} g_{k,j}^{1/G_{n_k}}, & \text{if } i = n_k + j \text{ for } (k, j) \in \mathbb{N}^2,\ j \leq k, \\ 0, & \text{otherwise.} \end{cases} \tag{44}$$

That is, x is of the form

$$x := \left(g_{1,1}^{1/G_{n_1}}, 0, \ldots, 0, g_{2,1}^{1/G_{n_2}}, g_{2,2}^{1/G_{n_2}}, 0, \ldots, 0, g_{3,1}^{1/G_{n_3}}, g_{3,2}^{1/G_{n_3}}, g_{3,3}^{1/G_{n_3}}, 0, \ldots\right), \tag{45}$$

where in each case $g_{k,j}^{1/G_{n_k}}$ denotes any of the roots of $z^{G_{n_k}} - g_{k,j}$. By (42), the N-linear orbit of the initial conditions $(x, Bx, \ldots, B^{N-1}x)$ under M is given by

$$u_n = \left(x_n^{G_n}, x_{n+1}^{G_n}, x_{n+2}^{G_n}, \ldots\right) \quad \forall n > N, \tag{46}$$

which is dense in ω. This last part can be proved in a similar way as we did with Example 9.

4. Hypercyclic N-Linear Operators on $\mathscr{H}(\mathbb{C})$

One may wonder whether lacking a continuous norm, such as ω does, is a requirement for a space to support hypercyclic N-linear operators. We answer this in the negative, with the following.

Theorem 12. *The space $\mathscr{H}(\mathbb{C})$ supports a hypercyclic bilinear operator.*

The proof of Theorem 12 relies on Lemmas 13 and 14. First, let us introduce some notation. Let us define the antiderivative operator on $\mathscr{H}(\mathbb{C})$ as $I(h)(z) = \int_0^z h(w)dw$ for every entire function $h(z)$. It is clear that, on the monomials z^n, $n \in \mathbb{N}_0$, this operator returns $I(z^n) = z^{n+1}/(n+1)$ and that $\lim_{k \to \infty} I^k(z^n) = 0$ in $H(\mathbb{C})$. Thus, the equicontinuity of any finite collection of iterates of the derivative operator on $H(\mathbb{C})$ immediately gives the following.

Lemma 13. *Let $P(z)$ be a complex polynomial. Let $R, \varepsilon > 0$ and let $j_0 \in \mathbb{N}$; there is some $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ and $Q(z) = I^k P(z)$, then*

$$\sup_{|z| \leq R} \left|Q^{(j)}(z)\right| < \varepsilon \quad \forall j = 1, \ldots, j_0. \tag{47}$$

If $\{F_k\}_k$ is a Fibonacci sequence of order $N \geq 2$ (i.e., $F_\ell = F_{\ell-1} + \cdots + F_{\ell-N}$ for $\ell > N$ where $F_1, \ldots, F_N \in \mathbb{N}$ are given), then

$$\varphi = \lim_{n \to \infty} \frac{F_n}{F_{n-1}}, \tag{48}$$

where φ is the real solution of $x + x^{-N} = 2$ that is closest to 2. Hence, by Lemma 13 we have the following.

Lemma 14. *Let $P(z)$ be a complex polynomial and let $\{F_n\}_n$ be the Fibonacci sequence defined in* (28). *Let $a_0, \ldots, a_s$ be a finite sequence of complex numbers satisfying*

$$\left|a_0^{\varphi^s} \cdot a_1^{\varphi^{s-1}} \cdots a_{s-1}^{\varphi} a_s\right| > 1, \tag{49}$$

where φ is the real solution of $x + x^{-N} = 2$ that is closest to 2. Let also $\varepsilon > 0$, $R > 0$, and $j_0 \in \mathbb{N}$. Then, there exists some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\sup_{|z|\leq R}\left|\frac{\left(I^k P\right)^{(j)}(z)}{a_0^{F_k} \cdot a_1^{F_{k-1}} \cdots a_s^{F_{k-s}}}\right| < \varepsilon, \tag{50}$$

for all $j = 1, \ldots, j_0$.

Proof of Theorem 12. Consider the bilinear operator $M : H(\mathbb{C}) \times H(\mathbb{C}) \to H(\mathbb{C})$,

$$M(g, f) = (\delta_0 \otimes D)(h, f) = g(0)\, Df, \tag{51}$$

where δ_0 is the evaluation at $z = 0$ and D is the operator of complex differentiation on $H(\mathbb{C})$. We seek a hypercyclic vector for M of the form (g, f), with $g \equiv 1$. Notice that for any $f(z) = \sum_{j=0}^{\infty} x_j (z^j/j!) \in H(\mathbb{C})$ the orbit $\{u_n\}_n = \mathrm{Orb}(M, (1, f))$ is given by $(u_1, u_2, u_3) = (1, f, Df)$ and

$$u_{3+n} = c_n D^{n+1} f, \tag{52}$$

where

$$c_n = \prod_{j=0}^{n-1} x_j^{F_{n-j}} \tag{53}$$

and where $\{F_n\}_n$ is the Fibonacci sequence in (28) given by $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for $n > 2$. The construction of the function $f(z)$ is inspired by the construction of a hypercyclic function for the derivative operator in [34]. We first consider a dense sequence of polynomials

$$\mathscr{P} = \left\{P_n(z) = \sum_{j=0}^{d_n} \frac{p_{n,j}}{j!} z^j\right\}_n \tag{54}$$

satisfying, for each $n \in \mathbb{N}$ that

(a) $d_{2n} = d_{2n-1} = n$, and $p_{2n-1,j} p_{2n,j} = 1$ $(j = 0, \ldots, n)$, and

(b) $0 < H_{2n} < 1 < H_{2n-1}$, where for each k the scalar H_k associated to P_k is defined as

$$H_k = H_k(P_k) = \prod_{j=0}^{d_k} \left|p_{k,j}\right|^{\varphi^{d_k - j}}. \tag{55}$$

(It is simple to see that such $\mathscr{P}$ exists: first get a dense sequence $\mathscr{P}_1 = \{Q_n(z) = \sum_{j=0}^{n} (q_{n,j}/j!) z^j\}_n$ and perturb if necessary the coefficients of each Q_n by at most 2^{-2n} so that $H_n \neq 0, 1$. The resulting sequence, which we call $\mathscr{P}_2$, is also dense. For each polynomial $Q_n(z) = \sum_{j=0}^{n} (q_{n,j}/j!) z^j$ in $\mathscr{P}_2$, the polynomial $\widehat{Q}_n(z) = \sum_{j=0}^{n} (1/j! q_{n,j}) z^j$ satisfies $H_n(\widehat{Q}_n) = 1/(H_n(Q_n))$. Then $\mathscr{P}$ results after reordering $\mathscr{P}_2 \cup \{\widehat{Q} : Q \in \mathscr{P}_2\}$.) We also consider another dense family of functions $\widetilde{\mathscr{P}} = \{\widetilde{P}_n(z)\}_n$ defined from the polynomials in $\mathscr{P}$ as

$$\widetilde{P}_n(z) := P_n(z) + \sum_{i=d_n+1}^{\infty} \frac{1}{i!} z^i. \tag{56}$$

Several Fibonacci sequences $\{F_{n,k}^{j,i}\}_k$ will be defined at each step n of the inductive construction of f, and so that $F_{n,k}^{j,i}$ denotes the power of the coefficient $p_{j,i}$ of P_j that appears in the factor c_n in the $(3+n)$-th iterate of the orbit of $(1, f)$ given by (53). All these Fibonacci sequences are of order $N = 2$, and thus we only define in the proof the first two non-zero terms of each such sequences. We also introduce the following notation:

$$\begin{aligned}
J^j &:= \left(p_{j,0}\right)^{\varphi^{d_j}} \cdot \left(p_{j,1}\right)^{\varphi^{d_j-1}} \cdots \left(p_{j,d_j-1}\right)^{\varphi} \cdot p_{j,d_j} \quad \text{for } k \in \mathbb{N},\\
J_{n,k}^j &:= \left(p_{j,0}\right)^{F_{n,k}^{j,0}} \cdot \left(p_{j,1}\right)^{F_{n,k}^{j,1}} \cdots \left(p_{j,d_j-1}\right)^{F_{n,k}^{j,d_j-1}} \cdot \left(p_{j,d_j}\right)^{F_{n,k}^{j,d_j}} \quad \text{for } j, n, k \in \mathbb{N},\\
H_{n,k}^j &:= \left|\left(p_{j,0}\right)^{F_{n,k}^{j,0}/F_{n,k}^{j,d_j}} \cdot \left(p_{j,1}\right)^{F_{n,k}^{j,1}/F_{n,k}^{j,d_j}} \cdots \left(p_{j,d_j-1}\right)^{F_{n,k}^{j,d_j-1}/F_{n,k}^{j,d_j}} \cdot \left(p_{j,d_j}\right)\right| \quad \text{for } j, n, k \in \mathbb{N}.
\end{aligned} \tag{57}$$

Clearly, the following relation holds

$$\left|J_{n,k}^j\right| = H_{n,k}^{j\ F_{n,k}^{j,d_j}} \quad \forall j, n, k \in \mathbb{N}. \tag{58}$$

Now, we are ready to construct $f(z)$.

Step 1. Let $k_1 = 0$, and define $Q_1(z) = P_1(z)$.

Step 2. We first let $\{F_{2,k}^{1,0}\}_k$ and $\{F_{2,k}^{1,1}\}_k$ denote the Fibonacci sequences whose first non-zero terms are $F_{2,k+i+1}^{1,i} = F_k$ for $i = 0, 1$ and $k = 1, 2$. Next we define $Q_2(z) = (I^{k_2+1}(P_2(z)))/J_{2,k_2}^1$, where $k_2 \in \mathbb{N}$ is chosen so that

(2.1) $k_2 > k_1 + d_1 + 2$,

(2.2) $|H_{2,k}^1 - H_1| < (H_1 - 1)/2$ for all $k \geq k_2 - 1$,

(2.3) $|J_{2,k}^1| > 1$ for all $k \geq k_2 - 1$, and

(2.4) $|Q_2(z)| < 1/2^2$ for all $|z| \leq 2$.

Conditions (2.2) and (2.3) are obtained thanks to (48) and the fact that $H_1 > 1$, and Condition (2.4) is obtained

from Lemma 14. We finish this step defining $\widetilde{Q}_1(z) = \sum_{i=d_1+1}^{k_2}(1/i!)z^i$. Notice that for any $h \in H(\mathbb{C})$ the orbit

$$\{\widetilde{u}_n\}_n = \mathrm{Orb}\left(M, \left(1, Q_1 + \widetilde{Q}_1 + Q_2 + I^{k_2+d_2}h\right)\right) \tag{59}$$

satisfies that $\Pi_{d_2}(\widetilde{u}_{3+k_2}) = P_2$, where for $k \geq 0$ we let $\Pi_k(g)(z) = \sum_{j=0}^{k} g^{(j)}(0)(z^j/j!)$ $(g \in H(\mathbb{C}))$.

Step 3. We let $\{F_{3,k}^{1,0}\}_k$ and $\{F_{3,k}^{1,1}\}_k$ denote the Fibonacci sequences whose first non-zero terms are given by

$$\begin{aligned} &F_{3,1}^{1,i} = F_{3,2}^{1,i} = F_{2,k_2-1}^{1,i} \quad \text{for } i = 0, 1, \\ &F_{3,k+i+1}^{2,i} = F_k \quad \text{for } i = 0, 1;\ k = 1, 2, \end{aligned} \tag{60}$$

and define $Q_3(z) = (I^{k_3+1}(P_3(z)))/(J_{3,k_3-k_2}^1 J_{3,k_3-k_2}^2)$, where $k_3 \in \mathbb{N}$ satisfies

(3.1) $k_3 + 1 > k_2 + d_2 + 2$,

(3.2) $|H_{3,k}^2 - H_2| < (1 - H_2)/2$ for all $k \geq k_3 - 1$.

(3.3) $|J_{3,k-k_2}^1 J_{3,k-k_2}^2| > 1$ for all $k \geq k_3 - 1$.

(3.4) $|Q_3(z)| < 1/2^3$, $|Q_3'(z)| < 1/2^3, \ldots, |Q_3^{(k_2)}(z)| < 1/2^3$ for all $|z| \leq 3$.

Condition (3.2) holds again thanks to (48). To see why Condition (3.3) can be obtained, recall that $0 < H_2 < 1 < H_1$ thanks to the order in which the polynomials $P_1 = p_{1,0} + p_{1,1}z$ and $P_2 = 1/p_{1,0} + (1/p_{1,1})z$ appear in the dense sequence $\mathscr{P}$ given in (54). Notice also that the quotients $F_{3,k}^{1,0}/F_{3,k}^{1,1}$ are eventually constant and coincide with $F_{2,k_2-1}^{1,0}/F_{2,k_2-1}^{1,1}$, so that $H_{3,k}^1$ coincides with H_{2,k_2-1}^1 and thus $H_{3,k}^1 > 1$ eventually by (2.2). On the other hand $H_{3,k}^2 < 1$ for all $k \geq k_2$ by Condition (3.2). All this together with (58) allow us to obtain (3.3), since $\lim_{k\to\infty} F_{3,k}^{1,d_1} - F_{3,k}^{2,d_2} = \infty$ by construction. Condition (3.4) can now be obtained as a combination of Lemma 13 and (3.3). Finally, we let $\widetilde{Q}_2(z) = (1/J_{2,k_2}^1)(\sum_{i=k_2+d_2+1}^{k_3}(1/i!)z^i)$, and as in Step 2 for any $h \in H(\mathbb{C})$ the orbit

$$\{\widetilde{u}_n\}_n = \mathrm{Orb}\left(M, \left(1, Q_1 + \widetilde{Q}_1 + Q_2 + \widetilde{Q}_2 + Q_3 + I^{k_3+d_3}h\right)\right) \tag{61}$$

satisfies that $\Pi_{d_3}(\widetilde{u}_{3+k_3}) = P_3$.

Step 4. We define the Fibonacci sequences needed in this step, $\{F_{4,k}^{j,i}\}_k$ with $j = 1, 2, 3$. The first two non-zero terms of each sequence are given by

$$\begin{aligned} &F_{4,1}^{j,i} = F_{4,2}^{j,i} = F_{3,k_3-1}^{j,i} \quad \text{for } j = 1, 2;\ i = 0, 1, \\ &F_{4,k+i+1}^{3,i} = F_k \quad \text{for } i = 0, 1, 2;\ k = 1, 2. \end{aligned} \tag{62}$$

We set $Q_4(z) = I^{k_4+1}(P_4(z))/J_{4,k_4-k_3}^1 J_{4,k_4-k_3}^2 J_{4,k_4-k_3}^3$, where $k_4 \in \mathbb{N}$ is chosen to satisfy the following:

(4.1) $k_4 + 1 > k_3 + d_3 + 2$,

(4.2) $|H_{4,k}^3 - H_3| < (H_3 - 1)/2$ for all $k \geq k_4 - 1$,

(4.3) $|J_{4,k-k_3}^1 J_{4,k-k_3}^2 J_{4,k-k_3}^3| > 1$ for all $k \geq k_4 - 1$, and

(4.4) $|Q_4(z)| < 1/2^4$, $|Q_4'(z)| < 1/2^4, \ldots, |Q_4^{(k_3)}(z)| < 1/2^4$ for all $|z| \leq 4$.

Condition (4.2) is given by (48). Let us proceed to check condition (4.3). The idea is very similar to the one used for condition (3.3). By the definition of the Fibonacci sequences in this step, the quotients $F_{4,k}^{1,0}/F_{4,k}^{1,1}$ are constant for every $k \in \mathbb{N}$ and coincide with $F_{3,k_3-1}^{1,0}/F_{3,k_3-1}^{1,1}$. Therefore $H_{4,k}^1 > 1$ for every $k \in \mathbb{N}$. Analogously, $F_{4,k}^{2,0}/F_{4,k}^{2,1}$ is constant for every $k \in \mathbb{N}$ and coincides with $F_{3,k_3-1}^{2,0}/F_{3,k_3-1}^{2,1}$. This gives $H_{4,k}^2 < 1$. Again, since $\lim_{k\to\infty} F_{4,k}^{1,d_1} - F_{4,k}^{2,d_2} = \infty$ by construction, then $|J_{4,k-k_3}^1 J_{4,k-k_3}^2| > 1$ for all $k \geq k_3$. In addition, $|J_{4,k-k_3}^3| > 1$ by condition (4.2) since H_3 is greater than 1 by definition. Combining these estimations we get (4.3). Lastly, condition (4.4) can be obtained from Lemma 13 using condition (4.3).

We also take $\widetilde{Q}_3(z) = (1/J_{2,k_3-k_2}^1 J_{2,k_3-k_2}^2)\sum_{i=k_3+d_3+1}^{k_4}(1/i!)z^i$ so that, as in Steps 2 and 3, for any $h \in H(\mathbb{C})$ the orbit

$$\begin{aligned} \{\widetilde{u}_n\}_n = \mathrm{Orb}\Big(M, \Big(&1, Q_1 + \widetilde{Q}_1 + Q_2 \\ &+ \widetilde{Q}_2 + Q_3 + \widetilde{Q}_3 + Q_4 + I^{k_4+d_4}h\Big)\Big) \end{aligned} \tag{63}$$

satisfies that $\Pi_{d_4}(\widetilde{u}_{3+k_4}) = P_4$.

Step n. Let us assume that we have done the previous $n - 1$ steps. Let us start with the definition of the Fibonacci sequences, $\{F_{n,k}^{j,i}\}_k$. The first non-zero terms of each sequence are the following ones:

$$\begin{aligned} &F_{n,1}^{j,i} = F_{n,2}^{j,i} = F_{n-1,k_{n-1}-1}^{j,i} \quad \text{for } j = 1, \ldots, n-1;\ i = 0, \ldots, d_j, \\ &F_{n,k+i+1}^{n-1,i} = F_k \quad \text{for } i = 0, \ldots, d_{n-1},\ k = 1, 2. \end{aligned} \tag{64}$$

We define $Q_n(z) = I^{k_n+1}(P_n(z))/\prod_{j=1}^{n-1} J_{n,k_n-k_{n-1}}^j$, where $k_n \in \mathbb{N}$ satisfies:

(n.1) $k_n + 1 > k_{n-1} + d_{n-1} + 2$,

(n.2.1) if n is odd, then $|H_{n,k}^{n-1} - H_{n-1}| < (1 - H_{n-1})/2$ for all $k \geq k_n - 1$,

(n.2.2) if n is even, then $|H_{n,k}^{n-1} - H_{n-1}| < (H_{n-1} - 1)/2$ for all $k \geq k_n - 1$,

(n.3) $|\prod_{j=1}^{n-1} J_{n,k-k_{n-1}}^j| > 1$ for all $k \geq k_n - 1$, and

(n.4) $|Q_n(z)| < 1/2^n$, $|Q_n'(z)| < 1/2^n, \ldots, |Q_n^{(k_{n-1})}(z)| < 1/2^n$ for all $|z| \leq n$.

Both conditions (n.2.1) and (n.2.2) can be deduced using the formula (48). For proving condition (n.3), we first prove

that $|J^{j_0}_{n,k-k_{n-1}} J^{j_0+1}_{n,k-k_{n-1}}| > 1$ for every odd number $j_0 \le n-2$ and $k \ge k_{n-1}$. This is due to the general fact that the quotients $F^{j,i}_{n,k}/F^{j,d_j}_{n,k}$ are constant for all $j \le n-1$ and for all $i = 0,\dots,d_j-1$, since each one coincides with $F^{j,i}_{n-1,k_n-1}/F^{j,d_j}_{n-1,k_n-1}$, respectively. So that $H^j_{n,k}$ coincides with $H^j_{n-1,k_{n-1}-1}$. By the previous steps $H^j_{n,k} > 1$ if j is odd and $H^j_{n,k} < 1$ if j is even. Taking this into account and since $\lim_{k\to\infty} F^{j_0,d_{j_0}}_{n,k} - F^{j_0+1,d_{j_0+1}}_{n,k} = \infty$, then $|J^{j_0}_{n,k-k_{n-1}} J^{j_0+1}_{n,k-k_{n-1}}| > 1$ for all k large enough.

Now, if $n-1$ is even, then condition (n.3) is obtained applying the previous argument $(n-1)/2$ with all the odd numbers $j_0 \le n-2$. If $n-1$ is odd, we apply the previous argument to all odd numbers $j_0 \le n-3$ and we have still to show that $|J^{n-1}_{n,k-k_{n-1}}| > 1$ for all k large enough, but this holds by condition (n.2.2) and (58) since $H_{n-1} > 1$ by definition. Combining these estimations we get (n.3). As before, condition (n.4) can be obtained from Lemma 13 using condition (n.3).

We finish this step defining $\widetilde{Q}_{n-1}(z) = (1/\prod_{j=1}^{n-1} J^j_{n-1,k_n-k_{n-1}}) \sum_{i=k_{n-1}+d_{n-1}+1}^{k_n} (1/i!) z^i$. Now, for any $h \in H(\mathbb{C})$ the orbit

$$\{\widetilde{u}_n\}_n = \operatorname{Orb}\Big(M, \Big(1, Q_1 + \widetilde{Q}_1 + Q_2 + \widetilde{Q}_2 + \dots Q_{n-1} + \widetilde{Q}_{n-1} + Q_n I^{k_n+d_n} h\Big)\Big) \tag{65}$$

satisfies that $\Pi_{d_n}(\widetilde{u}_{3+k_n}) = P_n$.

To sum up, our function $f(z)$ will be defined as $f(z) = \sum_{i=1}^{\infty} Q_i(z) + \widetilde{Q}_i(z)$. Clearly, it is well defined for every $z \in \mathbb{C}$ and converges uniformly on bounded sets of $\mathbb{C}$, because of statements $(*.4)$ and the fact that the sum $\sum_{i=1}^{\infty} |\widetilde{Q}_i(z)|$ is bounded in modulus by $e^{|z|}$.

Now, take an arbitrary function $g \in \mathscr{H}(\mathbb{C})$ and $R, \varepsilon > 0$. We choose $n_0 \in \mathbb{N}$ such that $R < n_0$ and $2^{-n_0} < \varepsilon/4$. Then we take $n \ge n_0$ such that

(i) $\max_{|z|\le n_0} |g(z) - \widetilde{P}_n(z)| < \varepsilon/2$,

(ii) $\sum_{i=d_{n+1}+1}^{\infty} (1/n!) < \varepsilon/8$ and

(iii) $\max_{|z|\le n_0} \sum_{i=k_{n+1}+d_{n+1}-k_n}^{\infty} (|z|^i/i!) < \varepsilon/8$.

We will see that $\max_{|z|\le n_0} |g(z) - u_{3+k_n}(z)| < \varepsilon$. This holds using the aforementioned estimations and the statements $(*.4)$.

$$\begin{aligned}\max_{|z|\le n_0} \left|g(z) - u_{3+k_n}(z)\right| &\le \max_{|z|\le n_0} \left|g(z) - \widetilde{P}_n(z)\right| \\ &\quad + \sum_{i=d_{n+1}+1}^{\infty} \frac{1}{n!} \\ &\quad + \max_{|z|\le n_0} \sum_{i=n+1}^{\infty} \left|Q_i^{(k_n)}(z) + \widetilde{Q}_i^{(k_n)}(z)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \varepsilon.\end{aligned} \tag{66}$$

□

Remark 15. The map $J : H(\mathbb{C}) \to \omega$, $J(f) = (f^{(n)}(0))_{n=0}^{\infty}$, is continuous, injective, of dense range, and satisfies that $BJ = JD$, where B is the backward shift on ω and D the derivative operator on $H(\mathbb{C})$. So for any $g \in H(\mathbb{C})$ the image of $\operatorname{Orb}(\delta_0 \otimes D, (1, g))$ under J is the orbit $\operatorname{Orb}(e_0' \otimes B, (e_0, J(g)))$. In particular, $(e_0, J(g)) \in \omega^2$ is hypercyclic for $e_0' \otimes B$ whenever $(1, g) \in H(\mathbb{C})^2$ is hypercyclic for $\delta_0 \otimes D$, what together with the example constructed to show Theorem 12 gives another proof of the hypercyclicity of the N-linear operator in Example 9 for the case $N = 2$.

5. Final Comments

We have not come up with examples of hypercyclic N-linear operators on Banach spaces. Hence, we pose the following.

Problem 16. Does any Banach space support a hypercyclic N-linear operator, for some $N \ge 2$?

We note that if $M : X^N \to X$ is an N-linear operator with $N \ge 2$ and $(X, \|\cdot\|)$ being a Banach space, the set of hypercyclic vectors for M must be nondense in X^N. Indeed, the continuity of M together with the fact that $N \ge 2$ ensures that any $(x_1, \dots, x_N)$ in X^N with

$$\max\{\|x_1\|, \dots, \|x_N\|\} \le \frac{1}{\|M\| + 1} \tag{67}$$

satisfies that the orbit $\operatorname{Orb}(L, (x_1, \dots, x_N))$ converges to zero, where

$$\|M\| = \sup\left\{\|L(z_1, \dots, z_N)\| : z_j \in X, \|z_j\| \le 1 \ (j = 1, \dots, N)\right\}. \tag{68}$$

We note that, by the above argument, a Banach space even lacks hypercyclic subspaces of dimension one for any N-linear operator with $N \ge 2$.

In view of Proposition 4 and Remark 10, it is also natural to ask.

Problem 17. Must the set of hypercyclic vectors of an N-linear operator be dense in X^N?

Of course, an affirmative answer to Problem 16 gives a negative answer to Problem 17 and an affirmative answer to Problem 17 gives a negative answer to Problem 16.

To conclude, we note that it is natural to seek extensions of results and notions in linear dynamics to N-linear dynamics. For instance, we may propose the following notion of chaos for N-linear operators, motivated by the notion of Devaney-chaos in linear dynamics [35].

Definition 18. A vector $x \in X^N$ is said to have a *periodic orbit* under an N-linear operator $M : X^N \to X$ whenever there exist some $y \in X$ and some $k > 1$ such that $u_1 = u_{nk} = y$ for all $n \in \mathbb{N}$. Such a vector is said to be *k-periodic* for M. We say that M is *N-linear Devaney chaotic* if it is a hypercyclic N-linear operator and it has a dense set of periodic orbits.

We note next that the N-linear operators of Examples 9 and 11 are also Devaney chaotic. Concerning Example 9, consider a dense sequence of eventually null elements of ω, namely, $\{g_n\}_{n\geq N+2}$ with $g_n(i) \neq 0$ if and only if $i \leq n$. Then, for every element g_n we define the corresponding vector

$$x_n(k) = \begin{cases} g_n(k), & \text{if } k \leq n, \\ \dfrac{g_n(k')}{(C_{n+N+1})^p}, & \text{if } k = np + k',\ 1 \leq k' \leq n,\ p \in \mathbb{N}, \end{cases} \tag{69}$$

where C_{n+N+1} is defined in (26). Let us also define the elements

$$x'_n(k) = \begin{cases} \dfrac{1}{g_n(1)}, & \text{if } k = 1, \\ 0, & \text{elsewhere}, \end{cases}$$

$$x''_n(k) = \begin{cases} 1, \\ \quad \text{if } k \leq n - N, \\ g_n(k), \\ \quad \text{if } k = n - N + k' \text{with } 1 \leq k' \leq n, \\ \dfrac{g_n(k')}{(C_{n+N+1})^{p-1}}, \\ \quad \text{if } k = pn - N + k',\ 1 \leq k' \leq n,\ 2 \leq p. \end{cases} \tag{70}$$

It follows that the initial conditions $(x_n, x'_n, e_1, \ldots, e_1, x''_n)$ force x_n to be an n-periodic point for the operator M defined on (23). Finally, the vectors $\{x_n\}_n$ are dense because of the denseness of $\{g_n\}_{n\geq N+2}$.

Let us proceed with Example 11. Consider a dense sequence of eventually null elements of ω, namely, $\{g_n\}_{n\geq N}$ with $g_n(i) \neq 0$ if and only if $i \leq n$. Then, for every element g_n, we define the corresponding vector

$$x_n(k) = \begin{cases} g_n(k), & \text{if } 1 \leq k \leq n, \\ g_n(k')^{1/G_{np+1}}, & \text{if } k = np + k',\ 1 \leq k' \leq n,\ p \in \mathbb{N}, \end{cases} \tag{71}$$

where $g_n(k')^{1/G_{np+1}}$ denotes one of the roots of $z^{G_{np+1}} - g_n(k')$. It follows that the initial conditions $(x_n, Bx_n, B^2x_n, \ldots, B^{N-1}x_n)$ force x_n to be an n-periodic point for the operator M defined on (42). To sum up, the set of periodic points for M is dense in ω.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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