Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 615351, 6 pages
http://dx.doi.org/10.1155/2014/615351

Research Article

On the Tumura-Clunie Theorem and Its Application

Gaixian Xue[1] and Jinjin Huang[2]

[1] *School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450046, China*
[2] *College of Economics and Management, Zhoukou Normal University, Zhoukou 466001, China*

Correspondence should be addressed to Gaixian Xue; qiaohuilei@163.com

Received 22 January 2014; Revised 15 March 2014; Accepted 26 March 2014; Published 22 April 2014

Academic Editor: Geraldo Botelho

We cast aside the restriction of the simple pole in the Tumura-Clunie type theorems for meromorphic functions and obtain a better result which improves the earlier results of Y. D. Ren. Furthermore, as an application, we improve a theorem given by B. Y. Su.

1. Introduction and Main Results

A meromorphic function will always mean meromorphic in the complex plane $\mathbb{C}$. We adopt the standard notation in the Nevanlinna value distribution theory of meromorphic functions such as $T(r,f)$, $m(r,f)$, $N(r,f)$, and $\overline{N}(r,f)$ as explained in [1, 2]. For any nonconstant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measures that is not necessarily the same at each occurrence.

Definition 1 (see [1]). A meromorphic function "$a(z)$" is said to be a small function of f if $T(r,a(z)) = S(r,f)$.

Definition 2. Throughout this paper one denotes by $a_j(z)$ meromorphic functions satisfying $(r,a_j(z)) = S(r,f)(j = 0,1,\ldots,n)$. If $a_n \not\equiv 0$, we call $P[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0$ a polynomial in f with degree n. If $n_0, n_1, \ldots, n_k$ are nonnegative integers, we call $M[f] = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$ a differential monomial in f of degree $\Upsilon_M = n_0+n_1+\cdots+n_k$ and of weight $\Gamma_M = n_0+2n_1+\cdots+(k+1)n_k$. If $M_1, M_2, \ldots, M_n$ are differential monomials in f, we call $Q[f] = \sum_{j=1}^{n} a_j(z)M_j[f]$ a differential polynomial in f and define the degree Υ_Q and the weight Γ_Q by $\Upsilon_Q = \max_{j=1}^{n}\Upsilon_{M_j}$ and $\Gamma_Q = \max_{j=1}^{n}\Gamma_{M_j}$, respectively.

Also $Q[f]$ is called a quasi-differential polynomial generated by f if, instead of assuming $T(r,a_j(z)) = S(r,f)$, we just assume that $m(r,a_j(z)) = S(r,f)$ for the coefficients $a_j(z)(j = 1,2,\ldots,n)$.

Definition 3. Let k be a positive integer; for any a in the complex plane, one denotes by $N_{k)}(r,1/(f-a))$ the counting function of a-points of f with multiplicity less than or equal to k, by $N_{(k}(r,1/(f-a))$ the counting function of a-points of f with multiplicity more than or equal to k, and by $N_k(r,1/(f-a))$ the counting function of a-points of f with multiplicity of k. Denote the reduced counting function by $\overline{N}_{k)}(r,1/(f-a))$, $\overline{N}_{(k}(r,1/(f-a))$, and $\overline{N}_k(r,(1/f-a))$, respectively.

Let f be a nonconstant meromorphic function and let

$$\digamma = f^n + Q[f] \tag{1}$$

be a differential polynomial, where $Q[f]$ is also a differential polynomial and $\Upsilon_Q \le n-1$.

Hua (see [3, page 69]) proved the following result.

Theorem A. *Let f be a nonconstant meromorphic function and let $\digamma$ be given by* (1) *with $\Upsilon_Q \le n-1$. If*

$$N(r,f) + N\left(r,\frac{1}{\digamma}\right) = S(r,f), \tag{2}$$

then

$$\digamma = \left(f + \frac{a(z)}{n}\right)^n, \tag{3}$$

where $a(z)$ is a small function of f.

Then $\digamma = g^n$, $g = f + (a(z)/n)$, and $a(z)g^{n-1}$ is obtained by substituting g for f, g' for f', and so forth in the terms of degree $n-1$ in $Q[f]$.

Remark 4. The conclusion still holds good if condition (2) is replaced with

$$N(r, f) + N\left(r, \frac{1}{F}\right) = S_o(r, f), \tag{4}$$

where $S_o(r, f)$ denotes any quantity which satisfies $S_o(r, f) = o(T(r, f))$ as $r \to +\infty$ through a set of r of infinite measure.

Hua (see [3]) improved Theorem A and obtained the following result.

Theorem B. *Let f be a nonconstant meromorphic function and let F be given by* (1) *with $\Upsilon_Q \leq n-1$. If*

$$N(r, f) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f), \tag{5}$$

then

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{6}$$

where $a(z)$ is a small function of f.

Another theorem is due to Zhang and Li (see [4]), which can be stated as follows.

Theorem C. *Let f be a nonconstant meromorphic function and let F be given by* (1), *where $n(\geq \Upsilon_Q + 1)$ is an integer. Then one of the following occurs.*

(i) *If $\Gamma_Q > n-1$, then*

$$\begin{aligned} T(r, f) \leq{}& \{1 + 2(\Gamma_Q - n + 1)\}\overline{N}(r, f) \\ &+ (\Gamma_Q - n + 2)\overline{N}\left(r, \frac{1}{F}\right) + S(r, f). \end{aligned} \tag{7}$$

Or there exists a small proximity function $a(z)$ of f such that

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{8}$$

and $N(r, a(z)) \leq (\Gamma_Q - n + 1)\{\overline{N}(r, f) + \overline{N}(r, 1/F)\} + S(r, f)$.

(ii) *If $\Gamma_Q \leq n-1$, then*

$$T(r, f) \leq 2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f), \tag{9}$$

or

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{10}$$

where $a(z)$ is a small function of f.

(iii) *In the special case, if $Q[f] = a_{n-1}f^{n-1} + P[f]$, where $\Gamma_P \leq n-2$, then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f), \tag{11}$$

or

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{12}$$

where $a(z)$ is a small function of f.

Corollary 5. *From Theorem C we know that if condition* (2) *is replaced with "$\overline{N}(r, f) + \overline{N}(r, 1/F) = S(r, f)$" in Theorem A, then the conclusion remains valid.*

In this direction Ren (see [5]) also generalized Tumura-Clunie's theorem concerning differential polynomials.

Combining the methods used in their proofs we show the following theorem.

Theorem 6. *Let f be a nonconstant meromorphic function and let F be given by* (1), *where $n(\geq \Upsilon_Q + 1)$ is an integer and $\Gamma_F(\neq 2)$ is the weight of F. If*

$$\overline{N}_{(2}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f), \tag{13}$$

then

$$F = \left(f + \frac{a(z)}{n}\right)^n, \tag{14}$$

where $a(z)$ is a small function of f.

It is easily seen from the following example that $\Gamma_F \neq 2$ in Theorem 6 is necessary.

Example 7. Let $f = \tan z$ and $F = f^2 + 1$. Obviously, (13) is obtained but (14) does not hold.

2. Some Lemmas

To prove our results, we need some lemmas.

Lemma 8 (see [1]). *Let f_1 and f_2 be two nonzero meromorphic functions in the complex plane; then*

$$\begin{aligned} &N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) \\ &\quad = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right). \end{aligned} \tag{15}$$

Lemma 9. *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting functions of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$\begin{aligned} N\left(r, 0; f^{(k)} \mid f \neq 0\right) \leq{}& k\overline{N}(r, f) + N(r, 0; f \mid < k) \\ &+ k\overline{N}(r, 0; f \mid \geq k) + S(r, f). \end{aligned} \tag{16}$$

Lemma 10. *Suppose that $Q[f]$ is given in Definition 2. Let z_0 be a pole of f of order p and neither a zero nor a pole of coefficients of $Q[f]$. Then z_0 is a pole of $Q[f]$ of order at most $p\Upsilon_Q + (\Gamma_Q - \Upsilon_Q)$.*

Lemma 11 (see [6]). *Let f be a nonconstant meromorphic function and let $Q[f]$ be given in Definition 2. Then*

$$\begin{aligned} m(r, Q[f]) &\leq \Upsilon_Q m(r, f) + \sum_{j=1}^{n} m\left(r, a_j\right) + S(r, f), \\ N(r, Q[f]) &\leq \Gamma_Q N(r, f) + \sum_{j=1}^{n} N\left(r, a_j\right) + S(r, f). \end{aligned} \tag{17}$$

Lemma 12. *Suppose that f is a nonconstant meromorphic function and $Q[f]$ is given in Definition 2. Then $S(r,Q) = S(r,f)$.*

Proof. It is straightforward by Lemma 11. □

Lemma 13 (see [7]). *Let f be a nonconstant meromorphic function in the complex plane and let $Q_1[f]$ and $Q_2[f]$ be quasi-differential polynomials in f. If $\Upsilon_{Q_2} \le n$ and $f^n Q_1[f] = Q_2[f]$, then $m(r, Q_1[f]) = S(r,f)$.*

Lemma 14. *Let f be a nonconstant meromorphic function and let $\digamma$ be given by (1). Then*

$$\left(\Gamma_\digamma - 2\right) N_1(r,f) \le 2\overline{N}_{(2}(r,f) + 2\overline{N}\left(r,\frac{1}{\digamma}\right) + S(r,f). \tag{18}$$

Proof. If $\Gamma_\digamma \le 2$, the conclusion of Lemma 14 holds obviously.

In the following we suppose that $\Gamma_\digamma > 2$.

With $\digamma = f^n + Q[f]$, we set

$$g(z) = \frac{\left\{\digamma'\right\}^{\Gamma_\digamma}}{\left\{\digamma\right\}^{\Gamma_\digamma + 1}}. \tag{19}$$

Let z_0 be a simple pole of f and not a zero of coefficients of $Q[f]$; then

$$f(z) = \frac{a}{z - z_0} + O(1), \quad a \ne 0 \text{ as } z \longrightarrow z_0. \tag{20}$$

From Lemma 10 we know that z_0 is a pole of $\digamma$ of order at most $\Gamma_\digamma$; then we have

$$\begin{aligned} \digamma(z) &= \frac{b}{(z - z_0)^{\Gamma_\digamma}} + O(1), \\ \digamma'(z) &= -\frac{b\Gamma_\digamma}{(z - z_0)^{\Gamma_\digamma + 1}} + O(1), \end{aligned} \tag{21}$$

where $b \ne 0$.

Then

$$\begin{aligned} \digamma(z) &= \frac{b}{(z - z_0)^{\Gamma_\digamma}} \left\{1 + O(z - z_0)^{\Gamma_\digamma}\right\}, \\ \digamma'(z) &= -\frac{b\Gamma_\digamma}{(z - z_0)^{\Gamma_\digamma + 1}} \left\{1 + O(z - z_0)^{\Gamma_\digamma + 1}\right\}, \\ g(z) &= \frac{(-1)^{\Gamma_\digamma} \Gamma_\digamma^{\Gamma_\digamma}}{b} \left\{1 + O(z - z_0)^{\Gamma_\digamma}\right\}. \end{aligned} \tag{22}$$

So $g(z_0) \ne 0, \infty$. But z_0 is a zero of $g'(z)$ of order at least $\Gamma_\digamma - 1$. Then

$$\left(\Gamma_\digamma - 1\right) N_1(r,f) \le N_0\left(r, \frac{1}{g'}\right), \tag{23}$$

where $N_0(r, 1/g')$ denotes the counting function of the zeros of g', not of g.

By Lemma 8 and Nevanlinna first fundamental theorem, we get

$$\begin{aligned} N\left(r, \frac{g}{g'}\right) &- N\left(r, \frac{g'}{g}\right) \\ &= N\left(r, \frac{1}{g'}\right) + N(r,g) - N(r,g') - N\left(r, \frac{1}{g}\right) \\ &= N_0\left(r, \frac{1}{g'}\right) - \overline{N}(r,g) - \overline{N}\left(r, \frac{1}{g}\right), \\ N\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g'}{g}\right) &= m\left(r, \frac{g'}{g}\right) - m\left(r, \frac{g}{g'}\right) + O(1). \end{aligned} \tag{24}$$

From (24), we have

$$\begin{aligned} N_0\left(r, \frac{1}{g'}\right) &\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g) + m\left(r, \frac{g'}{g}\right) + O(1) \\ &\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g) + S(r,f). \end{aligned} \tag{25}$$

From (19), we know that the poles and zeros of $g(z)$ can only occur at the multiple zeros of $f(z)$, the zeros of $\digamma$, and the zeros of $\digamma'$. Hence

$$\overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g}\right) \le \overline{N}_{(2}(r,f) + \overline{N}\left(r, \frac{1}{\digamma}\right) + N_0\left(r, \frac{1}{\digamma'}\right) + S(r,f), \tag{26}$$

where $N_0(r, 1/\digamma')$ denotes the counting function of the zeros of $\digamma'$, not of $\digamma$.

By Lemmas 9 and 12, we obtain

$$\begin{aligned} N_0\left(r, \frac{1}{\digamma'}\right) &\le \overline{N}(r,\digamma) + \overline{N}\left(r, \frac{1}{\digamma}\right) + S(r,\digamma) \\ &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{\digamma}\right) + S(r,f), \\ \overline{N}(r,f) &= N_1(r,f) + \overline{N}_{(2}(r,f). \end{aligned} \tag{27}$$

Combining (23), (25), (26), and (27), we obtain (18).

This completes the proof of Lemma 14. □

Proof of Theorem 6. We consider two cases.

Case 1. If $\Gamma_\digamma = 1$, (14) holds obviously.

Case 2. If $\Gamma_\digamma > 2$, by Lemma 14 and (13) we have

$$\begin{aligned} \overline{N}(r,f) &= N_1(r,f) + \overline{N}_{(2}(r,f) \\ &\le \frac{\Gamma_\digamma}{\Gamma_\digamma - 2} \overline{N}_{(2}(r,f) + \frac{2}{\Gamma_\digamma - 2} \overline{N}\left(r, \frac{1}{\digamma}\right) + S(r,f) \\ &\le S(r,f). \end{aligned} \tag{28}$$

This shows that

$$\overline{N}(r,f)=S(r,f). \tag{29}$$

Suppose that $F\equiv 0$.

So we have $f^n = -Q[f]$ and $Q[f] \not\equiv 0$; moreover $T(r,Q[f])=nT(r,f)+S(r,f)$.

By Lemma 11 we get $m(r,Q[f])\leq \Upsilon_Q m(r,f)+S(r,f)$.

On the other hand, we have

$$\begin{aligned} nm(r,f) &= m(r,f^n)=m(r,F-Q[f]) \\ &\leq m(r,F)+m(r,Q[f])+S(r,f) \\ &\leq \Upsilon_Q m(r,f)+S(r,f). \end{aligned} \tag{30}$$

It follows that $m(r,f)=S(r,f)$, which is impossible.

Therefore, $F\not\equiv 0$.

Then

$$\begin{aligned} T\left(r,\frac{F'}{F}\right) &\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+m\left(r,\frac{F'}{F}\right)+S(r,f) \\ &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f). \end{aligned} \tag{31}$$

From (29) and the condition of the theorem, we know $T(r,F'/F)=S(r,f)$.

By $F=f^n+Q[f]$, we have

$$F'=\frac{F'}{F}f^n+\frac{F'}{F}Q[f], \qquad F'=nf^{n-1}f'+Q'[f]. \tag{32}$$

And hence

$$f^{n-1}\left(f\frac{F'}{F}-nf'\right)=Q[f]\left(\frac{Q'[f]}{Q[f]}-\frac{F'}{F}\right). \tag{33}$$

Let

$$\begin{aligned} \Omega_1[f] &= f\frac{F'}{F}-nf', \\ \Omega_2[f] &= Q[f]\left(\frac{Q'[f]}{Q[f]}-\frac{F'}{F}\right). \end{aligned} \tag{34}$$

Then

$$f^{n-1}\Omega_1[f]=\Omega_2[f], \tag{35}$$

where $\Omega_1[f]$ and $\Omega_2[f]$ are quasi-differential polynomials.

By Lemma 13 we have

$$m(r,\Omega_1[f])=S(r,f). \tag{36}$$

By Lemma 10 and (35) we obtain

$$\begin{aligned} N(r,\Omega_1[f]) &= N(r,\Omega_2[f])-(n-1)N(r,f)+S(r,f) \\ &\leq \Upsilon_Q N(r,f)+(\Gamma_Q-\Upsilon_Q+1)\overline{N}(r,f) \\ &\quad -(n-1)N(r,f)+S(r,f) \\ &\leq (\Gamma_Q-\Upsilon_Q+1)\overline{N}(r,f)+S(r,f). \end{aligned} \tag{37}$$

Note that $\overline{N}(r,f)=S(r,f)$.

So $T(r,\Omega_1[f])=S(r,f)$.

From (34) we know that $Q[f]$ is a polynomial and $\Upsilon_Q\leq n-1$.

Set

$$Q[f]=b(z)f^{n-1}+P[f], \tag{38}$$

where $P[f]$ is a polynomial and $b(z)$ is a small function of f; moreover $\Upsilon_P\leq n-2$.

Set $g=f+(b(z)/n)$; we have

$$F=g^n+R[g], \tag{39}$$

where $R[g]$ is a polynomial and $\Upsilon_R\leq n-2$.

Now proceeding as the above proof, we get

$$g^{n-1}\left(g\frac{F'}{F}-ng'\right)=R[g]\left(\frac{R'[g]}{R[g]}-\frac{F'}{F}\right). \tag{40}$$

By Lemma 13 we obtain

$$\begin{aligned} m\left(r,\left(g\frac{F'}{F}-ng'\right)g\right) &= S(r,f), \\ m\left(r,g\frac{F'}{F}-ng'\right) &= S(r,f). \end{aligned} \tag{41}$$

Therefore we have

$$\begin{aligned} T\left(r,\left(g\frac{F'}{F}-ng'\right)g\right) &= S(r,f), \\ T\left(r,g\frac{F'}{F}-ng'\right) &= S(r,f). \end{aligned} \tag{42}$$

Notice that $T(r,g)=T(r,f)+S(r,f)\neq S(r,f)$.

We can get $g(F'/F)-ng'\equiv 0$.

So $F\equiv cg^n$, where c is a constant. Obviously $c=1$.

This proves Theorem 6. □

3. Application

Very recently, Yi (see [8, 9]) proved the following result.

Theorem D. *Let f be a transcendental meromorphic function and let $p(z)$ be a polynomial, $p(z)\not\equiv 0$. If f and f' share 0 in $\mathbb{C}$, then $f'-p(z)$ has infinitely many zeros.*

Remark 15. From the hypothesis of Theorem E, it can be easily seen that all zeros of f have multiplicity at least two.

Ren and Yang 2013 (see [10]) obtained the following result.

Theorem E. *Let f be a transcendental meromorphic function and let R be a rational function, $R\not\equiv 0$. Suppose that, with the exception of possibly finitely many, all zeros and poles of f are multiple. Then $f'-R$ has infinitely many zeros.*

It is natural to ask the following question: what can we say if f' is replaced by $f^{(k)}$ and $p(z)$ and R are replaced by a small function relative to f in Theorems D and E?

Later, Yang (see [11]) answered the above question and obtained the following result.

Theorem F. *Let f be a transcendental meromorphic function satisfying*

$$N\left(r, \frac{1}{f}\right) = S(r, f). \tag{43}$$

Then, for any $k \geq 1$ and any small function $a(z)(\not\equiv 0, \infty)$ of f,

$$N\left(r, \frac{1}{f^{(k)} - a(z)}\right) \neq S(r, f). \tag{44}$$

We supplement Theorems D and E, improve Theorem F, and obtain the following result.

Theorem 16. *Let h be a transcendental meromorphic function satisfying*

$$\overline{N}_{(2}\left(r, \frac{1}{h}\right) = S(r, h). \tag{45}$$

Then, for any $n \geq 2$ and any small function $a(z)(\not\equiv 0, \infty)$ of h,

$$N\left(r, \frac{1}{h^{(n)} - a(z)}\right) \neq S(r, h). \tag{46}$$

The method of our proof essentially belongs to Yang. For the completeness, we give the proof here.

Proof. Set

$$h = \frac{1}{f}. \tag{47}$$

Then

$$\begin{aligned} T(r, f) &= T(r, h) + O(1), \\ \overline{N}_{(2}\left(r, \frac{1}{h}\right) &= \overline{N}_{(2}(r, f). \end{aligned} \tag{48}$$

Obviously

$$S(r, f) = S(r, h). \tag{49}$$

Now

$$\begin{aligned} h'' &= \frac{-ff' + 2(f')^2}{f^3}, \\ h''' &= \frac{-6(f')^3 - f^2 f'' + 2f(f')^2 + 4ff'f''}{f^4} \cdots . \end{aligned} \tag{50}$$

Thus, in general,

$$h^{(n)} = \frac{Q_n(f)}{f^{n+1}}, \tag{51}$$

where $Q_n(f)$ denotes a homogeneous differential polynomial in f of degree n. So

$$h^{(n)} - a(z) = \frac{Q_n(f) - a(z) f^{n+1}}{f^{n+1}}. \tag{52}$$

If the assertion of the theorem was false, that is,

$$N\left(r, \frac{1}{h^{(n)} - a(z)}\right) = S(r, f), \tag{53}$$

then from (52) we have

$$F = f^{n+1} - \frac{Q_n(f)}{a(z)}. \tag{54}$$

Thus from (48), (53), and (54), we obtain

$$\overline{N}_{(2}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f). \tag{55}$$

Combining Theorem 6, (55) gives

$$F = \left(f + \frac{c}{n+1}\right)^{n+1}, \tag{56}$$

where c (a small function of f) is determined by the two equations: $g = f + (c/(n+1))$ and $cg^n = -(Q_n(g)/a(z))$.

We may claim that

(i) $S(r, f) = S(r, g)$;

(ii) $\overline{N}(r, g) = S(r, g)$;

(iii) $T(r, g^{(k)}/g) = S(r, g)$ for all $k \in \mathbb{N}$.

In fact, from the definition of g we know that the claim (i) above holds.

By (54) we have $\Gamma_F > 2$.

From $g = f + (c/(n+1))$, $\Gamma_F > 2$, and (29) we get

$$\overline{N}(r, g) = \overline{N}(r, f) + \overline{N}(r, c) = S(r, f) = S(r, g). \tag{57}$$

That is, the claim (ii) above holds.

Combining (53) and the claims (i) and (ii), we may deduce

$$\begin{aligned} T\left(r, \frac{g^{(k)}}{g}\right) &= N\left(r, \frac{g^{(k)}}{g}\right) + m\left(r, \frac{g^{(k)}}{g}\right) \\ &\leq k\overline{N}(r, g) + N\left(r, \frac{1}{g}\right) + S(r, g) \\ &\leq k\overline{N}(r, g) + N\left(r, \frac{1}{F}\right) + S(r, g) \\ &\leq S(r, g). \end{aligned} \tag{58}$$

Then the claim (iii) is true also.

Thus, by (54) and (56), we obtain

$$\begin{aligned} &\left(f + \frac{c}{n+1}\right)^{n+1} \\ &\quad = f^{n+1} + cf^n + \sum_{k=2}^{n+1} C_{n+1}^k \left(\frac{c}{n+1}\right)^k f^{n+1-k} \\ &\quad = f^{n+1} - \frac{Q_n(f)}{a(z)}. \end{aligned} \tag{59}$$

Since $cf^n \equiv -(Q_n(f)/a(z))$, it follows that

$$\sum_{k=2}^{n+1} C_{n+1}^{k}\left(\frac{c}{n+1}\right)^{k} f^{n+1-k} \equiv 0, \tag{60}$$

which is impossible unless $c \equiv 0$.

But then, from (59), $-(Q_n(f)/a(z)) \equiv 0$ and we have $h^{(n)} \equiv 0$ which contradicts the fact that h is a transcendental meromorphic function.

This completes the proof of Theorem 16. □

Remark 17. For $n = 1$, from the proof of Theorem 16 and Corollary 5, we know that if the condition "$\overline{N}_{(2}(r, 1/h) = S(r, h)$" is replaced with "$\overline{N}(r, 1/h) = S(r, h)$" in Theorem 16, then the conclusion still holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grant Nos. 11301140 and U1304102.

References

[1] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, UK, 1964.

[2] C. C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, vol. 557 of *Mathematics and Its Applications*, Kluwer Academic, New York, NY, USA, 2003.

[3] X. H. Hua, "Some extensions of the Tumura-Clunie theorem," *Complex Variables: Theory and Application*, vol. 16, no. 1, pp. 69–77, 1991.

[4] Z. L. Zhang and W. Li, "Tumura-Clunie's theorem for differential polynomials," *Complex Variables: Theory and Application*, vol. 25, no. 2, pp. 97–105, 1994.

[5] Y. D. Ren, "Solving integral representations problems for the stationary Schrödinger equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 715252, 5 pages, 2013.

[6] W. Doeringer, "Exceptional values of differential polynomials," *Pacific Journal of Mathematics*, vol. 98, no. 1, pp. 55–62, 1982.

[7] J. Clunie, "On integral and meromorphic functions," *Journal of the London Mathematical Society*, vol. 37, pp. 17–27, 1962.

[8] H. X. Yi, "On the theorem of Tumura-Clunie," *Kodai Mathematical Journal*, vol. 12, no. 1, pp. 49–55, 1989.

[9] H. X. Yi, "On a theorem of Tumura and Clunie for a differential polynomial," *Bulletin of the London Mathematical Society*, vol. 20, no. 6, pp. 593–596, 1988.

[10] Y. D. Ren and P. Yang, "Growth estimates for modified Neumann integrals in a half space," *Journal of Inequalities and Applications*, vol. 2013, article 572, 2013.

[11] C. C. Yang, "On the value distribution of a transcendental meromorphic function and its derivatives," *Indian Journal of Pure and Applied Mathematics*, vol. 35, no. 8, pp. 1027–1031, 2004.