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Research Article

Monotonicity and the Dominated Farthest Points Problem in Banach Lattice

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We introduce the dominated farthest points problem in Banach lattices. We prove that for two equivalent norms such that X becomes an STM and LLUM space the dominated farthest points problem has the same solution. We give some conditions such that under these conditions the Fréchet differentiability of the farthest point map is equivalent to the continuity of metric antiprojection in the dominated farthest points problem. Also we prove that these conditions are equivalent to strong solvability of the dominated farthest points problem. We prove these results in STM, reflexive STM, and UM spaces. Moreover, we give some applications of the stated results in Musielak-Orlicz spaces $L^{\phi}(\mu)$ and $E^{\phi}(\mu)$ over nonatomic measure spaces in terms of the function ϕ. We will prove that the Fréchet differentiability of the farthest point map and the conditions $\phi \in \Delta_2$ and $\phi > 0$ in reflexive Musielak-Orlicz function spaces are equivalent.

1. Introduction

The problem of farthest points in Banach spaces is studied with many authors (see [1–3]). An interesting question in this field is, under what conditions on the set A does the point $x_0 \in A$ farthest from point x in the spaces exist and is this unique? We recall that the mapping

$$f_A(x) = \sup\{\|x-y\| : y \in A\} \tag{1}$$

is called farthest point map and the mapping

$$F_A(x) = \{x_0 \in A : \|x-x_0\| = f_A(x)\} \tag{2}$$

is called a metric antiprojection. Fitzpatrick in [2] gives some conditions such that farthest point map is Fréchet differentiable and the metric antiprojection is continuous; in fact he showed that these conditions are equivalent. Balashov and Ivanov in [1] proved that in Hilbert spaces, the set of conditions for the existence, uniqueness, and Lipschitz dependence (on x) of the metric antiprojection of x on the set A for points x that are sufficiently far from the set A is equivalent to the strong convexity of the set A. Ivanov in [3] showed that the results of [1] generalized to uniformly convex Banach spaces with Fréchet differentiable norm.

Kurc in [4] introduces the dominated best approximation problem and examines the relations between monotonicity properties and the existence and uniqueness of the dominated best approximation problem. Hudzik and Kurc proved that strictly monotone and order continuity of the norm on X is equivalent to unique solvability of the dominated best approximation problem (e.g., [5]).

In this paper we introduce the dominated farthest point problem in a Banach lattice and try to examine the relation between the dominated farthest point problem and monotonicity, Fréchet differentiability of farthest point map, and the continuity of antiprojection map in Banach lattices.

In preliminaries section we recall main definitions and some lemmas that will be used in this context. In Section 3, we introduce the dominated farthest points problem and state some conditions such that guaranteed, existence and uniqueness of the dominated farthest points problem. We give some criteria for strict monotonicity, lower locally uniformly monotone, upper locally uniformly monotone and uniformly monotone. Also we prove that for two equivalent norms such that X becomes an STM space and LLUM space the dominated farthest point has the same solution. This note will prove that the Fréchet differentiability of farthest point map is equivalent to continuity of antiprojection map under some conditions. In fact these conditions are equivalent to strong solvability of the set A. We give some conditions such that it is proved that if $F_A(x)$ is a singleton set then A is a singleton set.

Finally we will say some application of the stated results in Musielak-Orlicz function space $L^{\phi}(\mu)$ and $E^{\phi}(\mu)$ over nonatomic measure spaces in terms of the function ϕ. Equivalency of the Fréchet differentiability of the farthest point map and the conditions $\phi \in \Delta_2$ and $\phi > 0$ in reflexive Musielak-Orlicz function spaces is the final result which will be proved.

2. Preliminaries

Let X be a Banach lattice and A a bounded sublattice in X. Suppose that $x \in X$ such that $x \geq A$ (i.e., $x \geq y$ for each $y \in A$); we define $F_A(x)$ as 1.1; we always refer to such problems as to the dominated Farthest points problem.

The dominated farthest points problem is called solvable if $F_A(x) \neq \emptyset$. The problem is said to be uniquely solvable if card $(F_A(x)) = 1$ and is to be stable if for every maximizing sequence $\{x_n\}$ in A, that is, a sequence in A such that $\lim_{n\to\infty}\|x - x_n\| = f_A(x)$, there holds $d(x_n, F_A(x)) \to 0$ as $n \to \infty$. Finally, the problem is said to be strongly solvable if it is uniquely solvable and stable. A sequence $\{x_n\}$ in A is a maximizing sequence for x_0 if $\lim_{n\to\infty}\|x_n - x_0\| = f_A(x_0)$.

In this section, we recall some definitions and lemmas which we need in main results.

Definition 1 (see [4]). A Banach lattice X is said to be strictly monotone ($X \in$ STM) if, for all $x, y \in X^+$, the conditions $x \geq y$, $y \neq 0$, and $\|x\| = \|y\|$ imply $x = y$.

Definition 2 (see [4]). A Banach lattice X is said to be uniformly monotone ($X \in$ UM) if, for all $y_n \geq x_n \geq 0$, such that $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\|$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Definition 3 (see [5]). A Banach lattice X is said to be upper (lower) locally uniformly monotone, $X \in$ ULUM ($X \in$ LLUM), if, for each $x, y_n \in X$, such that $y_n \geq x \geq 0$ ($y_n \leq x \leq 0$) and $\lim_{n\to\infty}\|x_n\| \to \|x\|$, then $\lim_{n\to\infty}\|x_n - x\| = 0$.

Definition 4 (see [6]). A Banach lattice X is said to be decreasing (increasing) uniformly monotone, $X \in$ DUM ($X \in$ IUM), if, for each $y_n, x_n \in X^+$, such that $y_n \geq x_n \downarrow$ ($x_n \leq y_n \uparrow$) and $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\|$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Definition 5 (see [5]). A Banach lattice X is said to be CWLLUM if for any nonnegative $x' \in X'$ with $\|x'\| = 1$ and any $x \in X$, with $x \geq 0$, $\|x\| = 1$ and any sequence $\{x_n\}$ in X satisfying $0 \leq x_n \leq x$ for all n the condition $x'(x - x_n) \to \|x\|$ implies $\|x_n\| \to 0$.

Definition 6 (see [7]). A lattice seminorm ρ on a Riesz space is said to be order continuous whenever $x_\alpha \downarrow 0$ implies $\rho(x_\alpha) \downarrow 0$. If the above condition holds for sequences, that is, $x_n \downarrow 0$ implies $\rho(x_n) \downarrow 0$, then ρ is said to be σ-order continuous. If ρ is a lattice norm then the norm is order continuous.

Definition 7 (see [7]). A Banach lattice X is said to be a Kantorovich-Banach space (or briefly a KB-space) whenever every increasing norm bounded sequence of X^+ is norm convergent.

Definition 8 (see [8]). We say that the norm of the Banach space X is Fréchet differentiable at $x_0 \in S(X)$ whenever

$$\lim_{\lambda\to 0}\frac{\|x_0 + \lambda y\| - \|x_0\|}{\lambda} \tag{3}$$

exists uniformly for $y \in S(X)$. If the norm of X is Fréchet differentiable at $x \in S(X)$, then we say that X has a Fréchet differentiable norm, where $S(X) = \{x \in X : \|x\| = 1\}$.

Definition 9 (see [9]). For a function f from a Banach space X into a Banach space Y the Gâteaux derivative at a point $x_0 \in X$ is by definition a bounded linear operator $T : X \to Y$ such that, for every $u \in X$,

$$\lim_{t\to 0}\frac{f(x_0 + tu) - f(x_0)}{t} = Tu. \tag{4}$$

The operator T is called the Fréchet derivative of f at x_0 if it is a Gâteaux derivative of f at x_0 and the limit in (4) holds uniformly in u in the unit ball (or unit sphere) in X.

Definition 10 (e.g., [10–12]). Let (T, Σ, μ) be a σ-finite, complete (nontrivial), positive measure space and $\phi(t, r) : T \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ a function such that for μ-a.e. $t \in T$, $\phi(t, 0) = 0$, and $\phi(t, \cdot)$ is nontrivial (continuous at zero with nonzero values), convex, and lsc.

Moreover, if $\phi(\cdot, r)$ is measurable, for all $r > 0$, then we call ϕ the Musielak-Orlicz function.

Definition 11. Musielak-Orlicz spaces $L^{\phi}(\mu)$ consist of all μ-measurable functions $f : T \to \overline{\mathbb{R}}$ such that

$$I_\phi(\alpha f) = \int_T \phi(\alpha|f(t)|, t)\, d\mu < +\infty \tag{5}$$

for some $\alpha > 0$ (depending on f).

Musielak-Orlicz spaces under the natural ordering, when endowed with each of the following norms, become a Banach lattice (e.g., [12, 13]). Luxemburg norm is

$$\|f\|_\phi = \inf\left\{\lambda > 0 : I_\phi\left(\frac{f}{\lambda}\right) \leq 1\right\}, \tag{6}$$

and Orlicz norm is

$$\|f\|_\phi^0 = \sup\left\{|\langle f, g\rangle| : I_{\phi^*}(g) \leq 1\right\}, \tag{7}$$

where $\langle f, g\rangle = \int_T f(t)g(t)d\mu$ and ϕ^* denote the young conjugate of ϕ. The amemiya norm (see [14–16] for the Orlicz spaces and [17] for the general case) is

$$\|f\|_\phi^A = \inf_{k>0}\frac{1}{k}\left(1 + I_\phi(kf)\right). \tag{8}$$

All norms defined above are lattice monotone norms and they are equivalent:

$$\|f\|_{\phi} \le \|f\|_{\phi}^{0} \le 2\|f\|_{\phi}, \qquad \|f\|_{\phi}^{0} \le \|f\|_{\phi}^{A}. \tag{9}$$

In the following we will write, for short, $\phi > 0$ or $\phi < +\infty$, if for μ-a.e. $t \in T$ the function $\phi(t,\cdot)$ is strictly positive (except zero) or assumes finite values only, respectively. In the case that ϕ is finitely valued and the ∞-condition is satisfied then $\|f\|_{\phi}^{0} = \|f\|_{\phi}^{A}$ for all $f \in L_{\phi}(\mu)$. We recall that ϕ is satisfied ∞-condition if $\phi(t,u)/u \to \infty$ as $u \to \infty$ for μ-a.e. $t \in T$.

The function ϕ is said to satisfy a Δ_2, condition ($\phi \in \Delta_2$), if there exist a set T_0 of zero measure, a constant $K > 0$, and an integrable (nonnegative) function h, such that, for all $t \in T \setminus T_0$ and $r > 0$, there holds

$$\phi(2r,t) \le K\phi(r,t) + h(t). \tag{10}$$

Suppose that $L_a^{\phi}(\mu)$ is a subspace of functions with order continuous norm

$$\begin{gathered} L_a^{\phi}(\mu) = \left\{ f \in L^{\phi}(\mu) : |f| \ge f_n \downarrow 0 \Longrightarrow \|f_n\|_{\phi} \downarrow 0 \right\}, \\ E^{\phi}(\mu) = \left\{ f \in L^{\phi}(\mu) : I_{\phi}(\alpha f) < \infty \ \forall \alpha > 0 \right\}. \end{gathered} \tag{11}$$

Then $E^{\phi}(\mu) \subset L_a^{\phi}(\mu) \subset L^{\phi}(\mu)$ as closed ideals (see [18], [13, p. 17], and [19]). If $\phi < +\infty$ then $E^{\phi}(\mu)$ is super order dense in $L^{\phi}(\mu)$ and $L_a^{\phi}(\mu) = E^{\phi}(\mu)$ [13, p. 19], and $L^{\phi}(\mu)$ has an order continuous norm precisely when $L_a^{\phi}(\mu) = L^{\phi}(\mu)$. Clearly the norm in $E^{\phi}(\mu)$ is order continuous.

Lemma 12 (see [20]). *The following assertions are equivalent:*

(i) *the norm on X is order continuous,*

(ii) *X is Dedekind complete (σ-Dedekind complete) satisfying $\|x_n\| \to 0$ as $n \to \infty$ for any decreasing sequence $\{x_n\} \subset X^+$ with $\inf\{x_n : n \in \mathbb{N}\} = 0$,*

(iii) *every monotone order bounded sequence of X is convergent,*

(iv) *every disjoint order bounded sequence of X^+ is convergent to zero,*

(v) *X is an ideal in X'',*

(vi) *every order interval of X is weakly compact,*

(vii) *$X' = X'_n$.*

Lemma 13 (see [20]). *The following assertions are equivalent:*

(i) *X is reflexive,*

(ii) *X and X' are KB-spaces,*

(iii) *X does not contain any subspace isomorphic to l_1 or to c_0,*

(iv) *X does not contain any sublattice isomorphic to l_1 or to c_0.*

Lemma 14 (see [21]). *A Banach lattice X has an order-continuous norm if and only if it has an equivalent locally uniformly convex lattice norm.*

Lemma 15 (see [22]). *If X is a Banach lattice, the following properties are equivalent:*

(i) *X and X' have order continuous norms,*

(ii) *there exists an equivalent lattice norm on X which is locally uniformly convexand Fréchet differentiable, such that its dual norm is also locally uniformly convex on X'.*

Lemma 16 (see [22]). *A Banach lattice is reflexive if and only if it can be given an equivalent lattice norm such that both the space and its dual are simultaneously locally uniformly convex and Fréchet differentiable.*

Lemma 17 (see [2]). *Suppose A is a closed subset of a Banach space X such that the norm of X' is Fréchet differentiable. If A is bounded and f_A is Fréchet differentiable at some $x \in X$, then every maximizing sequence for x converges; also F_A is continuous at x.*

Lemma 18 (see [2]). *Suppose that X is a Banach space such that the norms of X and X' are both Fréchet differentiable. If A is a closed bounded subset of X and x is a point of X, then the following are equivalent:*

(i) *the metric antiprojection is continuous at x,*

(ii) *every maximizing sequence in A for x converges,*

(iii) *the function f_A is Fréchet differentiable at x.*

Lemma 19 (see [23]). *Given a Banach lattice X the following hold true:*

(i) *if X^+ is rotund, then X is strictly monotone;*

(ii) *if X^+ is locally uniformly rotund, then X is upper and lower locally uniformly monotone;*

(iii) *if X^+ is uniformly rotund then X is uniformly monotone,*

(iv) *in the order intervals in the positive cone X^+ the inverse statement of each of the above is also true.*

Lemma 20 (see [24]). *The following statements are equivalent:*

(i) *$\phi \in \Delta_2$,*

(ii) *$\|f\|_{\phi} = 1$ implies $I_{\phi}(f) = 1$,*

(iii) *$\|f_n\|_{\phi} \uparrow 1$ implies $I_{\phi}(f_n) \to 1$,*

(iv) *$L^{\phi}(\mu)$ dose not contain an isometric copy of l_{∞},*

(v) *$L^{\phi}(\mu)$ dose not contain a lattice isometric copy of l_{∞},*

(vi) *$L^{\phi}(\mu) = L_a^{\phi}(\mu)$; that is, the norm $\|\cdot\|_{\phi}$ is order continuous on $L^{\phi}(\mu)$.*

3. Dominated Farthest Points Problem in Banach Lattices

In this section we introduce the dominated farthest points problem in Banach lattices and give some criteria for strict monotonicity, lower locally uniformly monotone, upper locally uniformly monotone, and uniformly monotone. We prove that if $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are two Banach lattices with the same order such that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms and $(X, \|\cdot\|_1)$ is an STM space and $(X, \|\cdot\|_2)$ is an LLUM space, then the problem of the dominated farthest points has the same solution in two spaces. We give some condition such that the Fréchet differentiability of farthest point map is equivalent to continuity of antiprojection map.

Lemma 21. *Let X be a Banach lattice, A a sublattice, and $x \in X$ such that $x \geq A$. If $a_1 \in F_A(x)$ and $a_2 \in A$ such that $a_1 \geq a_2$, then $a_2 \in F_A(x)$.*

Proof. Since $a_1 \geq a_2$ and $x \geq A$, so $0 \leq x - a_1 \leq x - a_2$ and thus $\|x - a_1\| \leq \|x - a_2\|$. Since $a_1 \in F_A(x)$ we have $\|x - a_1\| = \|x - a_2\|$. Therefore, $a_2 \in F_A(x)$. □

Theorem 22. *Let X be a Banach lattice. Then X is an STM space if and only if for every sublattice A in X and $x \in X$ such that $x \geq A$, card $(F_A(x)) \leq 1$.*

Proof. Suppose that X is an STM space and A a sublattice in X. Suppose that $x \in X$ such that $x \geq A$ and $s, t \in F_A(x)$. Since A is a sublattice $s \wedge t \in A$ and $s \wedge t \leq s$, from Lemma 21, $s \wedge t \in F_A(x)$, so $\|x - s \wedge t\| = \|x - s\|$; since X is an STM space $s \wedge t = s$; similarly $s \wedge t = t$; therefore $s = t$.

Conversely if X is not an STM space, then there exist $x, y \in X$ such that $x \geq y \geq 0$ and $\|x - y\| = \|x\|$. Define $A = [0, y]$; then A is a sublattice and $x \geq A$; since $x \geq x - t$ for each $t \in A$, so $\|x\| \geq \|x - t\|$ for each $t \in A$, and thus $0 \in F_A(x)$, by the assumption $\|x\| = \|x - y\|$; so $y \in F_A(x)$; therefore card $(F_A(x)) > 1$. □

Theorem 23. *A Banach lattice X is an STM space and has order continuous norm if and only if the dominated farthest points problem with respect to closed order bounded sublattices is uniquely solvable.*

Proof. Suppose that X is an STM space and has order continuous norm and A is a closed sublattice. We assume that $x \geq A$ and $\{y_n\}$ is a maximizing sequence; that is, $f_A(x) = \sup_{y \in A} \|x - y\| = \lim_{n \to \infty} \|x - y_n\|$. Put $z_n = \wedge_{k=1}^{n} y_k$. Since A is a sublattice and $0 \leq x - y_n \leq x - z_n$, $\{z_n\}$ is a decreasing maximizing sequence. Since there exists $z = \wedge_n y_n \leq x$, we have $0 \leq z_n - z \downarrow 0$, by the order continuity of the norm $\|z_n - z\| \to 0$. On the other hand A is a closed sublattice so $z \in A$ and, since $\|x - z\| = f_A(x)$, therefore $z \in F_A(x)$.

Suppose that the norm on X is not order continuous; then there exists a sequence $\{y_n\}$ such that $0 \leq y_n \downarrow 0$ but $\inf \|y_n\| > 0$. We can assume that $\|y_n\| > \|y_{n+1}\|$ for $n \in \mathbb{N}$, otherwise replacing y_n by $(1 + 1/n) y_n$, for $n \in \mathbb{N}$. Suppose that $x \in X$ such that $x > y_1$ and put $A = \{y_n\}$; then A is a sublattice and $F_A(x) = \emptyset$. On the other hand by Dini's theorem A is norm closed. Indeed if $\|y_{n_k} - z\| \to 0$, $y_{n_k} \in A$ and $z \notin A$, then we use the fact that if y_{n_k} is downward directed sequence which is weakly convergent to z then $z = \inf_k (y_{n_k})$, and thus $z = 0$; hence $\|y_{n_k}\| \to 0$; this is a contradiction and so $z \in A$. By the assumption $F_A(x) \neq \emptyset$ this contradiction completes the proof. □

Remark 24. In Theorem 23, if X is an LLUM, CWLLUM, IUM, or DUM space then the theorem is also true.

Theorem 25. *Let X be an order continuous Banach lattice with the ULUM property; then the dominated farthest points problem with respect to closed order bounded sublattices is strongly solvable.*

Proof. From Theorem 23 the dominated farthest points problem with respect to closed sublattices is uniquely solvable. The proof of stability is the same as the proof of Theorem 4.4 in [5]. □

Theorem 26. *Let X be an STM space and A a sublattice in X. If $x, y \in X$ such that $x, y \geq A$ (or $x, y \leq A$) and $F_A(x), F_A(y) \neq \emptyset$, then $F_A(x) = F_A(y)$.*

Proof. Suppose that $x, y \in X$ such that $x, y \geq A$ and $F_A(x) = \{x_0\}$, $F_A(y) = \{y_0\}$. Since A is a sublattice so $x_0 \wedge y_0 \in A$; from Lemma 21, $x_0 \wedge y_0 \in F_A(x)$. Similarly $x_0 \wedge y_0 \in F_A(y)$; therefore $x_0 \wedge y_0 = x_0 = y_0$. □

Corollary 27. *Let X be an STM space with order continuous norm. If A is a sublattice in X, $x \in X$ such that $x \geq A$ or $x \leq A$; then the metric antiprojection is in this form:*

$$F_A(x) = \begin{cases} \{x_0\}, & x \geq A, \\ \{y_0\}, & x \leq A, \end{cases} \tag{12}$$

for some $x_0, y_0 \in A$.

Proof. It is a consequence of Theorems 23 and 26. □

Theorem 28. *Let X be an STM space and A a sublattice in X. If $x, y \in X$ such that $x \geq A$ and $y \leq A$ and $F_A(x) = F_A(y) \neq \emptyset$, then A is singleton.*

Proof. Suppose that $F_A(x) = F_A(y) = \{x_0\}$ and $y_0 \in A$. Since A is a sublattice so $x_0 \wedge y_0 \in A$, from Lemma 21, $x - x_0 \wedge y_0 \geq x - x_0 \geq 0$, thus $x_0 \wedge y_0 \in F_A(x) = \{x_0\}$, thus $x_0 \leq y_0$, and again, from Lemma 21, we have $y_0 \in F_A(y) = \{x_0\}$. Therefore $A = \{x_0\}$. □

Lemma 29. *Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be two Banach lattices with the same order such that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norm. If $(X, \|\cdot\|_1)$ is an order continuous Banach lattice, then $(X, \|\cdot\|_2)$ is also a Banach lattice with order continuous norm.*

Proof. It is a consequence of equivalency of two norms. □

Example 30. (a) Every equivalent norm on $L^p(\mu)$ with $\|\cdot\|_p$ for $1 < p < \infty$ is order continuous.

(b) Every equivalent norm on $C(X)$ (X is not a finite set) with sup-norm is not order continuous.

Theorem 31. *Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be two Banach lattices with the same order such that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms. Suppose that $(X, \|\cdot\|_1)$ is an STM space and $(X, \|\cdot\|_2)$ is an LLUM space. If A is a sublattice of X and $x \in X$ such that $x \geq A$ (or $x \leq A$), then $F_A^1(x) = F_A^2(x)$, where $F_A^i(x)$ is the set of farthest points with respect to $\|\cdot\|_i$ for $i = 1, 2$.*

Proof. Since $(X, \|\cdot\|_2)$ is order continuous and $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, from Lemma 29, $(X, \|\cdot\|_1)$ is order continuous. From Theorem 23 and Remark 24, the dominated farthest points problem is uniquely solvable for two norms. Suppose that $x_0 \in F_A^2(x)$; if $x_0 \notin F_A^1(x)$, there exists $x_0 \neq y_0 \in A$ such that $\|x - y_0\|_1 > \|x - x_0\|_1$. Since A is a sublattice $x_0 \wedge y_0 \in A$; since $x_0 \neq y_0$, we have either $x_0 \wedge y_0 < x_0$ or $x_0 \wedge y_0 < y_0$.

If $x_0 \wedge y_0 < x_0$, then $x - x_0 \wedge y_0 > x - x_0 \geq 0$ and since $(X, \|\cdot\|_2)$ is an LLUM space $\|x - x_0 \wedge y_0\|_2 > \|x - x_0\|_2$ that is a contradiction.

If $x_0 \wedge y_0 < y_0$, then $x - x_0 \wedge y_0 > x - y_0$ and so $\|x - x_0 \wedge y_0\|_1 > \|x - y_0\|_1$; also $x - x_0 \wedge y_0 \geq x - x_0$ and so $\|x - x_0 \wedge y_0\|_2 \geq \|x - x_0\|_2$; since $\{x_0\} = F_A^2(x)$, we have $x_0 \wedge y_0 = x_0$ and so $\|x - x_0\|_1 = \|x - x_0 \wedge y_0\|_1 > \|x - y_0\|_1$ that is a contradiction; therefore $x_0 = y_0$; thus $F_A^1(x) = F_A^2(x)$. □

Remark 32. If we assume that $(X, \|\cdot\|_1)$ is an STM space with order continuous norm, then Lemmas 14 and 19 guaranteed the existence of $\|\cdot\|_2$, such that $(X, \|\cdot\|_2)$ is a LLUM space.

Theorem 33. *Let X be a Banach lattice and A a closed order bounded subset of X. If $x \in X$, $x \geq A$, the following statements are equivalent:*

(i) *any sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty}\|x - a_n\| = f_A(x)$ is convergent;*

(ii) *the metric antiprojection is uniquely defined on $x \geq A$ and, for any vector $x \geq A$, any maximizing sequences have a convergent subsequence;*

(iii) *dominated farthest point problem with respect to closed bounded sublattices is strongly solvable.*

Proof. (i) → (ii). Suppose that there exists an $x \geq A$, such that card $(F_A(x)) > 1$. We assume that $x_0, y_0 \in F_A(x)$, $x_0 \neq y_0$, and $\{a_n\}, \{b_n\}$ are two maximizing sequences in A convergent to x_0 and y_0, respectively.

We define

$$c_n = \begin{cases} a_n, & n \text{ is odd}, \\ b_n, & n \text{ is even}; \end{cases} \tag{13}$$

then c_n is not convergent, that is, a contradiction, so card $(F_A(x)) = 1$. Since every maximizing sequence is convergent the proof is complete.

(ii) → (i). Suppose that $F_A(x) = \{x_0\}$ and $\{x_n\}$ is a maximizing sequence. If the condition (i) is not true, then $\{x_n\}$ has a subsequent $\{x_{n_j}\}$ such that $\|x_0 - x_{n_j}\| \geq \varepsilon > 0$, for any $j \in \mathbb{N}$. Since $\lim_{j\to\infty}\|x - x_{n_j}\| = f_A(x)$, the sequence $\{x_{n_j}\}$ has a limit point of x_0', we have $x_0' \in F_A(x)$ and $x_0 \neq x_0'$ that is a contradiction; therefore (i) is true.

(ii) → (iii). By the assumption $F_A(x) = \{x_0\}$, suppose that $\{a_n\}$ is a maximizing sequence in A for x. From (ii), $\{a_n\}$ has a convergent subsequence to x_0, so $d(\{a_n\}, F_A(x)) = d(\{a_n\}, x_0) = 0$.

(iii) → (ii). From the definition of strongly solvable card $(F_A(x)) = 1$, put $F_A(x) = \{x_0\}$. Suppose that $\{a_n\}$ is a maximizing sequence in A for x. So $d(\{a_n\}, x_0) = 0$ so x_0 is a limit point of $\{a_n\}$; thus there exists a subsequence $\{a_{n_j}\}$ convergent to x_0; this completes the proof. □

Corollary 34. *Let X be a Banach lattice with ULUM property and order continuous norm and A a closed order bounded sublattice of X; then*

(i) *any sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty}\|x - a_n\| = f_A(x)$ is convergent;*

(ii) *the metric antiprojection is uniquely defined on $x \geq A$ and, for any vector $x \geq A$, any maximizing sequences have a convergence subsequence,*

(iii) *dominated farthest point problem with respect to closed order bounded sublattices is strongly solvable.*

Proof. It is a compound of Theorems 25 and 33. □

Theorem 35. *Let $(X, \|\cdot\|)$ be an STM space such that X' and X'' have order continuous norm, A a closed order bounded sublattice in X, and $x_0 \in X$, such that $x_0 \geq A$. If f_A is Fréchet differentiable at x_0 then*

(i) *card $(F_A(x_0)) = 1$,*

(ii) *any sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty}\|x_0 - a_n\| = f_A(x_0)$ is convergent,*

(iii) *if $x_n \to x_0$ and $\{a_k\} = F_A(x_k)$ for $k \in \mathbb{N} \cup \{0\}$ then $a_n \to a_0$,*

(iv) *the dominated farthest points problem is strongly solvable.*

Proof. Part (i). From Theorem 23, the dominated farthest points problem is uniquely solvable so card $(F_A(x_0)) = 1$.

Part (ii). From (i) $F_A(x_0) = \{a_0\}$, suppose that $\{a_n\}$ is a maximizing sequence in A, $\lim_{n\to\infty}\|x_0 - a_n\| = \|x_0 - a_0\|$. a_0 is a limit point of $\{a_n\}$ and so there is a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to a_0$ from Lemma 15, there exists an equivalent norm $\|\cdot\|_1$ such that X' and X'' are locally uniformly convex space and X' has Fréchet differentiable norm, from Theorem 31, and a_0 is unique farthest point in A from x_0 with $\|\cdot\|_1$. Since $\{a_n\}$ has a subsequence convergent to a_0, $\{a_n\}$ is also a maximizing sequence with $\|\cdot\|_1$ and hence $\lim_{n\to\infty}\|x_0 - a_n\|_1 = \|x_0 - a_0\|_1$. From Lemma 17, $\{a_n\}$ converges with $\|\cdot\|_1$, so it is convergent with $\|\cdot\|$ and this completes the proof.

Part (iii). From Lemma 17, F_A is continuous so if $x_n \to x_0$; then $F_A(x_n) \to F_A(x_0)$ or equivalently $a_n \to a_0$.

Part (iv). By part (i), card $(F_A(x)) = 1$. Suppose that $\{x_n\}$ is a maximizing sequence in A; that is, $\lim_{n\to\infty}\|x_0 - x_n\| = \|x_0 - a_0\|$. Since $x_n \to x_0$ we have $d(x_n, F_A(x_0)) \to 0$; therefore the dominated farthest points problem is strongly solvable. □

Theorem 36. *Let $(X, \|\cdot\|)$ be a reflexive, STM space A, a closed order bounded sublattice in X, and $x_0 \in X$, such that $x_0 \geq A$. The following statements are equivalent:*

(i) *the dominated farthest points problem is strongly solvable;*

(ii) *any sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty}\|x_0 - a_n\| = f_A(x_0)$ is convergent;*

(iii) *if $x_n \to x_0$ and $\{a_k\} = F_A(x_k)$ for $k \in \mathbb{N} \cup \{0\}$, then $a_n \to a_0$;*

(iv) *f_A is Fréchet differentiable at x_0.*

Proof. (i) $\leftrightarrow$ (ii). From Theorem 33 thus is true.

(iv) $\to$ (ii), (iv) $\to$ (iii). From Lemma 13, X and X' are KB-spaces and so from Lemma 12 they have order continuous norm. Therefore it is a part of Theorem 35.

(ii) $\to$ (iv), (iii) $\to$ (iv). From Lemma 16, there exists an equivalent norm $\|\cdot\|_1$ such that X' and X'' are locally uniformly convex space and X', X'' has Fréchet differentiable norm, and from Lemma 18, (ii), (iii), and (iv) are equivalent to norm $\|\cdot\|_1$, so f_A is Fréchet differentiable with norm $\|\cdot\|_1$ since $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ from [25, p. 3]; f_A is Fréchet differentiable with norm $\|\cdot\|$; this completes the proof. □

Theorem 37. *Let X be a uniformly convex Banach lattice space, A a closed order bounded sublattice in X, and $x_0 \in X$, such that $x_0 \geq A$. The following statements are equivalent:*

(i) *the dominated farthest points problem is strongly solvable;*

(ii) *any sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty}\|x_0 - a_n\| = f_A(x_0)$ is convergent;*

(iii) *if $x_n \to x_0$ and $\{a_k\} = F_A(x_k)$ for $k \in \mathbb{N} \cup \{0\}$, then $a_n \to a_0$;*

(iv) *f_A is Fréchet differentiable at x_0.*

Proof. Since X is uniformly convex, so it is a reflexive Banach lattice and from Lemma 19, it is a UM space; from Theorem 36 the proof is complete. □

Theorem 38. *Let X be a UM space with an order unit $\mathbf{1}$, A a closed order bounded sublattice in X, and $x_0 \in X$, such that $x_0 \geq A$. The following statements are equivalent:*

(i) *the dominated farthest points problem is strongly solvable;*

(ii) *any sequence $\{a_n\} \subset A$ such that $\lim_{n\to\infty}\|x - a_n\| = f_A(x)$ is convergent;*

(iii) *if $x_n \to x_0$ and $\{a_k\} = F_A(x_k)$ for $k \in \mathbb{N} \cup \{0\}$, then $a_n \to a_0$;*

(iv) *f_A is Fréchet differentiable at x_0.*

Proof. Since X is a UM space with order unit $\mathbf{1}$, $B_1(0) = [-\mathbf{1}, \mathbf{1}]$. So from Lemma 19, rotundity properties are equivalent to monotonicity properties on $B_1(0)$, so X is a uniformly convex Banach lattice from Theorem 37; the proof is complete. □

4. Some Applications of the Dominated Farthest Points Problem in Musielak-Orlicz Spaces

In this section we give some applications of the theorems that were proved in Section 3 in Musielak-Orlicz function space. The most important result that will be proved in this section is the equivalency of the Fréchet differentiability of farthest point map and the conditions $\phi \in \Delta_2$ and $\phi > 0$ in reflexive Musielak-Orlicz function spaces.

Theorem 39. *For the Musielak-Orlicz space $L^\phi(\mu)$ the following statements are equivalent:*

(i) *$\phi \in \Delta_2$;*

(ii) *for each closed order bounded sublattice A in $L^\phi(\mu)$ and $x \in L^\phi(\mu)$ such that $x \geq A$ (or $x \leq A$), card $(F_A(x)) \geq 1$.*

Moreover (ii) is true for $E^\phi(\mu)$ with $\phi < \infty$.

Proof. It is a consequence of Theorem 23. □

Remark 40. In Musielak-Orlicz space L^ϕ with $\mu(T) < \infty$ and nonatomic measure μ, if $\phi \in \Delta_2$ and $\phi > 0$ with any norm the sets of farthest points in dominated farthest points problem are coinciding; this means that $F_A^1(x) = F_A^2(x) = F_A^3(x)$, for every $x \in X$ and sublattice $A \subset X$ with $x \geq A$, and that $F_A^i(x)$ for $i = 1, 2, 3$ denote the set of farthest points with respect to Luxemburg norm, Orlicz norm, and Ammemyia norm, respectively.

Corollary 41. *The dominated farthest points problem in $L^\phi(\mu)$ ($E^\phi(\mu)$ with $\phi < \infty$) with respect to closed order bounded sublattices is unique if and only if $\phi > 0$ and $\phi \in \Delta_2$ (resp., $\phi > 0$).*

Proof. From [4, Theorem 2.7] (resp., [4, Theorem 2.8]) $L^\phi(\mu)$ has order continuous norm and it is an STM space (resp., $E^\phi(\mu)$); so from Theorem 22 the proof is complete. □

Theorem 42. *In the Musielak-Orlicz spaces the following statements are equivalent:*

(i) *$\phi \in \Delta_2$;*

(ii) *for all closed order bounded linear sublattices $A \subset L^\phi(\mu)$, $F_A(x) \neq \emptyset$ for each $x \in L^\phi(\mu)$.*

Moreover (ii) is true for $E^\phi(\mu)$ with $\phi < \infty$.

Proof. (ii) $\to$ (i). It is an immediate consequence of (ii) $\to$ (i), in Theorem 39.

(i) $\to$ (ii). Suppose that $\{x_n\}$ is a maximizing sequence in A; for example,

$$\lim_{n\to\infty}\|x - x_n\|_\phi = f_A(x). \tag{14}$$

Since A is a bounded subset of $L^\phi(\mu)$, so $\sup_n\{\|x_n\|_\phi\} < \infty$. $L^\phi(\mu)$ is a KB-space [4, Theorem 3.5] so $\{x_n\}$ is convergent to some $x_0 \in X$. Since A is a closed linear sublattice, so $x_0 \in A$; this complete the proof. □

Remark 43. Let X be a KB-space; with the same argument as Theorem 44, we can show that for all closed bounded linear sublattices $A \subset X$, $F_A(x) \neq \emptyset$ for each $x \in X$.

Theorem 44. *Let A be a closed order bounded sublattice in the reflexive Musielak-Orlicz spaces with the Luxemburg norm; the following statements are equivalent:*

(i) *the dominated farthest points problem is strongly solvable;*

(ii) *any sequence* $\{a_n\} \subset A$ *such that* $\lim_{n\to\infty}\|x - a_n\| = f_A(x)$ *is convergent;*

(iii) *if* $x_n \to x_0$ *and* $\{a_k\} = F_A(x_k)$ *for* $k \in \mathbb{N} \cup \{0\}$*, then* $a_n \to a_0$*;*

(iv) f_A *is Fréchet differentiable at* x_0*;*

(v) $\phi > 0$ *and* $\phi \in \Delta_2$*;*

(vi) $L^{\phi}(\mu)$ *(resp.* $E^{\phi}(\mu)$ *with* $\phi < \infty$*) is a UM (resp. STM) space.*

Proof. From Theorem 36 (i)–(iv) are equivalent. From [4, Theorem 2.7, 2.8] (v) and (vi) are equivalent. From Theorems 38 and 25 (i) and (v) are equivalent. □

Conflict of Interests

The authors declare that they have no conflict of interests.

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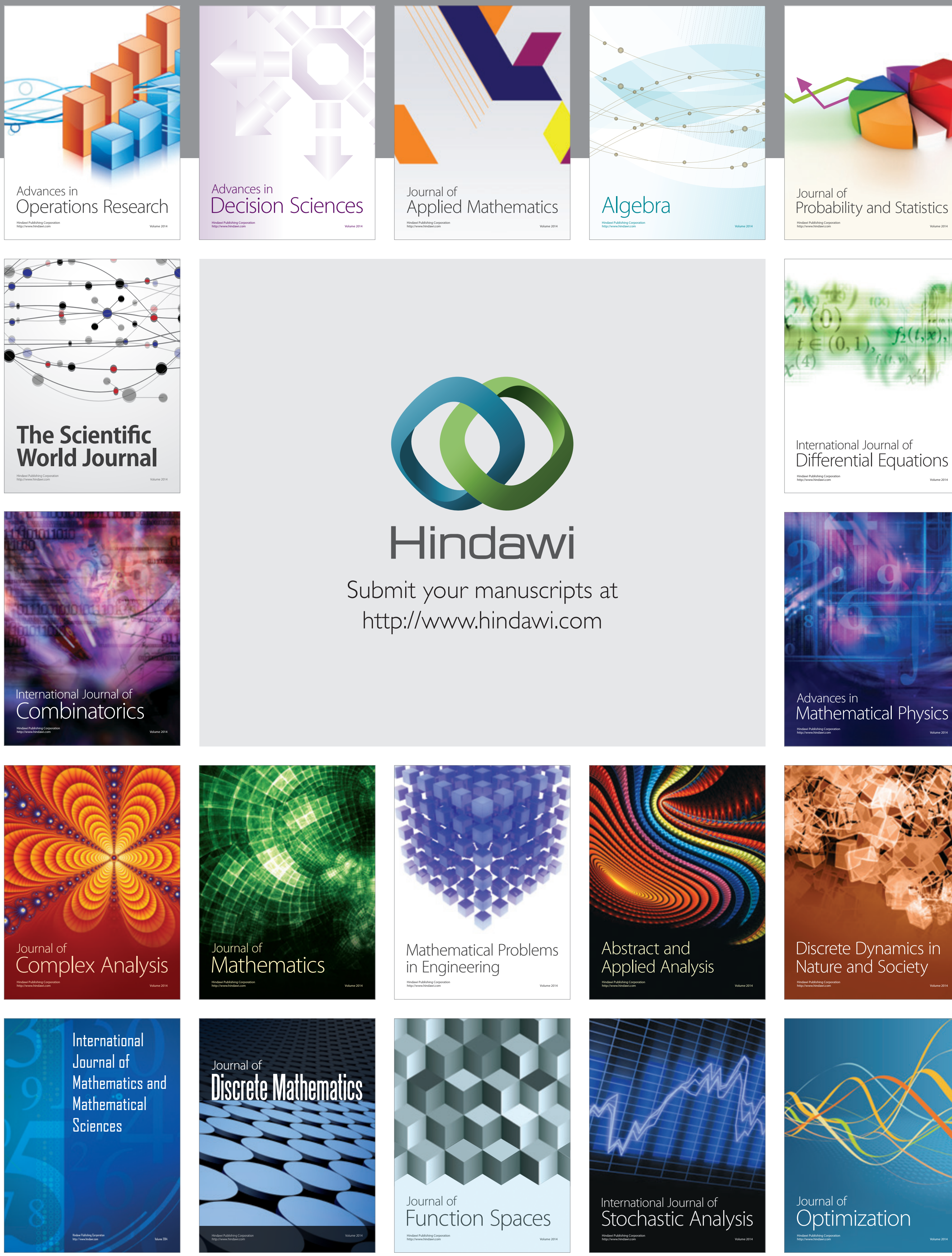
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