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Research Article

The Semimartingale Approach to Almost Sure Stability Analysis of a Two-Stage Numerical Method for Stochastic Delay Differential Equation

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Almost sure exponential stability of the split-step backward Euler (SSBE) method applied to an Itô-type stochastic differential equation with time-varying delay is discussed by the techniques based on Doob-Mayer decomposition and semimartingale convergence theorem. Numerical experiments confirm the theoretical analysis.

1. Introduction

In this paper we study the following nonlinear SDDE:

$$\begin{aligned} \mathrm{d}X(t) = {} & f(X(t), X(t-\tau(t)))\,\mathrm{d}t \\ & + g(X(t), X(t-\tau(t)))\,\mathrm{d}W(t), \end{aligned} \tag{1}$$

for every $t \geq 0$. Here $\tau(t)$ is a time-varying delay satisfying $\tau > 0$ and $-\tilde{\tau} := \inf\{t - \tau(t) : t \geq 0\}$. The initial function $X(t) = \psi(t)$ when $t \in [-\tilde{\tau}, 0]$. We further assume that the initial data is independent of Wiener measure driving the equation and $W(t)$ is a scalar Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ with a filtration satisfying the usual conditions. Moreover, $f, g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel-measurable functions.

Stability theory for numerical methods applied to stochastic differential equation (SDE) typically deals with mean-square behavior [1]. The mean-square stability analysis of numerical methods for SDDE has received a great deal of attention (see, e.g., [2, 3] and the references therein). Recently, the almost sure (a.s.) stability (or the trajectory stability) is becoming prevalent in the science literature [4–11]. However, the prior works concerned with SDDE are [7, 8, 10]. Rodkina et al. [7] studied almost sure stability of a drift-implicit θ-method applied to an SDE with memory. Using the martingale techniques, Wu and his coauthors [8, 10] discussed almost sure exponential stability of the Euler-Maruyama (EM) method for the SDE with a constant delay and stochastic functional differential equation. We note that the two above schemes are all single-stage method; this paper studies the almost sure stability of a two-stage scheme named split-step backward Euler (SSBE) method [12, 13] applied to the nonlinear SDDE (1) with time-varying delay.

Applying the SSBE method (see [12, 13]) to (1) yields

$$x_n^* = x_n + hf(x_n^*, \tilde{x}_n), \tag{2a}$$

$$x_{n+1} = x_n^* + g(x_n^*, \tilde{x}_n)\,\Delta w_n, \tag{2b}$$

where $\Delta w_n := W(t_{n+1}) - W(t_n)$ and for $0 \leq \mu \leq 1$, $q_n \in \mathbb{Z}^+$,

$$\tilde{x}_n = \begin{cases} \psi(t_n - \tau(t_n)), & t_n - \tau(t_n) < 0; \\ \mu x_{n-q_n+1} + (1-\mu)\,x_{n-q_n}, & 0 \leq t_n - \tau(t_n) \in [t_{n-q_n}, t_{n-q_n+1}). \end{cases} \tag{3}$$

Here h is the step size and x_n denotes the approximation of $X(t)$ at time $t_n = nh$ $(n = 0, 1, \ldots)$. We remark that μ in (3) depends on how memory values are handled on nongrid points. The almost sure convergence of SSBE method has been investigated by Guo and Tao [14]; the main aim of this paper is to study the almost sure stability of the SSBE method applied to (1).

2. Preliminary Results

Before stating the main results, we present the essential notation and definitions which are necessary for further consideration. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$ and $C([-\tilde{\tau},0];\mathbb{R}^d)$ the family of continuous functions φ from $[-\tilde{\tau},0]$ to $\mathbb{R}^d$, equipped with the supremum norm $\|\varphi\| = \sup_{-\tilde{\tau}\le\theta\le0}|\varphi(\theta)|$. Also, denote by $C^b_{\mathcal{F}_0}([-\tilde{\tau},0];\mathbb{R}^d)$ the family of bounded, $\mathcal{F}_0$-measurable, $C([-\tilde{\tau},0];\mathbb{R}^d)$-valued random variables. If A is a vector or matrix, its transpose is denoted by A^T. The inner product of $X, Y \in \mathbb{R}^d$ is denoted by $\langle X, Y\rangle$ or X^TY.

Now we give some definitions on the almost sure exponential stability of SDDEs and its numerical approximation.

Definition 1. The solution $X(t,\psi)$ to (1) is said to be almost surely exponentially stable if there exists a constant $\eta > 0$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\log\left|X(t,\psi)\right| \le -\eta \quad \text{a.s} \tag{4}$$

for any initial data $\psi \in C^b_{\mathcal{F}_0}([-\tilde{\tau},0];\mathbb{R}^d)$.

Definition 2. The solution x_n to (2a) and (2b) is said to be almost surely exponentially stable if there exists a constant $\gamma > 0$ such that

$$\limsup_{n\to\infty}\frac{1}{nh}\log\left|x_n\right| \le -\gamma \quad \text{a.s} \tag{5}$$

for any bounded variables $\psi(kh)$ when $kh \in [-\tilde{\tau},0]$.

For the purpose of stability, we assume that $f(0,0,t) = g(0,0,t) = 0$, which implies that (1) admits the equilibrium solution $X(t) = 0$ corresponding to the initial condition $\psi(t) = 0$ for $t \in [-\tilde{\tau},0]$. As a standing hypothesis, we will impose the following local Lipschitz condition (cf. [11, 12, 14]) on the coefficients f and g.

(A1) For each integer D, there exists a positive constant K_D such that, for all $x_1, x_2, y_1, y_2 \in R^d$ with $|x_1|\vee|x_2|\vee|y_1|\vee|y_2| \le D$, and all $t\ge0$, $|f(x_1,y_1)-f(x_2,y_2)|^2 \vee |g(x_1,y_1)-g(x_2,y_2)|^2 \le K_D(|x_1-x_2|^2+|y_1-y_2|^2)$, where $\vee$ is the maximal operator.

To guarantee the almost sure stability of the unique solution to (1), we need the following assumption for the time-varying delay $\tau(t)$.

(A2) Let the delay function $\tau(t) : [0,+\infty) \to [0,\overline{\tau}]$ be Borel measurable and bounded.

In what follows we introduce the result of almost sure stability of SDDEs (1). The proof of the following lemma can be found in [15].

Lemma 3. *Let Assumptions (A1) and (A2) hold. Assume that there are four nonnegative constants λ_1–λ_4 such that*

$$2x^T f(x,0) \le -\lambda_1|x|^2, \tag{6}$$

$$\left|f(x,y)-f(x,0)\right| \le \lambda_2\left|y\right|, \tag{7}$$

$$\left|g(x,y)\right|^2 \le \lambda_3|x|^2+\lambda_4\left|y\right|^2 \tag{8}$$

for all $t\ge t_0$ and $x, y\in\mathbb{R}^d$. If

$$\lambda_1 > 2\lambda_2+\lambda_3+\lambda_4, \tag{9}$$

then the trivial solution of (1) *is almost surely exponentially stable.*

To explain our idea, we cite the discrete semimartingale convergence theorem as follows.

Theorem 4 (see [8, 9]). *Let $Z = (Z_n)_{n\in N}$ be an almost sure nonnegative stochastic sequence of $(\mathcal{F}_n,\mathcal{B})$-measurable random variables Z_n on probability space $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n\in N},\mathbb{P})$. Assume that Z permits the decomposition*

$$Z_n \le Z_0 + A^1_n - A^2_n + \mathcal{M}_n, \quad n\in N, \tag{10}$$

where $A^1 = (A^1_n)_{n\in N}$ and $A^2 = (A^2_n)_{n\in N}$ are two non-decreasing, predictable processes with $A^i_0 = 0$ $(i = 1,2)$; $\mathcal{M} = (\mathcal{M}_n)_{n\in N}$ is local $(\mathcal{F}_n)_{n\in N}$-martingale with $\mathcal{M}_0 = 0$ on $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n\in N},\mathbb{P})$. Then, the requirement of $\lim_{n\to+\infty}A^1_n < +\infty$ (a.s.) implies that

$$\lim_{n\to+\infty}\sup Z_n < +\infty, \qquad \lim_{n\to+\infty}A^2_n < +\infty \tag{11}$$

for almost all $\omega\in\Omega$.

3. Almost Sure Asymptotic Exponential Stability of Numerical Solution

In this section, our aim is to examine if the SSBE method can reproduce the almost sure exponential stability of the exact solution of (1). Comparing to the existing results of single-stage methods [8, 10], we need to appropriately estimate the intermediate solution x^*_n, which also leads to more complex structure of the inner product of x_{n+1}, so that the discrete semimartingale convergence theorem is still valid for this case.

Now we give the main result of almost sure stability of the SSBE approximate solution x_n.

Theorem 5. *Suppose that conditions of Lemma 3 are satisfied and the drift coefficient f satisfies the linear growth condition; namely, there exists a constant $K > 0$ such that*

$$\left|f(x,y)\right|^2 \le K\left(|x|^2+\left|y\right|^2\right) \tag{12}$$

for all $x, y\in\mathbb{R}^n$ and $t\ge0$. Then there exists an h_0 such that if $h < h_0$, the SSBE approximate solution x_n is almost surely exponentially stable.

Proof. Note that

$$\begin{aligned}|x_{n+1}|^2 &= |x_n|^2 + 2hx_n^T f(x_n^*, \tilde{x}_n) + |hf(x_n^*, \tilde{x}_n)|^2 \\ &\quad + |g(x_n^*, \tilde{x}_n)|(\Delta w_n)^2 \\ &\quad + 2\langle x_n + hf(x_n^*, \tilde{x}_n), g(x_n^*, \tilde{x}_n)\Delta w_n\rangle\end{aligned} \tag{13}$$

from the SSBE method (2a) and (2b).

By using (6) and (7), we have

$$\begin{aligned}x_n^T f(x_n^*, \tilde{x}_n) &= (x_n^T - x_n^{*T}) f(x_n^*, \tilde{x}_n) + x_n^{*T} f(x_n^*, \tilde{x}_n) \\ &\le h|f(x_n^*, \tilde{x}_n)|^2 + x_n^{*T} f(x_n^*, 0) \\ &\quad + x_n^{*T}(f(x_n^*, \tilde{x}_n) - f(x_n^*, 0)) \\ &\le h|f(x_n^*, \tilde{x}_n)|^2 - \frac{\lambda_1}{2}|x_n^*|^2 \\ &\quad + |x_n^*||f(x_n^*, \tilde{x}_n) - f(x_n^*, 0)|.\end{aligned} \tag{14}$$

Equation (13), together with (14), shows that

$$\begin{aligned}|x_{n+1}|^2 &\le |x_n|^2 + 3h^2|f(x_n^*, \tilde{x}_n)|^2 \\ &\quad - \lambda_1 h|x_n^*|^2 + 2\lambda_2 h|x_n^*||\tilde{x}_n| + |g(x_n^*, \tilde{x}_n)|(\Delta w_n)^2 \\ &\quad + 2\langle x_n + hf(x_n^*, \tilde{x}_n), g(x_n^*, \tilde{x}_n)\Delta w_n\rangle.\end{aligned} \tag{15}$$

Therefore, by conditions (8) and (12), we have

$$\begin{aligned}|x_{n+1}|^2 &\le |x_n|^2 + 3h^2(K|x_n^*|^2 + K|\tilde{x}_n|^2) \\ &\quad - \lambda_1 h|x_n^*|^2 + 2\lambda_2 h|x_n^*||\tilde{x}_n| \\ &\quad + \lambda_3(\Delta w_n)^2|x_n^*|^2 + \lambda_4(\Delta w_n)^2|\tilde{x}_n|^2 \\ &\quad + 2\langle x_n + hf(x_n^*, \tilde{x}_n), g(x_n^*, \tilde{x}_n)\Delta w_n\rangle \\ &\le |x_n|^2 + (3Kh^2 - \lambda_1 h + \lambda_2 h + \lambda_3(\Delta w_n)^2)|x_n^*|^2 \\ &\quad + (3Kh^2 + \lambda_2 h + \lambda_4(\Delta w_n)^2)|\tilde{x}_n|^2 \\ &\quad + 2\langle x_n + hf(x_n^*, \tilde{x}_n), g(x_n^*, \tilde{x}_n)\Delta w_n\rangle.\end{aligned} \tag{16}$$

Similarly, under conditions (6), (7), and (12),

$$\begin{aligned}|x_n^*|^2 &= \langle x_n + hf(x_n^*, \tilde{x}_n), x_n + hf(x_n^*, \tilde{x}_n)\rangle \\ &= |x_n|^2 + 2hx_n^T f(x_n^*, \tilde{x}_n) + |hf(x_n^*, \tilde{x}_n)|^2 \\ &\le |x_n|^2 + 3h^2|f(x_n^*, \tilde{x}_n)|^2 - \lambda_1 h|x_n^*|^2 + 2\lambda_2 h|x_n^*||\tilde{x}_n| \\ &\le |x_n|^2 + 3Kh^2(|x_n^*|^2 + |\tilde{x}_n|^2) \\ &\quad - \lambda_1 h|x_n^*|^2 + \lambda_2 h(|x_n^*|^2 + |\tilde{x}_n|^2),\end{aligned} \tag{17}$$

which implies that

$$(1 - 3Kh^2 + \lambda_1 h - \lambda_2 h)|x_n^*|^2 \le |x_n|^2 + (3Kh^2 + \lambda_2 h)|\tilde{x}_n|^2. \tag{18}$$

By Vieta theorem, because the discriminant of the quadratic equation $1 - 3Kh^2 + \lambda_1 h - \lambda_2 h = 0$ is positive and $-3K < 0$, there must exist an $h_1 > 0$ such that $1 - 3Kh^2 + \lambda_1 h - \lambda_2 h > 0$ for any $0 < h < h_1$; then

$$\begin{aligned}|x_n^*|^2 &\le \frac{1}{1 - 3Kh^2 + \lambda_1 h - \lambda_2 h}|x_n|^2 \\ &\quad + \frac{3Kh^2 + \lambda_2 h}{1 - 3Kh^2 + \lambda_1 h - \lambda_2 h}|\tilde{x}_n|^2.\end{aligned} \tag{19}$$

For simplicity, in what follows, the formula $1 - 3Kh^2 + \lambda_1 h - \lambda_2 h$ is denoted by G. Combining (16) and (19) leads us to

$$\begin{aligned}|x_{n+1}|^2 &\le |x_n|^2 + \frac{1 - G + \lambda_3(\Delta w_n)^2}{G}|x_n|^2 \\ &\quad + \frac{(1 - G + \lambda_3(\Delta w_n)^2)(3Kh^2 + \lambda_2 h)}{G}|\tilde{x}_n|^2 \\ &\quad + (3Kh^2 + \lambda_2 h + \lambda_4(\Delta w_n)^2)|\tilde{x}_n|^2 \\ &\quad + 2\langle x_n + hf(x_n^*, \tilde{x}_n), g(x_n^*, \tilde{x}_n)\Delta w_n\rangle.\end{aligned} \tag{20}$$

For any positive constant $C > 1$, we have

$$\begin{aligned}&C^{(i+1)h}|x_{i+1}|^2 - C^{ih}|x_i|^2 \\ &\quad = C^{(i+1)h}(|x_{i+1}|^2 - |x_i|^2) + (C^{(i+1)h} - C^{ih})|x_i|^2,\end{aligned} \tag{21}$$

which yields

$$\begin{aligned}&C^{(i+1)h}|x_{i+1}|^2 - C^{ih}|x_i|^2 \\ &\quad \le C^{(i+1)h}\left[1 - C^{-h} + \frac{1 - G + \lambda_3(\Delta w_i)^2}{G}\right]|x_i|^2 \\ &\quad + C^{(i+1)h}\left[\frac{(1 - G + \lambda_3(\Delta w_i)^2)(3Kh^2 + \lambda_2 h)}{G}\right. \\ &\qquad \left. + 3Kh^2 + \lambda_2 h + \lambda_4(\Delta w_i)^2\right]|\tilde{x}_i|^2 \\ &\quad + 2C^{(i+1)h}\langle x_i + hf(x_i^*, \tilde{x}_i), g(x_i^*, \tilde{x}_i)\Delta w_i\rangle\end{aligned} \tag{22}$$

by using (20). Summing up both sides of inequality (22) from $i = 0$ to $n-1$ ($n \geq 1$), we get

$$\begin{aligned} C^{nh}|x_n|^2 \leq{}& |x_0|^2 + \left[1 - C^{-h} + \frac{1-G}{G}\right]\sum_{i=0}^{n-1} C^{(i+1)h}|x_i|^2 \\ &+ \left(\frac{\lambda_3}{G}\right)\sum_{i=0}^{n-1} C^{(i+1)h}(\Delta w_i)^2|x_i|^2 \\ &+ \frac{3Kh^2+\lambda_2 h}{G}\sum_{i=0}^{n-1} C^{(i+1)h}|\widetilde{x}_i|^2 \\ &+ \left[\frac{(3Kh^2+\lambda_2 h)\lambda_3}{G} + \lambda_4\right]\sum_{i=0}^{n-1} C^{(i+1)h}(\Delta w_i)^2|\widetilde{x}_i|^2 \\ &+ 2\sum_{i=0}^{n-1} C^{(i+1)h}\langle x_i + hf(x_i^*, \widetilde{x}_i), g(x_i^*, \widetilde{x}_i)\Delta w_i\rangle. \end{aligned} \tag{23}$$

Let $\mathscr{M}_n^{(1)} = \sum_{i=0}^{n-1} C^{(i+1)h}|x_i|^2((\Delta w_i)^2 - h)$. Since $E((\Delta w_n)^2 - h) = 0$ and x_n is $\mathscr{F}_{nh}$-measurable, we obtain

$$\begin{aligned} E\left[\mathscr{M}_n^{(1)} \mid \mathscr{F}_{(n-1)h}\right] &= \mathscr{M}_{n-1}^{(1)} + E\left[C^{nh}|x_{n-1}|^2\left((\Delta w_{n-1})^2 - h\right) \mid \mathscr{F}_{(n-1)h}\right] \\ &= \mathscr{M}_{n-1}^{(1)} + C^{nh}|x_{n-1}|^2 E\left[\left((\Delta w_{n-1})^2 - h\right) \mid \mathscr{F}_{(n-1)h}\right] \\ &= \mathscr{M}_{n-1}^{(1)}, \end{aligned} \tag{24}$$

which implies that $\mathscr{M}_n^{(1)}$ is a martingale.

Similarly,

$$\begin{aligned} \mathscr{M}_n^{(2)} &= \sum_{i=0}^{n-1} C^{(i+1)h}|\widetilde{x}_i|^2\left((\Delta w_i)^2 - h\right), \\ \mathscr{M}_n^{(3)} &= 2\sum_{i=0}^{n-1} C^{(i+1)h}\langle x_i + hf(x_i^*, \widetilde{x}_i), g(x_i^*, \widetilde{x}_i)\Delta w_i\rangle \end{aligned} \tag{25}$$

are also martingales. Therefore,

$$\mathscr{M}_n = \frac{\lambda_3}{G}\mathscr{M}_n^{(1)} + \left[\frac{\lambda_3(3Kh^2+\lambda_2 h)}{G} + \lambda_4\right]\mathscr{M}_n^{(2)} + \mathscr{M}_n^{(3)} \tag{26}$$

is a martingale with $\mathscr{M}_0 = 0$. Then we have

$$\begin{aligned} C^{nh}|x_n|^2 \leq{}& |x_0|^2 + \left[-C^{-h} + \frac{1+\lambda_3 h}{G}\right]\sum_{i=0}^{n-1} C^{(i+1)h}|x_i|^2 \\ &+ \left[\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h\right] \\ &\times \sum_{i=0}^{n-1} C^{(i+1)h}|\widetilde{x}_i|^2 + \mathscr{M}_n. \end{aligned} \tag{27}$$

Noting that there are two approximating cases of the time dependent delay term $X(t_n - \tau(t_n))$ in (3), the following analysis will be divided into two situations. First, we have

$$\begin{aligned} C^{nh}|x_n|^2 \leq{}& |x_0|^2 + \left[-C^{-h} + \frac{1+\lambda_3 h}{G}\right]\sum_{i=0}^{n-1} C^{(i+1)h}|x_i|^2 \\ &+ \left[\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h\right] \\ &\times \sum_{i=0}^{n-1} C^{(i+1)h}|\psi(t_i - \tau(t_i))|^2 + \mathscr{M}_n \end{aligned} \tag{28}$$

under condition $t_n < \tau(t_n)$. There exists an h_2 such that, for any $0 < h < h_1 \wedge h_2$, $C^{-h} - (1+\lambda_3 h)/G > 0$, where $\wedge$ is the minimal operator. Further, we set $A_n^1 = 0$ for any nonnegative integer n, $A_0^2 = 0$,

$$A_n^2 = \left[C^{-h} - \frac{1+\lambda_3 h}{G}\right]\sum_{i=0}^{n-1} C^{(i+1)h}|x_i|^2 \tag{29}$$

for $n \geq 1$, and

$$\begin{aligned} Z_n &= C^{nh}|x_n|^2, \\ Z_0 &= |x_0|^2 + \left[\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h\right] \\ &\quad\times \sum_{i=0}^{n-1} C^{(i+1)h}|\psi(t_i - \tau(t_i))|^2. \end{aligned} \tag{30}$$

Therefore, a direct application of Theorem 4 to the sequence Z_n yields that

$$\limsup_{n\to\infty} C^{nh}|x_n|^2 \leq +\infty. \tag{31}$$

Choose the $\gamma > 0$, such that $C = e^{\gamma}$ and hence

$$\limsup_{n\to\infty} e^{\gamma nh}|x_n|^2 \leq +\infty. \tag{32}$$

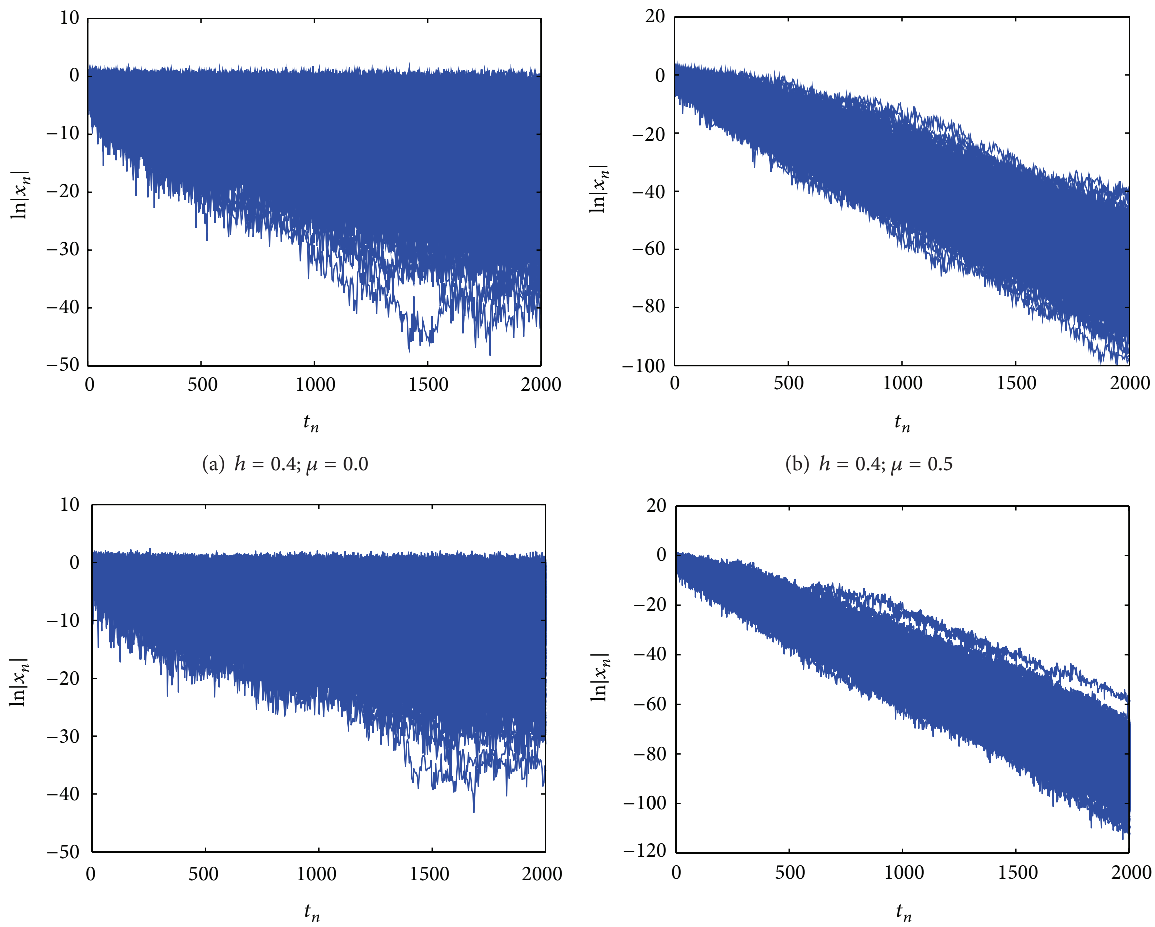
We therefore obtain that, for any $0 < h < h_1 \wedge h_2$,

$$\limsup_{n\to\infty} \frac{1}{nh}\log|x_n| \leq -\frac{\gamma}{2}, \quad \text{a.s} \tag{33}$$

as required.

Now, let us discuss the second situation: $t_n \geq \tau(t_n)$. Inequality (27) gives

$$\begin{aligned} C^{nh}|x_n|^2 \leq{}& |x_0|^2 + \left[-C^{-h} + \frac{1+\lambda_3 h}{G}\right]\sum_{i=0}^{n-1} C^{(i+1)h}|x_i|^2 \\ &+ 2\left[\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h\right] \\ &\times \sum_{i=0}^{n-1} C^{(i+1)h}\left(\mu^2|x_{i-q_i+1}|^2 + (1-\mu)^2|x_{i-q_i}|^2\right) \\ &+ \mathscr{M}_n. \end{aligned} \tag{34}$$



(a) $h = 0.4$; $\mu = 0.0$ (b) $h = 0.4$; $\mu = 0.5$ (c) $h = 0.4$; $\mu = 1.0$ (d) $h = 0.2$; $\mu = 1.0$

Figure 1: Almost sure stability of SSBE method applied to (1) with $\alpha = -10$, $\sigma = 1$, $\lambda = 1$, and $\beta = 2$.

Since

$$\sum_{i=0}^{n-1} C^{(i+1)h} \left|x_{i-q_i}\right|^2 = \sum_{i=-q_i}^{-1} C^{(i+q_i+1)h} |x_i|^2 + \sum_{i=0}^{n-1} C^{(i+q_i+1)h} |x_i|^2 - \sum_{i=n-q_i}^{n-1} C^{(i+q_i+1)h} |x_i|^2,$$

$$\sum_{i=0}^{n-1} C^{(i+1)h} \left|x_{i-q_i+1}\right|^2 = \sum_{i=-q_i+1}^{-1} C^{(i+q_i)h} |x_i|^2 + \sum_{i=0}^{n-1} C^{(i+q_i)h} |x_i|^2 - \sum_{i=n-q_i+1}^{n-1} C^{(i+q_i)h} |x_i|^2, \quad (35)$$

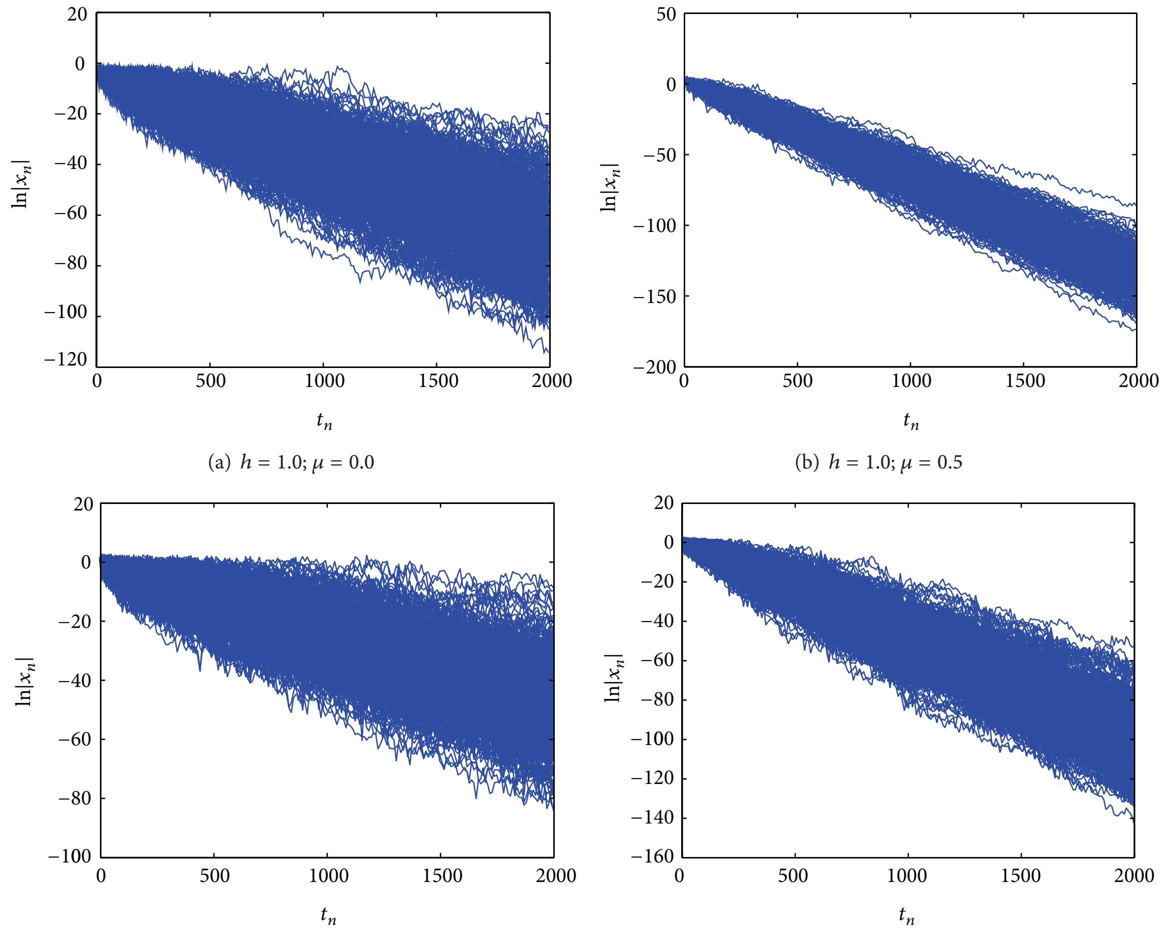
we have

$$Z_n \le Z_0 - A_n^2 + \mathscr{M}_n, \quad (36)$$

where

$$\begin{aligned} Z_n = {} & C^{nh} |x_n|^2 + 2\mu^2 \left[\frac{(1+\lambda_3 h)(3Kh^2 + \lambda_2 h)}{G} + \lambda_4 h \right] \times \sum_{i=n-q_i+1}^{n-1} C^{(i+q_i)h} |x_i|^2 \\ & + 2(1-\mu)^2 \left[\frac{(1+\lambda_3 h)(3Kh^2 + \lambda_2 h)}{G} + \lambda_4 h \right] \times \sum_{i=n-q_i}^{n-1} C^{(i+q_i+1)h} |x_i|^2, \end{aligned}$$

$$A_n^2 = \left[C^{-h} - \frac{1+\lambda_3 h}{G} - 2\mu^2 \times \left(\frac{(1+\lambda_3 h)(3Kh^2 + \lambda_2 h)}{G} + \lambda_4 h \right) C^{q_i - 1} \right.$$



(a) $h = 1.0$; $\mu = 0.0$

(b) $h = 1.0$; $\mu = 0.5$

(c) $h = 1.0$; $\mu = 1.0$

(d) $h = 0.9$; $\mu = 1.0$

Figure 2: Almost sure stability of SSBE method applied to (1) with $\alpha = -10$, $\sigma = 1$, $\lambda = 3$, and $\beta = 1$.

$$- 2(1-\mu)^2 \left(\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h \right) C^{q_i} \Bigg] \times \sum_{i=0}^{n-1} C^{(i+1)h} |x_i|^2. \tag{37}$$

There exists an h_3 such that, for any $0 < h < h_1 \wedge h_3$,

$$C^{-h} - \frac{1+\lambda_3 h}{G} - 2\mu^2 \left(\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h \right) C^{q_i - 1} - 2(1-\mu)^2 \left(\frac{(1+\lambda_3 h)(3Kh^2+\lambda_2 h)}{G} + \lambda_4 h \right) C^{q_i} > 0. \tag{38}$$

Similarly, the solution x_n is almost surely exponentially stable by using Theorem 4.

Consequently, we conclude that, for any $0 < h < h_1 \wedge h_2 \wedge h_3$, the SSBE approximate solution x_n is almost surely exponentially stable. □

4. Numerical Experiments

In this section, we present some numerical examples to illustrate our theoretical analysis. We calculated 500 sample paths of the approximate solution and plotted them along the time t (see, e.g., Figure 1(a)). Figures 1 to 2 depict the results by SSBE method in the log-scaled vertical axis. Here we set $d = 1$, drift coefficient $f = \alpha X(t) + \sigma \sin(X(t-\tau))$, diffusion coefficient $g = \lambda X(t) + \beta \sin(X(t-\tau))$, initial function $\psi(t) = t + 1$, and delay function $\tau(t) = 3(\cos(t/2))^2 + 3$.

Figures 1 to 2 show that the SSBE approximate solution x_n has better almost sure stability in the case of choosing the parameter $\mu = 0.5$ in (3). Comparing Figures 1(c) and 1(d), the almost sure stability of approximate solution can be obtained by reducing the step size h.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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