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Research Article

Tri-Integrable Couplings of the Giachetti-Johnson Soliton Hierarchy as well as Their Hamiltonian Structure

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Based on zero curvature equations from semidirect sums of Lie algebras, we construct tri-integrable couplings of the Giachetti-Johnson (GJ) hierarchy of soliton equations and establish Hamiltonian structures of the resulting tri-integrable couplings by the variational identity.

1. Introduction

Soliton theory is a power tool in expanding and describing the nonlinear phenomena in the fields of nonlinear optics, plasma physics, magnetic fluid, and so on. Searching for new integrable systems is an interesting and significant event and the subject of the integrable coupling is a new and important direction in soliton theory. Recently, various examples of bi-integrable couplings and tri-integrable couplings were introduced, which bring us inspiring thoughts and ideas to classify integrable systems with multicomponents and can generate even more diverse recursion operators in block matrix form.

For a given integrable system of evolution type [1]:

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots), \tag{1}$$

where u is a column vector of dependent variables. It has an integrable coupling as follows:

$$\overline{u}_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix} = \overline{K}(\overline{u}) = \begin{bmatrix} K(u) \\ S(u, v) \end{bmatrix}, \tag{2}$$

where u and v denote two column vectors of additional dependent variables. The earliest paper on integrable couplings obtained by Lie algebras and the Tu scheme is the [2], which gave a direct method for establishing integrable couplings and the integrable couplings of TD hierarchy. Many papers have been dedicated to this topic [3–10]. And there are other ways to construct integrable couplings such as by using perturbations [11], enlarging spectral problems [12], and creating new loop algebras [13]. Professor Yu, especially, shows that the Kronecker product is an important and effective method to construct the discrete integrable couplings in [14] and presents a scheme for constructing real nonlinear integrable couplings of continuous soliton hierarchy in [15]. In 2012, we know that bi-integrable couplings were introduced and developed in [16]. Recently, bi-integrable couplings were further extended to tri-integrable couplings. The following enlarged triangular integrable system:

$$\overline{u}_t = \begin{bmatrix} u_t \\ u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix} = \overline{K}(\overline{u}) = \begin{bmatrix} K(u) \\ S_1(u, u_1) \\ S_2(u, u_1, u_2) \\ S_3(u, u_1, u_2, u_3) \end{bmatrix}, \tag{3}$$

is called a tri-integrable coupling of the system (1) in [17, 18]. If at least one of $S_1(u, u_1)$, $S_2(u, u_1, u_2)$, and $S_3(u, u_1, u_2, u_3)$ is nonlinear with respect to any subvectors u_1, u_2, and u_3 of new dependent variables, we call this system (3) a nonlinear integrable coupling.

To construct tri-integrable couplings, we need a class of triangular 4×4 block matrices $M(A_1, A_2, A_3, A_4)$ with A_i $(i = 1, \ldots, 4)$ being square matrices of the same order. Therefore the Lie algebra $\overline{g}$ has a semidirect sum decomposition:

$$\overline{g} = g \oplus g_c, \tag{4}$$

in which $g = \{M(A_1, 0, 0, 0) \mid A_1\text{-arbitrary}\}$, $g_c = \{M(0, A_2, A_3, A_4) \mid A_2, A_3, A_4\text{-arbitrary}\}$. $\overline{g}$ is non-semisimple because of g_c being a nontrivial ideal of $\overline{g}$. The block A_1 corresponds to the original integrable system, and the other three blocks A_2, A_3, and A_4 are used to generate the supplementary vector fields S_1, S_2, and S_3 in (3) that we are looking for. Such presented Lie algebras establish a basis for generating nonlinear Hamiltonian tri-integrable couplings, while many other existing Lie algebras lead to linear Hamiltonian integrable couplings [5, 19–22].

Four classes of block matrices were introduced in [17] and the Hamiltonian tri-integrable couplings of the AKNS hierarchy were constructed based on one of the four triangular block matrices. While in this paper, we would like to construct tri-integrable couplings of the Giachetti-Johnson (GJ) hierarchy based on other triangular block matrices as follows:

$$M\left(A_1, A_2, A_3, A_4\right) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ \mathbf{0} & A_1 & \alpha A_2 & \beta A_2 + \mu A_3 \\ \mathbf{0} & \mathbf{0} & A_1 & \mu A_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_1 \end{bmatrix}, \quad (5)$$

where A_i $(i = 1, \ldots, 4)$ are square matrices of the same order and α, β, μ are arbitrary constants. Moreover, we also hope to generate the Hamiltonian structure of the resulting tri-integrable couplings.

The rest of the paper is organized as follows. In the next section, we first recall the GJ soliton hierarchy; then we construct a kind of tri-integrable couplings of the Giachetti-Johnson (GJ) soliton hierarchy and furnish Hamiltonian structures for the resulting tri-integrable couplings by the corresponding variational identity. Moreover, we will show that the resulting tri-integrable couplings have a recursion relation. In the final section, conclusions will be given.

2. Tri-Integrable Couplings of the Giachetti-Johnson (GJ) Hierarchy

2.1. The Giachetti-Johnson (GJ) Hierarchy. We first recall the GJ soliton hierarchy as follows [23]:

$$\phi_x = U\phi, \quad U = U(u, \lambda) = \begin{bmatrix} -\lambda + s & q \\ r & \lambda - s \end{bmatrix}, \quad u = \begin{pmatrix} q \\ r \\ s \end{pmatrix}, \quad (6)$$

where λ is the spectral parameter, q, r, and s are three dependent variables. Upon setting

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\ge0} W_i \lambda^{-i} = \sum_{i\ge0} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}, \quad (7)$$

and choosing the initial data $a_0 = -1$, $b_0 = c_0 = 0$, the stationary zero curvature equation $W_x = [U, W]$ generates

$$\begin{aligned} b_{i+1} &= -\frac{1}{2} b_{i,x} + s b_i - q a_i, \\ c_{i+1} &= \frac{1}{2} c_{i,x} + s c_i - r a_i, \\ a_{i+1,x} &= -r b_{i+1} + q c_{i+1}, \\ & \qquad i \ge 0. \end{aligned} \quad (8)$$

Using the compatibility conditions

$$U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0, \quad m \ge 0, \quad (9)$$

with

$$\begin{aligned} V^{[m]} &= \left(\lambda^m W\right)_+ + \Delta_m \\ &= \sum_{i\ge0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{m-i} + \begin{bmatrix} a_{m+1} & 0 \\ 0 & -a_{m+1} \end{bmatrix}, \end{aligned} \quad (10)$$

we have the GJ hierarchy of soliton equations:

$$\begin{aligned} u_{t_m} &= \begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_m} = K_m(u) = \begin{pmatrix} 2q a_{m+1} - 2b_{m+1} \\ -2r a_{m+1} + 2c_{m+1} \\ a_{m+1,x} \end{pmatrix} \\ &= J \frac{\delta H_m}{\delta u} = J L^m \begin{pmatrix} r \\ q \\ 0 \end{pmatrix}, \quad m \ge 0. \end{aligned} \quad (11)$$

The Hamiltonian operator J, the recursion operator L, and the Hamiltonian functionals in (11) are given by

$$J = \begin{pmatrix} 0 & -2 & q \\ 2 & 0 & -r \\ -q & r & \partial \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}, \quad (12)$$

$$L = \begin{pmatrix} \frac{1}{2}\partial + s & 0 & -\frac{1}{2} r \\ 0 & -\frac{1}{2}\partial + s & -\frac{1}{2} q \\ \partial^{-1} q \partial + 2\partial^{-1} q s & \partial^{-1} r \partial - 2\partial^{-1} r s & 0 \end{pmatrix}, \quad (13)$$

$$H_m = \int \frac{2 a_{m+2}}{m+1} dx, \quad m \ge 0. \quad (14)$$

Note JL is not antisymmetric; therefore, the system (11) does not possess bi-Hamiltonian structures (the method of the verification is the same as the Appendix A of [24]) and is not Liouville integrable.

2.2. Tri-Integrable Couplings. Based on the special non-semisimple Lie algebra $\overline{g}$, we choose the enlarged spectral matrix

$$\overline{U} = \overline{U}(\overline{u}, \lambda) = M\left(U, U_1, U_2, U_3\right) \in \overline{g}, \quad \overline{u} = \left(u^T, u_1^T, u_2^T, u_3^T\right)^T, \quad (15)$$

with U being defined as in (6) and

$$U_1 = \begin{bmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} v_4 & v_5 \\ v_6 & -v_4 \end{bmatrix},$$
$$U_3 = \begin{bmatrix} v_7 & v_8 \\ v_9 & -v_7 \end{bmatrix},$$
$$u_1 = (v_2, v_3, v_1)^T, \qquad u_2 = (v_5, v_6, v_4)^T,$$
$$u_3 = (v_8, v_9, v_7)^T, \tag{16}$$

where v_i, $1 \le i \le 9$ are new dependent variables.

To solve the enlarged stationary zero curvature equation

$$\overline{W}_x = [\overline{U}, \overline{W}], \tag{17}$$

we take a solution of the following type:

$$\overline{W} = \overline{W}(\overline{u}, \lambda) = M(W, W_1, W_2, W_3) \in \overline{g}, \tag{18}$$

where W is defined by (7), and

$$W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i\ge 0} W_{1,i}\lambda^{-i},$$
$$W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix} = \sum_{i\ge 0} W_{2,i}\lambda^{-i}, \tag{19}$$
$$W_3 = W_3(u, u_1, u_2, u_3, \lambda) = \begin{bmatrix} e'' & f'' \\ g'' & -e'' \end{bmatrix} = \sum_{i\ge 0} W_{3,i}\lambda^{-i}.$$

Then, from (17), we immediately get

$$\begin{aligned} W_x &= [U, W], \\ W_{1x} &= [U, W_1] + [U_1, W], \\ W_{2x} &= [U, W_2] + [U_2, W] + \alpha [U_1, W_1], \\ W_{3x} &= [U, W_3] + [U_3, W] + \beta [U_1, W_1] \\ &\quad + \mu [U_1, W_2] + \mu [U_2, W_1], \end{aligned} \tag{20}$$

with the help of Maple, which leads to

$$\begin{aligned} b_x &= -2(\lambda - s) b - 2qa, \\ c_x &= 2(\lambda - s) c + 2ra, \\ a_x &= -rb + qc, \\ f_x &= -2(\lambda - s) f - 2qe - 2v_2 a + 2v_1 b, \\ g_x &= 2(\lambda - s) g + 2re + 2v_3 a - 2v_1 c, \\ e_x &= -rf + qg - v_3 b + v_2 c, \\ f'_x &= 2\alpha v_1 f - 2\alpha v_2 e - 2(\lambda - s) f' \\ &\quad - 2qe' - 2v_5 a + 2v_4 b, \\ g'_x &= -2\alpha v_1 g + 2\alpha v_3 e + 2(\lambda - s) g' \\ &\quad + 2re' + 2v_6 a - 2v_4 c, \\ e'_x &= -\alpha v_3 f + \alpha v_2 g - rf' + qg' - v_6 b + v_5 c, \\ f''_x &= 2(\beta v_1 + \mu v_4) f - 2(\beta v_2 + \mu v_5) e + 2\mu v_1 f' \\ &\quad - 2\mu v_2 e' - 2(\lambda - s) f'' - 2qe'' - 2v_8 a + 2v_7 b, \\ g''_x &= -2(\beta v_1 + \mu v_4) g + 2(\beta v_3 + \mu v_6) e - 2\mu v_1 g' \\ &\quad + 2\mu v_3 e' + 2(\lambda - s) g'' + 2re'' + 2v_9 a - 2v_7 c, \\ e''_x &= -(\beta v_3 + \mu v_6) f + (\beta v_2 + \mu v_5) g - \mu v_3 f' \\ &\quad + \mu v_2 g' - rf'' + qg'' - v_9 b + v_8 c. \end{aligned} \tag{21}$$

The corresponding recursion relations are

$$\begin{aligned} f_{i,x} &= -2f_{i+1} + 2sf_i - 2qe_i - 2v_2 a_i + 2v_1 b_i, \\ g_{i,x} &= 2g_{i+1} - 2sg_i + 2re_i + 2v_3 a_i - 2v_1 c_i, \\ e_{i,x} &= -rf_i + qg_i - v_3 b_i + v_2 c_i; \\ f'_{i,x} &= 2\alpha v_1 f_i - 2\alpha v_2 e_i - 2f'_{i+1} + 2sf'_i - 2qe'_i \\ &\quad - 2v_5 a_i + 2v_4 b_i, \\ g'_{i,x} &= -2\alpha v_1 g_i + 2\alpha v_3 e_i + 2g'_{i+1} - 2sg'_i + 2re'_i \\ &\quad + 2v_6 a_i - 2v_4 c_i, \\ e'_{i,x} &= -\alpha v_3 f_i + \alpha v_2 g_i - rf'_i + qg'_i - v_6 b_i + v_5 c_i; \\ f''_{i,x} &= 2(\beta v_1 + \mu v_4) f_i - 2(\beta v_2 + \mu v_5) e_i + 2\mu v_1 f'_i \\ &\quad - 2\mu v_2 e'_i - 2f''_{i+1} + 2sf''_i - 2qe''_i - 2v_8 a_i + 2v_7 b_i, \\ g''_{i,x} &= -2(\beta v_1 + \mu v_4) g_i + 2(\beta v_3 + \mu v_6) e_i - 2\mu v_1 g'_i \\ &\quad + 2\mu v_3 e'_i + 2g''_{i+1} - 2sg''_i + 2re''_i + 2v_9 a_i - 2v_7 c_i, \\ e'_{i,x} &= -(\beta v_3 + \mu v_6) f_i + (\beta v_2 + \mu v_5) g_i - \mu v_3 f'_i + \mu v_2 g'_i \\ &\quad - rf''_i + qg''_i - v_9 b_i + v_8 c_i, \end{aligned} \tag{22}$$

together with (8), where $i \geq 0$. We select the initial data to be

$$e_0 = e_0' = e_0'' = -1, \qquad f_0 = g_0 = f_0' = g_0' = f_0'' = g_0'' = 0. \tag{23}$$

Then the recursion relations (22) uniquely determine the sequence of $f_i, g_i, e_i, f_i', g_i', e_i', f_i'', g_i'', e_i'', i \geq 1$, recursively. It is direct to compute the first two sets of functions by Maple:

$$\begin{gathered}
b_1 = q, \qquad c_1 = r, \qquad a_1 = 0; \\
b_2 = -\frac{1}{2}q_x + sq, \qquad c_2 = \frac{1}{2}r_x + sr, \qquad a_2 = \frac{1}{2}qr; \\
f_1 = q + v_2, \qquad g_1 = r + v_3, \qquad e_1 = 0; \\
f_2 = -\frac{1}{2}(q + v_2)_x + s(q + v_3) + v_1 q, \\
g_2 = \frac{1}{2}(r + v_3)_x + s(r + v_3) + v_1 r, \\
e_2 = \frac{1}{2}[(r + v_3)q + rv_2]; \\
f_1' = q + \alpha v_2 + v_5, \qquad g_1' = r + \alpha v_3 + v_6, \qquad e_1' = 0; \\
f_2' = -\frac{1}{2}(q + \alpha v_2 + v_5)_x + \alpha v_1(q + v_2) \\
+ s(q + \alpha v_2 + v_5) + v_4 q, \\
g_2' = \frac{1}{2}(r + \alpha v_3 + v_6)_x + \alpha v_1(r + v_3) \\
+ s(r + \alpha v_3 + v_6) + v_4 r, \\
e_2' = \frac{1}{2}[r(q + \alpha v_2 + v_5) + (\alpha v_3 + v_6)q + \alpha v_2 v_3]; \\
f_1'' = (\beta + \mu)v_2 + \mu v_5 + v_8 + q, \\
g_1'' = (\beta + \mu)v_3 + \mu v_6 + v_9 + r, \qquad e_1'' = 0; \\
f_2'' = -\frac{1}{2}[(\beta + \mu)v_2 + \mu v_5 + v_8 + q]_x \\
+ (\beta v_1 + \mu v_4)(q + v_2) + \mu v_1(q + \alpha v_2 + v_5) \\
+ s[(\beta + \mu)v_2 + \mu v_5 + v_8 + q] + v_7 q, \\
g_2'' = \frac{1}{2}[(\beta + \mu)v_3 + \mu v_6 + v_9 + r]_x \\
+ (\beta v_1 + \mu v_4)(r + v_3) + \mu v_1(r + \alpha v_3 + v_6) \\
+ s[(\beta + \mu)v_3 + \mu v_6 + v_9 + r] + v_7 r, \\
e_2'' = \frac{1}{2}\{q[(\beta + \mu)v_3 + \mu v_6 + v_9 + r] \\
+ r[(\beta + \mu)v_2 + \mu v_5 + v_8] \\
+ (\beta + \alpha\mu)v_2 v_3 + \mu(v_2 v_6 + v_3 v_5)\}.
\end{gathered} \tag{24}$$

For each integer $m \geq 0$, let us further introduce the enlarged Lax matrices:

$$\overline{V}^{[m]} = M\left(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}\right) \in \overline{g}, \tag{25}$$

with $V^{[m]}$ being defined as in (10), and

$$V_i^{[m]} = (\lambda^m W_i)_+ + \Delta_{mi}, \quad i = 1, 2, 3, \tag{26}$$

in which

$$\begin{gathered}
\Delta_{m1} = \begin{bmatrix} e_{m+1} & 0 \\ 0 & -e_{m+1} \end{bmatrix}, \qquad \Delta_{m2} = \begin{bmatrix} e_{m+1}' & 0 \\ 0 & -e_{m+1}' \end{bmatrix}, \\
\Delta_{m3} = \begin{bmatrix} e_{m+1}'' & 0 \\ 0 & -e_{m+1}'' \end{bmatrix}.
\end{gathered} \tag{27}$$

Then the enlarged zero curvature equation

$$\overline{U}_{t_m} - \overline{V}_x^{[m]} + \left[\overline{U}, \overline{V}^{[m]}\right] = 0, \tag{28}$$

generates

$$\begin{gathered}
U_{1,t_m} - V_{1,x}^{[m]} + \left[U, V_1^{[m]}\right] + \left[U_1, V^{[m]}\right] = 0, \\
U_{2,t_m} - V_{2,x}^{[m]} + \left[U, V_2^{[m]}\right] + \left[U_2, V^{[m]}\right] + \alpha\left[U_1, V_1^{[m]}\right] = 0, \\
U_{3,t_m} - V_{3,x}^{[m]} + \left[U, V_3^{[m]}\right] + \left[U_3, V^{[m]}\right] + \beta\left[U_1, V_1^{[m]}\right] \\
+ \mu\left[U_1, V_2^{[m]}\right] + \mu\left[U_2, V_1^{[m]}\right] = 0,
\end{gathered} \tag{29}$$

together with (9). This presents the supplementary systems:

$$\begin{gathered}
\overline{v}_{t_m} = S_m(\overline{u}) = \begin{bmatrix} S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix}, \\
\overline{v} = (v_2, v_3, v_1, v_5, v_6, v_4, v_8, v_9, v_7)^T, \quad m \geq 0,
\end{gathered} \tag{30}$$

where

$$S_{1,m}(u, u_1) = \begin{bmatrix} -2f_{m+1} + 2qe_{m+1} + 2v_2 a_{m+1} \\ 2g_{m+1} - 2re_{m+1} - 2v_3 a_{m+1} \\ e_{m+1,x} \end{bmatrix}, \tag{31}$$

$$\begin{aligned}
&S_{2,m}(u, u_1, u_2) \\
&= \begin{bmatrix} -2f_{m+1}' + 2qe_{m+1}' + 2\alpha v_2 e_{m+1} + 2v_5 a_{m+1} \\ 2g_{m+1}' - 2re_{m+1}' - 2\alpha v_3 e_{m+1} - 2v_6 a_{m+1} \\ e_{m+1,x}' \end{bmatrix},
\end{aligned} \tag{32}$$

$$\begin{aligned}
&S_{3,m}(u, u_1, u_2, u_3) \\
&= \begin{bmatrix} -2f_{m+1}'' + 2qe_{m+1}'' + 2\mu v_2 e_{m+1}' \\ +2(\beta v_2 + \mu v_5)e_{m+1} + 2v_8 a_{m+1} \\ 2g_{m+1}'' - 2re_{m+1}'' - 2\mu v_3 e_{m+1}' \\ -2(\beta v_3 + \mu v_6)e_{m+1} - 2v_9 a_{m+1} \\ e_{m+1,x}'' \end{bmatrix}.
\end{aligned} \tag{33}$$

In this way, the hierarchy from enlarged zero curvature equations can be written as

$$\begin{aligned}
\overline{u}_{t_m} &= \overline{K}_m(\overline{u}) \\
&= \left(q_{t_m}, r_{t_m}, s_{t_m}, v_{2,t_m}, v_{3,t_m}, v_{1,t_m}, v_{5,t_m}, v_{6,t_m},\right. \\
&\qquad \left. v_{4,t_m}, v_{8,t_m}, v_{9,t_m}, v_{7,t_m}\right)^T \\
&= \begin{bmatrix} K_m(u) \\ S_{1,m}(u,u_1) \\ S_{2,m}(u,u_1,u_2) \\ S_{3,m}(u,u_1,u_2,u_3) \end{bmatrix} \\
&= \begin{bmatrix} 2qa_{m+1} - 2b_{m+1} \\ -2ra_{m+1} + 2c_{m+1} \\ a_{m+1,x} \\ -2f_{m+1} + 2qe_{m+1} + 2v_2 a_{m+1} \\ 2g_{m+1} - 2re_{m+1} - 2v_3 a_{m+1} \\ e_{m+1,x} \\ -2f'_{m+1} + 2qe'_{m+1} + 2\alpha v_2 e_{m+1} + 2v_5 a_{m+1} \\ 2g'_{m+1} - 2re'_{m+1} - 2\alpha v_3 e_{m+1} - 2v_6 a_{m+1} \\ e'_{m+1,x} \\ -2f''_{m+1} + 2qe''_{m+1} + 2\mu v_2 e'_{m+1} \\ +2(\beta v_2 + \mu v_5) e_{m+1} + 2v_8 a_{m+1} \\ 2g''_{m+1} - 2re''_{m+1} - 2\mu v_3 e'_{m+1} \\ -2(\beta v_3 + \mu v_6) e_{m+1} - 2v_9 a_{m+1} \\ e''_{m+1,x} \end{bmatrix},
\end{aligned} \tag{34}$$

for the given hierarchy (11).

Obviously, taking $v_i = 0$ ($i = 1, \ldots, 9$), the system (34) reduces to the system (11). Therefore, the system (34) is a tri-integrable coupling of the system (11).

2.3. Hamiltonian Structures. As we all know, when an integrable system is generated, one of our primary tasks is to construct Hamiltonian structures of the resulting integrable system. In this subsection, we will generate Hamiltonian structures for the tri-integrable couplings (34) by applying the associated variational identity [25]:

$$\frac{\delta}{\delta \overline{u}} \int \left\langle \frac{\partial \overline{U}}{\partial \lambda}, \overline{W} \right\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left\langle \frac{\partial \overline{U}}{\partial \overline{u}}, \overline{W} \right\rangle. \tag{35}$$

For the sake of convenience, we transform the Lie algebra $\overline{g}$ into a vector form by the mapping:

$$\begin{gathered}
\delta : \overline{g} \longrightarrow R^{12}, \qquad A \longmapsto (a_1, \ldots, a_{12})^T, \\
A = M(A_1, A_2, A_3, A_4) \in \overline{g}, \\
A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix}, \quad 1 \le i \le 4.
\end{gathered} \tag{36}$$

The mapping δ induces a Lie algebraic structure and the commutator $[\cdot,\cdot]$ on R^{12} reads

$$\begin{gathered}
[a,b]^T = a^T R(b), \qquad a = (a_1, \ldots, a_{12})^T, \\
b = (b_1, \ldots, b_{12})^T \in R^{12},
\end{gathered} \tag{37}$$

where

$$\begin{gathered}
R(b) = M(R_1, R_2, R_3, R_4), \\
R_i = \begin{bmatrix} 0 & 2b_{3i-1} & -2b_{3i} \\ b_{3i} & -2b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2b_{3i-2} \end{bmatrix}, \quad 1 \le i \le 4.
\end{gathered} \tag{38}$$

A bilinear form on R^{12} can be defined by $\langle a,b\rangle = a^T F b$, where F is a constant matrix. The symmetric property $\langle a,b\rangle = \langle b,a\rangle$ and the Lie product $\langle a,[b,c]\rangle = \langle [a,b],c\rangle$ mean that $F^T = F$ and $F(R(b))^T = -R(b)F$ for all $b \in R^{12}$. This matrix equation leads to a linear system of equations on the elements of F. Solving the resulting system by Maple yields

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\eta_3 + \beta\eta_4 & \mu\eta_4 & 0 \\ \eta_3 & \mu\eta_4 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{39}$$

where η_i, $1 \le i \le 4$, are arbitrary constants. Therefore, a bilinear form on the semidirect sum $\overline{g}$ of Lie algebras can be determined by

$$\begin{aligned}
\langle A,B\rangle_{\overline{g}} &= \langle \delta(A), \delta(B)\rangle_{R^{12}} \\
&= (a_1, a_2, \ldots, a_{12}) F (b_1, b_2, \ldots, b_{12})^T,
\end{aligned} \tag{40}$$

with $A, B \in \overline{g}$. It is easy to compute $\det(F) = 16\eta_4^{12}\mu^6$; obviously, when η_4 and μ are nonzero constants, the bilinear form (40) is nondegenerate. But η_1, η_2, η_3 can be arbitrary constants. Simply, we take $\eta_1 = \eta_2 = \eta_3 = 0$; therefore, to apply the variational identity (35), we compute that

$$\begin{gathered}
\left\langle \overline{W}, \frac{\partial \overline{U}}{\partial \lambda} \right\rangle = -2e''\eta_4, \\
\begin{aligned}
&\left\langle \overline{W}, \frac{\partial \overline{U}}{\partial \overline{u}} \right\rangle \\
&\quad = \left(g''\eta_4, f''\eta_4, 2e''\eta_4, (\beta g + \mu g')\eta_4, (\beta f + \mu f')\eta_4,\right. \\
&\qquad \left. 2(\beta e + \mu e')\eta_4, \mu g\eta_4, \mu f\eta_4, 2\mu e\eta_4, c\eta_4, b\eta_4, 2a\eta_4\right)^T,
\end{aligned} \\
\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda} \ln\left|\left\langle \overline{W}, \overline{W}\right\rangle\right| = 0.
\end{gathered} \tag{41}$$

Thus by (35), we obtain

$$\begin{aligned}
&\frac{\delta}{\delta \overline{u}} \int \frac{2e''_{m+1}\,\eta_4}{m} dx \\
&\quad = \left(g''_m\eta_4, f''_m\eta_4, 2e''_m\eta_4, (\beta g_m + \mu g'_m)\eta_4, (\beta f_m + \mu f'_m)\eta_4,\right. \\
&\qquad 2(\beta e_m + \mu e'_m)\eta_4, \mu g_m\eta_4, \mu f_m\eta_4, \\
&\qquad \left. 2\mu e_m\eta_4, c_m\eta_4, b_m\eta_4, 2a_m\eta_4\right)^T.
\end{aligned} \tag{42}$$

Therefore, the tri-integrable couplings of the GJ hierarchy in (34) possess the following Hamiltonian structures:

$$\overline{u}_{t_m} = \overline{K}_m(\overline{u}) = \overline{J}\frac{\delta \overline{H}_m}{\delta \overline{u}}, \quad m \ge 0, \tag{43}$$

where the Hamiltonian operator $\bar{J}$ is given by

$$\bar{J} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{\eta_4}J \\ \mathbf{0} & \mathbf{0} & \frac{1}{\mu\eta_4}J & J_1 \\ \mathbf{0} & \frac{1}{\mu\eta_4}J & J_2 & J_3 \\ \frac{1}{\eta_4}J & J_1 & J_3 & J_4 \end{bmatrix}, \quad (44)$$

with J being the same as (12), and

$$J_1 = \frac{1}{\eta_4}\begin{bmatrix} 0 & 0 & v_2 \\ 0 & 0 & -v_3 \\ -v_2 & v_3 & 0 \end{bmatrix},$$
$$J_2 = \frac{1}{\mu^2\eta_4}\begin{bmatrix} 0 & 2\beta & \alpha\mu v_2 - \beta q \\ -2\beta & 0 & -(\alpha\mu v_3 - \beta r) \\ -(\alpha\mu v_2 - \beta q) & \alpha\mu v_3 - \beta r & -\beta\partial \end{bmatrix},$$
$$J_3 = \frac{1}{\eta_4}\begin{bmatrix} 0 & 0 & v_5 \\ 0 & 0 & -v_6 \\ -v_5 & v_6 & 0 \end{bmatrix},$$
$$J_4 = \frac{1}{\eta_4}\begin{bmatrix} 0 & 0 & v_8 \\ 0 & 0 & -v_9 \\ -v_8 & v_9 & 0 \end{bmatrix}, \quad (45)$$

and the Hamiltonian functionals are determined by

$$\bar{H}_m = \int \frac{2e''_{m+2}\,\eta_4}{m+1}d_x, \quad m \geq 0. \quad (46)$$

The hierarchy (43) can be rewritten as

$$\bar{u}_{t_m} = \bar{K}_m = \bar{J}\frac{\delta\bar{H}_m}{\delta\bar{u}} = \bar{J}\,\bar{L}\frac{\delta\bar{H}_{m-1}}{\delta\bar{u}}$$
$$= \bar{J}\,\bar{L}^m \begin{bmatrix} [(\beta+\mu)v_3 + \mu v_6 + v_9 + r]\eta_4 \\ [(\beta+\mu)v_2 + \mu v_5 + v_8 + q]\eta_4 \\ 0 \\ [\beta(r+v_3) + \mu(r + \alpha v_3 + v_6)]\eta_4 \\ [\beta(q+v_2) + \mu(q + \alpha v_2 + v_5)]\eta_4 \\ 0 \\ \mu(r+v_3)\eta_4 \\ \mu(q+v_2)\eta_4 \\ 0 \\ r\eta_4 \\ q\eta_4 \\ 0 \end{bmatrix}, \quad m \geq 1, \quad (47)$$

where the recursion operator $\bar{L}$ is given by

$$\bar{L} = \begin{bmatrix} L & L_1 & L_2 & L_3 \\ \mathbf{0} & L & \alpha L_1 & \beta L_1 + \mu L_2 \\ \mathbf{0} & \mathbf{0} & L & \mu L_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & L \end{bmatrix}, \quad (48)$$

with L being the same as (13), and

$$L_1 = \begin{bmatrix} v_1 & 0 & -\frac{1}{2}v_3 \\ 0 & v_1 & -\frac{1}{2}v_2 \\ 2\partial^{-1}(qv_1 + v_2 s) + \partial^{-1}v_2\partial & -2\partial^{-1}(sv_3 + rv_1) + \partial^{-1}v_3\partial & 0 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} v_4 & 0 & -\frac{1}{2}v_6 \\ 0 & v_4 & -\frac{1}{2}v_5 \\ 2\partial^{-1}(\alpha v_1 v_2 + qv_4 + sv_5) + \partial^{-1}v_5\partial & -2\partial^{-1}(\alpha v_1 v_3 + rv_4 + sv_6) + \partial^{-1}v_6\partial & 0 \end{bmatrix}, \quad (49)$$
$$L_3 = \begin{bmatrix} v_7 & 0 & -\frac{1}{2}v_9 \\ 0 & v_7 & -\frac{1}{2}v_8 \\ L_{3,1} & L_{3,2} & 0 \end{bmatrix},$$

with

$$L_{3,1} = 2\partial^{-1}[v_1(\beta v_2 + \mu v_5) + \mu v_2 v_4 + qv_7 + sv_8] + \partial^{-1}v_8\partial,$$
$$L_{3,2} = -2\partial^{-1}[v_1(\beta v_3 + \mu v_6) + \mu v_3 v_4 + rv_7 + sv_9] + \partial^{-1}v_9\partial. \quad (50)$$

Therefore, the hierarchy (34) possesses a recursion relation:

$$K_{m+1} = \Phi K_m, \quad m \geq 0, \quad (51)$$

where $\Phi = \bar{J}\,\bar{L}\,\bar{J}^{-1}$. But $\bar{J}\,\bar{L}$ is not antisymmetric; therefore, the system (11) does not have bi-Hamiltonian structures (the way of the verification is the same as the Appendix A in [24]) and is not Liouville integrable.

3. Conclusions

In this paper, tri-integrable couplings for the Giachetti-Johnson hierarchy of continuous soliton equations were generated by using semidirect sums of Lie algebras. Moreover, we established their Hamiltonian structures through the variational identities. Clearly, mathematical structures behind integrable couplings are indeed rich and interesting, though complicated. It is worthy to mention that the method proposed in this paper can also be applied to other soliton hierarchy.

Note that we can generate more diverse tri-integrable couplings because the enlarged spectral matrix $\overline{U}$ has more other forms. For instance, we can specify it in either one of the following forms:

$$\overline{U} = \begin{bmatrix} U & U_1 & U_2 & U_3 \\ \mathbf{0} & U & \alpha U_1 + \beta U_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & U & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & U \end{bmatrix},$$
$$\overline{U} = \begin{bmatrix} U & U_1 & U_2 & U_3 \\ \mathbf{0} & U & \mathbf{0} & \alpha U_1 + \beta U_2 \\ \mathbf{0} & \mathbf{0} & U & \zeta U_1 + \mu U_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & U \end{bmatrix}, \tag{52}$$

which were introduced in [17].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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