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Research Article

Weak and Strong Limit Theorems for Stochastic Processes under Nonadditive Probability

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This paper extends laws of large numbers under upper probability to sequences of stochastic processes generated by linear interpolation. This extension characterizes the relation between sequences of stochastic processes and subsets of continuous function space in the framework of upper probability. Limit results for sequences of functional random variables and some useful inequalities are also obtained as applications.

1. Introduction

Laws of large numbers are the cornerstones of theory of probability and statistics. As we know, under appropriate assumptions, the well-known strong law of large numbers (SLLN for short) states that for a sequence of random variables $\{X_n\}_{n=1}^{\infty}$, its sample mean $S_n/n := \sum_{i=1}^{n} X_i/n$ converges to a unique constant almost surely in the framework of probability. But many empirical analyses and theoretical works show us that nonadditive probability and nonlinear expectation are very probably faced in economics, finance, number theory, statistics, and many other fields, such as capacity, Choquet integral (see Choquet [1]), (nontrivial) g-probability, (nontrivial) g-expectation (see El Karoui et al. [2]), and G-expectation (see Peng [3]). And for each nonadditive probability, say c, we can define many different expectations related to c, denoted by E^c. For nonlinear E^c, random variables X_n may have mean uncertainty; that is, $E^c[X_n] \neq -E^c[-X_n]$, or variance uncertainty; that is, $E^c[X_n^2] \neq -E^c[-X_n^2]$. In such cases, there are many scholars that investigate the limit theorems under E^c or c, such as the laws of large numbers, laws of iterated logarithm, central limit theorems under either E^c or c, and other related problems. One can refer to Peng [4, 5], Chen and Hu [6], Wu and Chen [7], the papers mentioned in the following, and some references therein.

When $\{X_n\}_{n=1}^{\infty}$ has mean uncertainty, sample mean S_n/n probably cannot converge to a unique constant almost everywhere (shortly a.e., which should be well defined) under a nonadditive probability or a set of probabilities. Marinacci [8], Teran [9], and some of the references therein investigate the SLLN via Choquet integrals related to completely monotone capacity c. They suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed random variables under capacity c and prove that all the limit points of convergent subsequences of sample mean S_n/n belong to an interval $[E^c[X_1], -E^c[-X_1]]$ with probability 1 (w.p. 1 for short) under c; that is,

$$c\left(E^c[X_1] \le \liminf_{n\to\infty} \frac{S_n}{n} \le \limsup_{n\to\infty} \frac{S_n}{n} \le -E^c[-X_1]\right) = 1. \tag{1}$$

Recently, Chen [10] and Chen et al. [11] prove the SLLN via a sublinear expectation $\mathbb{E}$. They suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent random variables under $\mathbb{E}$ (see Peng [4]) and prove that

$$v\left(\underline{\mu} \le \liminf_{n\to\infty} \frac{S_n}{n} \le \limsup_{n\to\infty} \frac{S_n}{n} \le \overline{\mu}\right) = 1, \tag{2}$$

where v is the lower probability (see Halpern [12]) corresponding to $\mathbb{E}$, $\underline{\mu} := -\mathbb{E}[-X_1]$ and $\overline{\mu} := \mathbb{E}[X_1]$. It is obvious

that $[\underline{\mu}, \overline{\mu}]$ is a subset of $[E^c[X_1], -E^c[-X_1]]$. On the other hand, Chen [10] and Chen [13] prove that any element of $[\underline{\mu}, \overline{\mu}]$ is the limit of certain convergent subsequence of sample mean w.p. 1 under upper probability V corresponding to $\mathbb{E}$.

This paper is motivated by the problem of limit theorems of sequences of stochastic processes in the framework of nonadditive probabilities and the estimation of expectations of functionals of stock prices with ambiguity. If there is no mean uncertainty, they are trivial. But if there is mean uncertainty, then as the SLLN of random variables under nonadditive probability behaves, limit theorems related to stochastic processes become interesting and different from classical case. Chen [14] investigates a limit theorem for G-quadratic variational process in the framework of G-expectation. More generally, for random variables $\{X_n\}_{n=1}^{\infty}$ with mean uncertainty, in the framework of upper and lower probabilities (V, v), we consider a simple sequence of stochastic processes $\{\eta_n(t); t \in [0,1]\}_{n=1}^{\infty}$ generated by linearly interpolating S_i/n at i/n. X. Chen and Z. Chen [15] prove that all the limit points of subsequences of $\{\eta_n\}_{n=1}^{\infty}$ are elements of $J(\underline{\mu}, \overline{\mu})$ w.p. 1 under lower probability v; namely,

$$v\left(\text{Clust}\{\eta_n\} \subset J\left(\underline{\mu}, \overline{\mu}\right)\right) = 1, \tag{3}$$

where $J(\underline{\mu}, \overline{\mu})$ is a subset of continuous function space on $[0,1]$ (see Section 2). Conversely, for any element of $J(\underline{\mu}, \overline{\mu})$, is it a limit point of certain subsequence of $\{\eta_n\}_{n=1}^{\infty}$ w.p. 1 under upper probability V? In other words, does the following statement

$$V\left(x \in \text{Clust}\{\eta_n\}\right) = 1? \tag{4}$$

hold true?

In this paper we will employ the independence condition of Peng [4] to investigate this problem and prove that under certain conditions it holds true. We will see that this strong form can be implied by a weak form (see Section 4). Under continuous upper probability our strong limit theorem becomes weaker than our weak one. From the face of this meaning it is different from classical framework. But in fact, it coincides with the classical case. We also extend our strong limit theorem to functional random variables and show some useful inequalities under continuous upper probability V.

The remaining part of this paper is organized as follows. In Section 2 we recall some basic definitions and properties of lower and upper probabilities. And we will also give basic assumptions for all of the subsequent sections. Some auxiliary lemmas are proved in Section 3. In Section 4, we prove a weak limit theorem under general upper probability. Section 5 mainly presents a strong limit theorem under continuous upper probability and its extension to functional random variables. In Section 6 we give a simple example as applications in finance.

2. Basic Settings

Let Ω be a nonempty set. $\mathscr{F}$ denotes a σ-algebra of subsets of Ω. Let (V, v) be a pair of nonadditive probabilities, related to a set of probabilities $\mathscr{P}$ on measurable space $(\Omega, \mathscr{F})$, given by

$$V(A) = \sup_{Q \in \mathscr{P}} Q(A), \qquad v(A) = \inf_{Q \in \mathscr{P}} Q(A), \quad \forall A \in \mathscr{F}. \tag{5}$$

It is obvious that upper probability V and lower probability v are conjugate capacities (see Choquet [1]); that is, (1) *normalization*: $V(\Omega) = v(\Omega) = 1$, $V(\emptyset) = v(\emptyset) = 0$; (2) *monotonicity:* for all $A, B \in \mathscr{F}$, if $A \subseteq B$, then $V(A) \le V(B)$ and $v(A) \le v(B)$; (3) *conjugation:* for all $A \in \mathscr{F}$, $v(A) = 1 - V(A^c)$, where A^c denotes the complementary set of A.

Moreover, we can easily get the following properties which are useful in this paper (see also Chen et al. [11]).

Proposition 1. *For any sequence of sets $A_n \in \mathscr{F}$, $n \ge 1$, we have the following.*

(i) *Subadditivity of V: $V(\sum_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} V(A_n)$.*

(ii) *Lower continuity of V: if $A_n \uparrow A$, then $V(A) = \lim_{n\to\infty} V(A_n)$.*

(iii) *Upper continuity of v: if $A_n \downarrow A$, then $v(A) = \lim_{n\to\infty} v(A_n)$.*

(iv) *If $v(A_n) = 1$ for all $n \ge 1$, then $v(\bigcap_{n=1}^{\infty} A_n) = 1$.*

We say upper probability V (resp., lower probability v) is continuous if and only if it is upper and lower continuous. Obviously, upper probability V is continuous if and only if lower probability v is continuous.

The corresponding pair of upper and lower expectations $(\mathbb{E}, \mathscr{E})$ of (V, v) is given as follows:

$$\mathbb{E}[X] = \sup_{Q \in \mathscr{P}} E_Q[X], \qquad \mathscr{E}[X] = \inf_{Q \in \mathscr{P}} E_Q[X], \quad \forall X \in \mathscr{M}, \tag{6}$$

where $\mathscr{M}$ denotes the set of all real-valued random variables X on $(\Omega, \mathscr{F})$ such that $\sup_{Q \in \mathscr{P}} E_Q[X] < \infty$. Obviously, $\mathbb{E}$ is a sublinear expectation (see Peng [16]).

Definition 2 (see Peng [16]). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables on $(\Omega, \mathscr{F})$ in $\mathscr{M}$. We say it is a sequence of independent random variables under upper expectation $\mathbb{E}$, if for all real-valued continuous functions φ on $\mathbb{R}^n$, denoted by $\varphi \in C(\mathbb{R}^n)$, with linear growth condition; that is, there exists a constant $C > 0$ s.t.

$$\left|\varphi(x)\right| \le C(1 + |x|), \quad \forall x \in \mathbb{R}^n, \tag{7}$$

we have

$$\begin{aligned} &\mathbb{E}\left[\varphi(X_1, X_2, \ldots, X_n)\right] \\ &\quad = \mathbb{E}\left[\mathbb{E}[\varphi(y, X_n)]_{y=(X_1, X_2, \ldots, X_{n-1})}\right], \quad \forall n \ge 2. \end{aligned} \tag{8}$$

Throughout this paper we assume (unless otherwise specified) that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent random variables under upper expectation $\mathbb{E}$ satisfying

$$\mathbb{E}[X_n] = \overline{\mu}, \qquad \mathscr{E}[X_n] = \underline{\mu}, \qquad \mathbb{E}\left[\sup_{n\geq 1}|X_n|^2\right] < \infty, \quad (9)$$

for all $n \geq 1$, respectively, where $-\infty < \underline{\mu} \leq \overline{\mu} < \infty$.

Set $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for any $n \geq 1$. We define a sequence of stochastic processes $\{\eta_n\}_{n=1}^{\infty}$ by linearly interpolating S_i/n at i/n for each $n \geq 1$ and $1 \leq i \leq n$; that is,

$$\eta_n(t) = \frac{1}{n}(1 + [nt] - nt) S_{[nt]} + \frac{1}{n}(nt - [nt]) S_{[nt]+1}, \quad \forall t \in [0,1], \quad (10)$$

where $[x]$ denotes the greatest integer which is less or equal to a nonnegative number x.

Let $C[0,1]$ be a linear space of all real-valued continuous functions on $[0,1]$ with supremum as its norm, denoted by $\|\cdot\|$. Let $J(\underline{\mu}, \overline{\mu})$ be a subset of $C[0,1]$ such that all the functions $x \in J(\underline{\mu}, \overline{\mu})$ are absolutely continuous on $[0,1]$ with $x(0) = 0$ and $\underline{\mu} \leq x'(t) \leq \overline{\mu}$ almost everywhere on $[0,1]$. Thus, we can easily have the following.

Proposition 3. *$J(\underline{\mu}, \overline{\mu})$ is compact.*

3. Auxiliary Lemmas

Before investigating the convergence problem of sequence $\{\eta_n\}_{n=1}^{\infty}$ under upper probability, in this section we first give some useful lemmas.

Definition 4. A set $A \in \mathscr{F}$ is said to be a polar set if $V(A) = 0$. We say an event holds quasisurely (q.s. for short) if it holds outside a polar set.

We first give the following property.

Lemma 5. *The sequence $\{\eta_n\}_{n=1}^{\infty}$ of functions on $[0,1]$ is relatively compact w.p. 1 under lower probability v.*

Proof. For each $n \geq 1$, function η_n can be rewritten as

$$\eta_n(t) = \left\{\frac{S_{i-1}}{n} + (S_i - S_{i-1})\left(t - \frac{i-1}{n}\right)\right\} \times I_{[(i-1)/n, i/n)}(t) + \frac{S_n}{n} I_{\{1\}}(t), \quad \forall t \in [0,1]. \quad (11)$$

Obviously, for each $n \geq 1$, $\eta_n(0) = 0$, and for any $1 \leq i \leq n$, the first-order derivative of η_n with respect to t for every $\omega \in \Omega$ is

$$\eta_n'(t) = S_i - S_{i-1} = X_i, \quad \forall t \in \left(\frac{i-1}{n}, \frac{i}{n}\right). \quad (12)$$

Then the difference of η_n with respect to t follows that for any $s, t \in [0,1]$ with $s \leq t$,

$$\begin{aligned}\eta_n(t) - \eta_n(s) &= \int_s^t \eta'(r)\,dr \\ &= \int_{[ns]/n}^{[nt]/n} \eta'(r)\,dr + \int_{[nt]/n}^{t} \eta'(r)\,dr \\ &\quad - \int_{[ns]/n}^{s} \eta'(r)\,dr \\ &= \frac{S_{[nt]} - S_{[ns]}}{n} + X_{[nt]+1}\left(t - \frac{[nt]}{n}\right) \\ &\quad - X_{[ns]+1}\left(s - \frac{[ns]}{n}\right).\end{aligned} \quad (13)$$

From $\mathbb{E}[\sup_{n\geq 1}|X_n|^2] < \infty$, we have $M := \sup_{n\geq 1}|X_n| < \infty$, *q.s.* Thus, we can get an upper bound of the norm of η_n as follows:

$$\|\eta_n\| = \sup_{t\in[0,1]} |\eta_n(t)| \leq \frac{\sum_{i=1}^{n}|X_i|}{n} \leq M, \quad q.s. \quad (14)$$

In addition, for any $s, t \in [0,1]$ such that $|t - s| \leq 1/n$, we can get from (13) that

$$|\eta_n(t) - \eta_n(s)| \leq M|t - s|, \quad q.s. \quad (15)$$

In fact, without loss of generality, we assume that $s \leq t$, if $[nt] > ns$; thus, $[nt] = [ns] + 1$; then from (13) it follows that

$$\begin{aligned}&|\eta_n(t) - \eta_n(s)| \\ &= \left|\frac{X_{[nt]}}{n} + X_{[nt]+1}\left(t - \frac{[nt]}{n}\right) - X_{[ns]+1}\left(s - \frac{[ns]}{n}\right)\right| \\ &= \left|X_{[nt]}\left(\frac{[nt]}{n} - s\right) + X_{[nt]+1}\left(t - \frac{[nt]}{n}\right)\right| \leq M(t - s), \quad q.s.\end{aligned} \quad (16)$$

Otherwise if $[nt] \leq ns$, thus $[ns] \leq [nt] < [ns] + 1 \leq [nt] + 1$, which implies that $[nt] = [ns]$; then from (13) we have

$$\begin{aligned}&|\eta_n(t) - \eta_n(s)| \\ &= \left|X_{[nt]+1}\left(t - \frac{[nt]}{n}\right) - X_{[ns]+1}\left(s - \frac{[ns]}{n}\right)\right| \\ &= |X_{[nt]+1}|(t - s) \leq M(t - s), \quad q.s.\end{aligned} \quad (17)$$

Hence, from (16) and (17) we know that (15) holds true. Thus, we can easily get that $\{\eta_n\}_{n=1}^{\infty}$ is equicontinuous with respect to t w.p. 1 under lower probability v from property (iv) of Proposition 1. Together with (14) this sequence $\{\eta_n\}_{n=1}^{\infty}$ is relatively compact in $C[0,1]$ w.p. 1 under v. We get the desired result. □

The following lemma is very useful in the proofs of our main theorems and its proof is similar as Theorem 3.1 of Hu [17]. Here we omit its proof.

Lemma 6. *Given a sequence of independent random variables $\{Y_n\}_{n=1}^{\infty}$ under $\mathbb{E}$, we assume that there exist two constants $a < b$ such that $\mathscr{E}[Y_n] = a$ and $\mathbb{E}[Y_n] = b$ for all $n \geq 1$, and we also assume that $\sup_{n\geq 1}\mathbb{E}[|Y_n|^2] < \infty$. Then for any increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ satisfying $n_k - n_{k-1}$ converges to ∞ as n tends to ∞, and for any $\varphi \in C(R)$ with linear growth, we have*

$$\lim_{k\to\infty}\mathbb{E}\left[\varphi\left(\frac{S_{n_k}-S_{n_{k-1}}}{n_k-n_{k-1}}\right)\right] = \sup_{a\leq u\leq b}\varphi(u), \tag{18}$$

where $S_m = \sum_{i=1}^{m} Y_i$ for all $m \geq 1$.

4. Weak Limit Theorem

In this section we will investigate the weak convergence problem of $\{\eta_n\}_{n=1}^{\infty}$ under general upper probability.

Theorem 7. *For any $x \in J(\underline{\mu}, \overline{\mu})$ and $\epsilon > 0$, there exists a subsequence $\{\eta_{n_m}\}_{m=1}^{\infty}$ such that*

$$\lim_{m\to\infty} V\left(\left\|\eta_{n_m} - x\right\| \leq \epsilon\right) = 1, \tag{19}$$

where $\{n_m\}_{m=1}^{\infty}$ is an increasing subsequence of $\mathbb{N}$ and depends on $\underline{\mu}$, $\overline{\mu}$, and ϵ.

Proof. For any $x \in J(\underline{\mu}, \overline{\mu})$ and $\epsilon > 0$, by Lemma 5 we only need to find a subsequence $\{\eta_{n_m}\}_{m=1}^{\infty}$ satisfying (19). Set

$$A_m = \left\{\omega \in \Omega : \left\|\eta_{n_m} - x\right\| \leq \epsilon\right\}, \quad \forall m \geq 1. \tag{20}$$

Note that for any integer $l \geq 1$,

$$\begin{aligned}
&V(A_m)\\
&= V\left(\sup_{t\in[(i-1)/l,\, i/l],\, 1\leq i\leq l}\left|\left\{\eta_{n_m}(t) - \eta_{n_m}\left(\frac{i-1}{l}\right)\right\}\right.\right.\\
&\qquad + \left\{\eta_{n_m}\left(\frac{i-1}{l}\right) - x\left(\frac{i-1}{l}\right)\right\}\\
&\qquad \left.\left. + \left\{x\left(\frac{i-1}{l}\right) - x(t)\right\}\right| \leq \epsilon\right)\\
&\geq V\left(\sup_{t\in[(i-1)/l,\, i/l],\, 1\leq i\leq l}\left|\left\{\eta_{n_m}(t) - \eta_{n_m}\left(\frac{i-1}{l}\right)\right\}\right.\right.\\
&\qquad \left. + \left\{\eta_{n_m}\left(\frac{i-1}{l}\right) - x\left(\frac{i-1}{l}\right)\right\}\right|\\
&\qquad \left. + \sup_{t\in[(i-1)/l,\, i/l],\, 1\leq i\leq l}\left|x\left(\frac{i-1}{l}\right) - x(t)\right| \leq \epsilon\right).
\end{aligned} \tag{21}$$

Denoting $D = \max\{|\underline{\mu}|, |\overline{\mu}|\}$, since $x \in J(\underline{\mu}, \overline{\mu})$, thus, for all $1 \leq i \leq l$, $|x((i-1)/l) - x(t)| \leq D/l$, for all $t \in [(i-1)/l,\, i/l]$. Hence, taking $l \geq 3D/\epsilon$ we have

$$\begin{aligned}
V(A_m) \geq V\Bigg(&\left|\left\{\eta_{n_m}\left(\frac{i-1}{l}\right) - \eta_{n_m}\left(\frac{i-2}{l}\right)\right\}\right.\\
&\left. - \left\{x\left(\frac{i-1}{l}\right) - x\left(\frac{i-2}{l}\right)\right\}\right| \leq \frac{\epsilon}{3l},\ 2 \leq i \leq l\Bigg).
\end{aligned} \tag{22}$$

Let n_m/l be a positive integer for any $m \geq 1$; then by the definition of η_{n_m} (see (10)), it follows that for $2 \leq i \leq l$, $l \geq 3D/\epsilon$ and $m \geq 1$,

$$\eta_{n_m}\left(\frac{i-1}{l}\right) - \eta_{n_m}\left(\frac{i-2}{l}\right) = \frac{S_{(i-1)n_m/l} - S_{(i-2)n_m/l}}{n_m}. \tag{23}$$

In addition, since $x \in J(\underline{\mu}, \overline{\mu})$, we know that

$$\begin{aligned}
&\frac{a_i}{l} \triangleq x\left(\frac{i-1}{l}\right) - x\left(\frac{i-2}{l}\right) \in \left[\frac{\underline{\mu}}{l}, \frac{\overline{\mu}}{l}\right],\\
&\qquad \forall 2 \leq i \leq l,\ l \geq \frac{3D}{\epsilon}.
\end{aligned} \tag{24}$$

Then it follows that

$$\begin{aligned}
V(A_m) \geq V\left(\bigcap_{i=2}^{l}\left\{\left|\frac{S_{(i-1)n_m/l} - S_{(i-2)n_m/l}}{n_m/l} - a_i\right| \leq \frac{\epsilon}{3}\right\}\right),\\
\forall m \geq 1.
\end{aligned} \tag{25}$$

For $2 \leq i \leq l$ and $\delta \in (0, \epsilon/3)$, we set

$$g^{\delta}(y) = \begin{cases} 1, & y \in \left[-\frac{\epsilon}{3} + \delta, \frac{\epsilon}{3} - \delta\right];\\ \frac{y + (\epsilon/3)}{\delta}, & y \in \left(-\frac{\epsilon}{3}, -\frac{\epsilon}{3} + \delta\right);\\ \frac{(\epsilon/3) - \delta - y}{\delta}, & y \in \left(\frac{\epsilon}{3} - \delta, \frac{\epsilon}{3}\right);\\ 0, & y \in \left[\frac{\epsilon}{3}, +\infty\right) \cup \left(-\infty, -\frac{\epsilon}{3}\right]. \end{cases} \tag{26}$$

Obviously, $\Pi_{i=2}^{l} g^{\delta}(y_i)$ is a continuous function on $\mathbb{R}^{l-1}$ satisfying linear growth condition. Since $\{X_n\}_{n=1}^{\infty}$ is independent under $\mathbb{E}$ (see Definition 2), from (25), we have

$$\begin{aligned}
V(A_m) &\geq \mathbb{E}\left[\prod_{i=2}^{l} g^{\delta}\left(\frac{S_{(i-1)n_m/l} - S_{(i-2)n_m/l}}{n_m/l} - a_i\right)\right]\\
&= \prod_{i=2}^{l}\mathbb{E}\left[g^{\delta}\left(\frac{S_{(i-1)n_m/l} - S_{(i-2)n_m/l}}{n_m/l} - a_i\right)\right].
\end{aligned} \tag{27}$$

Since, for all $2 \le i \le l$ with $l \ge 3D/\epsilon$ and $n \ge 1$, $\mathbb{E}[X_n - a_i] = \overline{\mu} - a_i$, $\mathscr{E}[X_n - a_i] = \underline{\mu} - a_i$ and $\sup_{n\ge1}\mathbb{E}[|X_n - a_i|^2] < \infty$, let n_m tend to ∞ as m tends to ∞; then by Lemma 6 we have

$$\begin{aligned}\lim_{m\to\infty}\mathbb{E}\left[g^{\delta}\left(\frac{S_{(i-1)n_m/l} - S_{(i-2)n_m/l}}{n_m/l} - a_i\right)\right] \\ = \sup_{\underline{\mu}-a_i\le u\le\overline{\mu}-a_i} g^{\delta}(u) = 1,\end{aligned} \tag{28}$$

since $a_i \in [\underline{\mu}, \overline{\mu}]$ for $2 \le i \le l$. Thus from (27) and (28) it follows that $\liminf_{m\to\infty} V(A_m) \ge 1$. Obviously, $V(A_m) \le 1$ for all $m \ge 1$. Hence this theorem follows. □

Corollary 8. *Let φ be a real-valued continuous functional on $C[0,1]$; then for any $x \in J(\underline{\mu}, \overline{\mu})$ and $\epsilon > 0$, there exists a subsequence $\{\eta_{n_m}\}_{m=1}^{\infty}$ such that*

$$\lim_{m\to\infty} V\left(\left|\varphi\left(\eta_{n_m}\right) - \varphi(x)\right| \le \epsilon\right) = 1, \tag{29}$$

where $\{n_m\}_{m=1}^{\infty}$ is an increasing subsequence of $\mathbb{N}$ and depends on $\underline{\mu}$, $\overline{\mu}$, and ϵ.

In particular, if we assume that $\varphi(x) = x(1)$ for all $x \in C[0,1]$, then we have

$$\lim_{m\to\infty} V\left(\left|\frac{S_{n_m}}{n_m} - x(1)\right| \le \epsilon\right) = 1, \tag{30}$$

where $x(1) \in [\underline{\mu}, \overline{\mu}]$.

5. Strong Limit Theorem under Continuous Upper Probability

In the previous Sections 2–4, we consider the general upper probability V. For the sake of technique, in this section we further assume that V is continuous and investigate a strong limit theorem of $\{\eta_n\}_{n=1}^{\infty}$ under such a continuous upper probability V and its extension.

5.1. Strong Limit Theorem

Theorem 9. *Any $x \in J(\underline{\mu}, \overline{\mu})$ is a limit point of some subsequence of $\{\eta_n\}_{n=1}^{\infty}$ w.p. 1 under V; that is,*

$$V\left(x \in Clust\{\eta_n\}\right) = 1, \tag{31}$$

where $Clust\{x_n\}$ denotes the cluster set of all the limit points of real sequence $\{x_n\}_{n=1}^{\infty}$.

Proof. From Lemma 5, since V is continuous, we only need to prove that for any $x \in J(\underline{\mu}, \overline{\mu})$ and any $\epsilon > 0$,

$$\begin{aligned}V\left(\liminf_{n\to\infty}\|\eta_n - x\| \le \epsilon\right) \\ = V\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\{\|\eta_m - x\| \le \epsilon\}\right) = 1.\end{aligned} \tag{32}$$

Let $\{A_m\}_{m=1}^{\infty}$ and D be defined the same as in the proof of Theorem 7. Then it is sufficient to prove that for any fixed $\epsilon > 0$ we can find a subsequence $\{n_m\}_{m=1}^{\infty}$ of $\mathbb{N}$ such that

$$V\left(\liminf_{m\to\infty}\|\eta_{n_m} - x\| \le \epsilon\right) = V\left(\bigcap_{m=1}^{\infty}\bigcup_{j=m}^{\infty}A_j\right) = 1. \tag{33}$$

Take $n_m = l^m$ for $m \ge 1$, where $l \ge 3D/\epsilon$ is an integer. From Theorem 7 and the continuity of V we can get

$$V\left(\bigcap_{m=1}^{\infty}\bigcup_{j=m}^{\infty}A_j\right) = \lim_{m\to\infty}V\left(\bigcup_{j=m}^{\infty}A_j\right) \ge \lim_{m\to\infty}V(A_m) = 1. \tag{34}$$

Thus this theorem is proved. □

Remark 10. From the proof of Theorem 9 we can see that it is implied by weak limit Theorem 7 under continuous upper probability. It seems that "weak limit theorem" is stronger than "strong limit theorem" under continuous upper probability. If $\mathscr{P}$ is a singleton, thus we have $\underline{\mu} = \overline{\mu}$. Then our "strong limit theorem" is not the same form as the strong law of large numbers for sequences of random variables, since the former form is related to inferior limit and the latter one is related to limit.

5.2. Extension to Functional Random Variables.

By Theorem 9 we can easily get the following limit result for functional random variables.

Corollary 11. *Let φ be a real-valued continuous functional defined on $C[0,1]$; then we have, for any $x \in J(\underline{\mu}, \overline{\mu})$,*

$$V\left(\varphi(x) \in Clust\{\varphi(\eta_n)\}\right) = 1. \tag{35}$$

In particular,

$$\begin{aligned}V\left(\sup_{x\in J(\underline{\mu},\overline{\mu})}\varphi(x) \le \limsup_{n\to\infty}\varphi(\eta_n)\right) \\ = V\left(\inf_{x\in J(\underline{\mu},\overline{\mu})}\varphi(x) \ge \liminf_{n\to\infty}\varphi(\eta_n)\right) = 1.\end{aligned} \tag{36}$$

From the proof of Theorem 3.1 and Corollary 3.2 of Chen et al. [11] the following lemma can be easily obtained.

Lemma 12. *Supposing f is a real-valued continuous function on $\mathbb{R}$, then*

$$\begin{aligned}v\left(\inf_{y\in[\underline{\mu},\overline{\mu}]}f(x) \le \liminf_{n\to\infty}f\left(\frac{S_n}{n}\right)\right. \\ \left.\le \limsup_{n\to\infty}f\left(\frac{S_n}{n}\right) \le \sup_{y\in[\underline{\mu},\overline{\mu}]}f(x)\right) = 1.\end{aligned} \tag{37}$$

Corollary 13. *Let f be defined the same as Lemma 12, then*

$$V\left(\limsup_{n\to\infty} f\left(\frac{S_n}{n}\right) = \sup_{y\in[\underline{\mu},\overline{\mu}]} f(x)\right) = V\left(\liminf_{n\to\infty} f\left(\frac{S_n}{n}\right) = \inf_{y\in[\underline{\mu},\overline{\mu}]} f(x)\right) = 1. \tag{38}$$

Especially, if we assume $f(x) = x$, for all $x \in \mathbb{R}$, then

$$V\left(\limsup_{n\to\infty} \frac{S_n}{n} = \overline{\mu}\right) = V\left(\liminf_{n\to\infty} \frac{S_n}{n} = \underline{\mu}\right) = 1. \tag{39}$$

Proof. Take $\varphi(x) = f(x(1))$, $\forall x \in C[0,1]$. It is easy to check that φ is a continuous functional on $C[0,1]$, and obviously $\varphi(x) \in [\underline{\mu},\overline{\mu}]$. For any $n \geq 1$, $\varphi(\eta_n) = f(\eta_n(1)) = f(S_n/n)$. Thus, from Corollary 11 it follows that

$$V\left(\limsup_{n\to\infty} f\left(\frac{S_n}{n}\right) \geq \sup_{y\in[\underline{\mu},\overline{\mu}]} f(x)\right) = V\left(\liminf_{n\to\infty} f\left(\frac{S_n}{n}\right) \leq \inf_{y\in[\underline{\mu},\overline{\mu}]} f(x)\right) = 1. \tag{40}$$

Then this corollary follows from (37) of Lemma 12 and (40). □

5.3. Inequalities. In this subsection we will give some useful examples as applications in inequalities.

Example 14. Let f be a Lebesgue integrable function defined from $[0,1]$ to $\mathbb{R}$; we denote $F(t) = \int_t^1 f(s)ds$, $t \in [0,1]$. Then

$$\liminf_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{S_i}{n^2} \leq \int_0^1 F(t)\, g_1(F(t))\, dt, \tag{41}$$

$$\limsup_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{S_i}{n^2} \geq \int_0^1 F(t)\, g_2(F(t))\, dt \tag{42}$$

hold w.p. 1 under V, respectively, where

$$g_1(y) = \begin{cases} \underline{\mu}, & y \geq 0; \\ \overline{\mu}, & y < 0; \end{cases} \qquad g_2(y) = \begin{cases} \overline{\mu}, & y \geq 0; \\ \underline{\mu}, & y < 0. \end{cases} \tag{43}$$

Especially for $f \equiv 1$, we have w.p. 1 under V, respectively,

$$\liminf_{n\to\infty} \sum_{i=1}^{n} \frac{S_i}{n^2} \leq \frac{\underline{\mu}}{2}, \qquad \limsup_{n\to\infty} \sum_{i=1}^{n} \frac{S_i}{n^2} \geq \frac{\overline{\mu}}{2}. \tag{44}$$

Proof. Observe that $\varphi(x) = \int_0^1 f(t)x(t)dt$ for all $x \in C[0,1]$ is a continuous functional defined from $C[0,1]$ to $\mathbb{R}$. And it is easy to check that w.p. 1 under v,

$$\liminf_{n\to\infty} \varphi(\eta_n) = \liminf_{n\to\infty} \int_0^1 f(t)\,\eta_n(t)\, dt = \liminf_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{S_i}{n^2}. \tag{45}$$

By Corollary 11 we know that w.p. 1 under V

$$\liminf_{n\to\infty} \varphi(\eta_n) \leq \inf_{x\in J(\underline{\mu},\overline{\mu})} \int_0^1 f(t)\, x(t)\, dt. \tag{46}$$

Since for any $x \in J(\underline{\mu},\overline{\mu})$, $x'(t) \in [\underline{\mu},\overline{\mu}]$ almost everywhere for $t \in [0,1]$, then note that, for all $x \in J(\underline{\mu},\overline{\mu})$,

$$\inf_{x\in J(\underline{\mu},\overline{\mu})} \int_0^1 f(t)\, x(t)\, dt = \inf_{x\in J(\underline{\mu},\overline{\mu})} \int_0^1 F(t)\, x'(t)\, dt \leq \int_0^1 F(t)\, g_1(t)\, dt. \tag{47}$$

Thus, inequality (41) holds w.p. 1 under V. The proof of inequality (42) is similar to inequality (41) and inequalities (44) are obvious. We complete the whole proof. □

Example 15. For any integer $k \geq 1$, we have that

$$\liminf_{n\to\infty} \frac{|S_i|^k}{k+1} \leq \frac{\min\left\{|\underline{\mu}|^k, |\overline{\mu}|^k\right\}}{k+1}, \qquad \limsup_{n\to\infty} \frac{|S_i|^k}{k+1} \geq \frac{\max\left\{|\underline{\mu}|^k, |\overline{\mu}|^k\right\}}{k+1} \tag{48}$$

hold w.p. 1 under V, respectively.

Proof. It is easy to check that $\varphi(x) = \int_0^1 |x(t)|^k dt$ is a continuous functional on $C[0,1]$. Thus, this example can be similarly proved as Example 14. □

6. Applications in Finance

We consider a capital market with ambiguity which is characterized by a set of probabilities, denoted the same as previous sections by $\mathscr{P}$ such that the corresponding upper probability V is continuous. For simplicity, let risk free rate be zero. We will investigate the stock price $\widehat{S}_t$ over time interval $[0,1]$ on the measurable space $(\Omega, \mathscr{F})$, and we assume that the increments $\Delta\widehat{S}_t := \widehat{S}_{t+\Delta t} - \widehat{S}_t$ of stock price $\widehat{S}_t$ in time period $[t, t+\Delta t]$ is independent from $\widehat{S}_t$ for all t, $t+\Delta t \in [0,1]$; that is, for each probability $Q \in \mathscr{P}$, $\Delta\widehat{S}_t$ and $\widehat{S}_t$ are mutually independent under Q for all t, $t+\Delta t \in [0,1]$. We also assume that the price of the stock is uniformly bounded with respect to $(t,\omega) \in [0,1]\times\Omega$ and the largest and smallest expected average return of this stock over time interval $[t, t+\Delta t]$ are $\overline{\mu}$ and $\underline{\mu}$, respectively; that is,

$$\mathbb{E}\left[\frac{\Delta\widehat{S}_t}{\Delta t}\right] = \sup_{Q\in\mathscr{P}} E_Q\left[\frac{\Delta\widehat{S}_t}{\Delta t}\right] = \overline{\mu}, \qquad -\mathbb{E}\left[-\frac{\Delta\widehat{S}_t}{\Delta t}\right] = \inf_{Q\in\mathscr{P}} E_Q\left[\frac{\Delta\widehat{S}_t}{\Delta t}\right] = \underline{\mu}, \tag{49}$$

where $-\infty < \underline{\mu} \leq \overline{\mu} < \infty$, and t, $t+\Delta t \in [0,1]$.

For any $n \geq 1$, take $\Delta t = 1/n$, and let $X_k = (\Delta \widehat{S}_{(k-1)/n})/(1/n)$ and $S_k = \sum_{i=1}^{k} X_i$ for $1 \leq k \leq n$. Then it is obvious that $\{X_k\}_{k=1}^{n}$ is a sequence of independent random variables in $\mathscr{M}$ under upper probability $\mathbb{E}$, with supermean $\mathbb{E}[X_k] = \overline{\mu}$ and submean $-\mathbb{E}[-X_k] = \underline{\mu}$ for all $1 \leq k \leq n$. Denote the average stock price of $\{\widehat{S}_{k/n}\}_{k=1}^{n}$ by $\int_0^1 \widehat{S}_t^n dt$; then

$$\begin{aligned}\int_0^1 \widehat{S}_t^n(\omega)\, dt &:= \sum_{k=1}^{n} \frac{\widehat{S}_{k/n}(\omega)}{n} = \sum_{k=1}^{n} \frac{\sum_{i=1}^{k} \Delta \widehat{S}_{(i-1)/n}(\omega)}{n} \\ &= \sum_{k=1}^{n} \frac{\sum_{i=1}^{k} X_i(\omega)}{n^2} = \sum_{k=1}^{n} \frac{S_k(\omega)}{n^2}, \quad \forall \omega \in \Omega.\end{aligned} \tag{50}$$

Then by inequalities (41) and (42) it follows that

$$\liminf_{n\to\infty} \int_0^1 \widehat{S}_t^n dt \leq \frac{\underline{\mu}}{2}, \qquad \limsup_{n\to\infty} \int_0^1 \widehat{S}_t^n dt \geq \frac{\overline{\mu}}{2} \tag{51}$$

hold, respectively, w.p. 1 under continuous upper probability V. (Together with X. Chen and Z. Chen [15] we will see that these two inequalities can become equalities in the future.)

7. Concluding Remarks

This paper proves that any element of subset $J(\underline{\mu}, \overline{\mu})$ of continuous function space on $[0, 1]$ is a limit point of certain subsequence of stochastic processes η_n in upper probability V and with probability 1 under continuous upper probability. It is an extension of strong law of large numbers from random variables to stochastic processes in the framework of upper probability. The limit theorem for functional random variables also is proved. It is very useful in finance when there is ambiguity. But the constraint conditions in this paper are very strong, such as the condition $\mathbb{E}[\sup_{n\geq 1} X_n^2] < \infty$ and independence under sublinear expectation. How can we weaken the constraint conditions? Does the strong limit theorem under upper probability still hold without continuity of V? We will investigate them in the future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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