

Research Article

Explicit Determinants of the RFP*r***L***r***R Circulant and RLP***r***F***r***L Circulant Matrices Involving Some Famous Numbers**

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Circulant matrices may play a crucial role in solving various differential equations. In this paper, the techniques used herein are based on the inverse factorization of polynomial. We give the explicit determinants of the RFPrLrR circulant matrices and RLPrFrL circulant matrices involving Fibonacci, Lucas, Pell, and Pell-Lucas number, respectively.

1. Introduction

It has been found out that circulant matrices play an important role in solving differential equations in various fields such as Lin and Yang discretized the partial integrodifferential equation (PIDE) in pricing options with the preconditioned conjugate gradient (PCG) method, where constructed the circulant preconditioners. By using the FFT, the cost for each linear system is $O(n \log n)$ where n is the size of the system in [1]. Lei and Sun [2] proposed the preconditioned CGNR (PCGNR) method with a circulant preconditioner to solve such Toeplitz-like systems. Kloeden et al. adopted the simplest approximation schemes for (1) in [3] with the Euler method, which reads (5) in [3]. They exploited that the covariance matrix of the increments can be embedded in a circulant matrix. The total loops can be done by fast Fourier transformation, which leads to a total computational $\cot O(m \log m) = O(n \log n)$. By using a Strang-type block-circulant preconditioner, Zhang et al. [4] speeded up the convergent rate of boundary-value methods. In [5], the resulting dense linear system exhibits so much structure that it can be solved very efficiently by a circulant preconditioned conjugate gradient method. Ahmed et al. used coupled map lattices (CML) as an alternative approach to include spatial effects in FOS. Consider the 1-system CML (10) in [6]. They claimed that the system is stable if all the eigenvalues of the circulant matrix satisfy (2) in [6]. Wu and Zou in [7] discussed the existence and approximation of solutions of asymptotic or periodic boundary-value problems of mixed functional differential equations. They focused on (5.13) in [7] with a circulant matrix, whose principal diagonal entries are zeroes.

Circulant matrix family have important applications in various disciplines including image processing, communications, signal processing, encoding, and preconditioner. They have been put on firm basis with the work of Davis [8] and Jiang and Zhou [9]. The circulant matrices, long a fruitful subject of research, have in recent years been extended in many directions [10–13]. The f(x)-circulant matrices are another natural extension of this well-studied class and can be found in [14–20]. The f(x)-circulant matrix has a wide application, especially on the generalized cyclic codes in [14]. The properties and structures of the $x^n - rx - r$ -circulant matrices, are better than those of the general f(x)-circulant matrices, so there are good algorithms for determinants.

There are many interests in properties and generalization of some special matrices with famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [21]. Dazheng gave the determinant of the Fibonacci-Lucas quasicyclic matrices [22]. Lind presented the determinants of circulant and skew circulant involving Fibonacci numbers in [23]. Shen et al. [24] discussed the determinant of circulant matrix involving Fibonacci and Lucas numbers. Akbulak and Bozkurt [25] gave the norms of Toeplitz involving Fibonacci and Lucas numbers. The authors [26, 27] discussed some properties of Fibonacci and Lucas matrices. Stanimirović et al. gave generalized Fibonacci and Lucas matrix in [28]. Z. Zhang and Y. Zhang [29] investigated the Lucas matrix and some combinatorial identities.

Firstly, we introduce the definitions of the RFPrLrR circulant matrices and RLPrFrL circulant matrices and properties of the related famous numbers. Then, we present the main results and the detailed process.

2. Definition and Lemma

Definition 1. A row first-plus-rlast *r*-right (RFPrLrR) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by RFPrLRcirc_{*r*} fr $(a_0, a_1, \ldots, a_{n-1})$, means a square matrix of the form

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & \cdots & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & \cdots & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ ra_1 & ra_2 + ra_1 & \cdots & a_0 + ra_{n-1} \end{pmatrix}.$$
 (1)

Note that the RFPrLrR circulant matrix is a $x^n - rx - r$ circulant matrix, which is neither an extension nor special case of the circulant matrix [8]. They are two completely different kinds of special matrices.

We define $\Theta_{(r,r)}$ as the basic RFPrLrR circulant matrix; that is,

$$\Theta_{(r,r)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ r & r & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}$$
(2)

$$= \operatorname{RFP} r \operatorname{LRcirc}_r \operatorname{fr} (0, 1, 0, \dots, 0).$$

Both the minimal polynomial and the characteristic polynomial of $\Theta_{(r,r)}$ are $g(x) = x^n - rx - r$, which has only simple roots, denoted by $\varepsilon_k (k = 1, 2, ..., n)$. In addition, $\Theta_{(r,r)}$ satisfies $\Theta_{(r,r)}^j = \text{RFPrLRcirc}_r \text{fr}(\underbrace{0, \ldots, 0}_{j}, 1, \underbrace{0, \ldots, 0}_{n-j-1})$ and $\Theta_{(r,r)}^n = rI_n + r\Theta_{(r,r)}$. Then a matrix *A* can be written in the

$$A = f(\Theta_{(r,r)}) = \sum_{i=0}^{n-1} a_i \Theta_{(r,r)}^i,$$
(3)

if and only if A is a RFPrLrR circulant matrix, where the polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ is called the representer of the RFPrLrR circulant matrix A.

Since $\Theta_{(r,r)}$ is nonderogatory, then *A* is a RFMrLrR circulant matrix if and only if *A* commutes with $\Theta_{(r,r)}$; that is, $A\Theta_{(r,r)} = \Theta_{(r,r)}A$. Because of the representation, RFMrLrR circulant matrices have very nice structure and the algebraic properties also can be easily attained. Moreover, the product of two RFMrLrR circulant matrices and the inverse A^{-1} are again RFMrLrR circulant matrices.

Definition 2. A row last-plus-*r*first *r*-left (RLP*r*F*r*L) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by RLP*r*FLcirc_{*r*}fr $(a_0, a_1, \ldots, a_{n-1})$, means a square matrix of the form

$$B = \begin{pmatrix} a_0 & \cdots & a_{n-2} & a_{n-1} \\ a_1 & \cdots & a_{n-1} + ra_0 & ra_0 \\ a_2 & \cdots & ra_0 + ra_1 & ra_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} + ra_0 & \cdots & ra_{n-3} + ra_{n-2} & ra_{n-2} \end{pmatrix}.$$
 (4)

Let $A = \text{RLPrFLcirc}_r \text{fr}(a_0, a_1, \dots, a_{n-1})$ and $B = \text{RFPrLRcirc}_r \text{fr}(a_{n-1}, a_{n-2}, \dots, a_0)$. By explicit computation, we find

$$A = B\widehat{I}_n,\tag{5}$$

where \hat{I}_n is the backward identity matrix of the form

$$\hat{I}_{n} = \begin{pmatrix} & & 1 \\ & 1 \\ & \ddots & \\ & 1 \\ & 1 \end{pmatrix}.$$
(6)

The Fibonacci, Lucas, Pell, and the Pell-Lucas sequences [30–36] are defined by the following recurrence relations, respectively:

$$F_{n+1} = F_n + F_{n-1}, \text{ where } F_0 = 0, \ F_1 = 1,$$

$$L_{n+1} = L_n + L_{n-1}, \text{ where } L_0 = 2, \ L_1 = 1,$$

$$P_{n+1} = 2P_n + P_{n-1}, \text{ where } P_0 = 0, \ P_1 = 1,$$

$$Q_{n+1} = 2Q_n + Q_{n-1}, \text{ where } Q_0 = 2, \ Q_1 = 2.$$
(7)

The first few values of these sequences are given by the following table ($n \ge 0$):

The sequences $\{F_n\}$, $\{L_n\}$, $\{P_n\}$, and $\{Q_n\}$ are given by the Binet formulae

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n,$$

$$P_n = \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1}, \qquad Q_n = \alpha_1^n + \beta_1^n,$$
(9)

where α , β are the roots of the characteristic equation $x^2 - x - 1 = 0$ and α_1 , β_1 are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

By Proposition 5.1 in [14], we deduce the following lemma.

Lemma 3. Let $A = RFPrLRcirc_r fr(a_0, ..., a_{n-1})$; then the eigenvalues of A are

$$f(\varepsilon_k) = \sum_{i=0}^{n-1} \left(a_i \varepsilon_k^i \right), \tag{10}$$

and in addition,

$$\det A = \prod_{k=1}^{n} \sum_{i=0}^{n-1} \left(a_i \varepsilon_k^i \right),$$
 (11)

where ε_k (k = 1, 2, ..., n) are the roots of the equation

$$x^{n} - rx - r = 0. (12)$$

Lemma 4. Consider

$$\prod_{k=1}^{n} \left(c + \varepsilon_k b + \varepsilon_k^2 a \right)$$

= $c^n - rc \left[(as)^{n-1} + (at)^{n-1} \right]$
 $- r \left[(as)^n + (at)^n \right] + r^2 a^{n-1} (c - b + a),$ (13)

where

$$s = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad t = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$
 (14)

and ε_k (k = 1, 2, ..., n) satisfy (12), $a, b, c \in \mathbf{R}$, $a \neq 0$.

Proof. Consider

$$\prod_{k=1}^{n} \left(c + \varepsilon_k b + \varepsilon_k^2 a \right) = a^n \prod_{k=1}^{n} \left(\varepsilon_k^2 + \frac{b}{a} \varepsilon_k + \frac{c}{a} \right)$$
$$= a^n \prod_{k=1}^{n} \left(\varepsilon_k - s \right) \left(\varepsilon_k - t \right) \qquad (15)$$
$$= a^n \prod_{k=1}^{n} \left(s - \varepsilon_k \right) \left(t - \varepsilon_k \right),$$

while

$$s + t = -\frac{b}{a}, \qquad st = \frac{c}{a},$$

$$s = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad t = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
(16)

Since ε_k (k = 1, 2, ..., n) satisfy (12), we must have

$$x^{n} - rx - r = \prod_{k=1}^{n} \left(x - \varepsilon_{k} \right).$$
(17)

So

$$\prod_{k=1}^{n} \left(c + \varepsilon_{k}b + \varepsilon_{k}^{2}a \right)$$

$$= a^{n} \left(s^{n} - sr - r \right) \left(t^{n} - tr - r \right)$$

$$= a^{n} \left[\left(st \right)^{n} - rst \left(s^{n-1} + t^{n-1} \right) - r \left(s^{n} + t^{n} \right) \right]$$

$$+ a^{n} \left[r^{2} \left(s + t + st + 1 \right) \right]$$

$$= a^{n} \left[\left(\frac{c}{a} \right)^{n} - r\frac{c}{a} \left(s^{n-1} + t^{n-1} \right) - r \left(s^{n} + t^{n} \right) \right]$$

$$+ a^{n} \left[r^{2} \left(\frac{c}{a} - \frac{b}{a} + 1 \right) \right]$$

$$= c^{n} - rc \left[\left(as \right)^{n-1} + \left(at \right)^{n-1} \right] - r \left[\left(as \right)^{n} + \left(at \right)^{n} \right]$$

$$+ r^{2} a^{n-1} \left(c - b + a \right).$$

3. Determinant of the RFP*r*L*r***R and RLP***r*F*r*L Circulant Matrices with the Fibonacci Numbers

Theorem 5. Let $\mathbb{A} = RFPrLRcirc_r fr(F_0, F_1, \dots, F_{n-1})$. Then

$$\det \mathbb{A} = \frac{\left(-rF_{n}\right)^{n} - (-r)^{n+1}F_{n-1}^{n-1}}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}} + \frac{\left(-r\right)^{n+1}F_{n-1}^{n-1}F_{n}\left(g_{1}^{n-1} + h_{1}^{n-1}\right)}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}} + \frac{\left(-r\right)^{n+1}F_{n-1}^{n}\left(g_{1}^{n} + h_{1}^{n}\right)}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}},$$
(19)

where

$$g_{1} = \frac{(rF_{n} + rF_{n-1} - 1)}{-2rF_{n-1}} + \frac{\sqrt{r^{2}(F_{n} - F_{n-1})^{2} - 2r(F_{n} + F_{n+1})}}{-2rF_{n-1}},$$

$$h_{1} = \frac{(rF_{n} + rF_{n-1} - 1)}{-2rF_{n-1}} - \frac{1}{\sqrt{r^{2}(F_{n} - F_{n-1})^{2} - 2r(F_{n} + F_{n+1})}}$$
(20)

 $-2rF_{n-1}$

Proof. The matrix A can be written as

$$\mathbb{A} = \begin{pmatrix} F_0 & F_1 & \cdots & F_{n-1} \\ rF_{n-1} & F_0 + rF_{n-1} & \cdots & F_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ rF_2 & rF_3 + rF_2 & \cdots & F_1 \\ rF_1 & rF_2 + rF_1 & \cdots & F_0 + rF_{n-1} \end{pmatrix}_{n \times n}$$
(21)

Using Lemma 3, the determinant of \mathbb{A} is

$$\det \mathbb{A} = \prod_{k=1}^{n} \left(F_0 + F_1 \varepsilon_k + \dots + F_{n-1} \varepsilon_k^{n-1} \right)$$
$$= \prod_{k=1}^{n} \left(\frac{\alpha - \beta}{\alpha - \beta} \varepsilon_k + \dots + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \varepsilon_k^{n-1} \right)$$
$$= \prod_{k=1}^{n} \frac{-rF_{n-1} \varepsilon_k^2 + (1 - rF_{n-1} - rF_n) \varepsilon_k - rF_n}{1 - \varepsilon_k - \varepsilon_k^2}.$$
(22)

Using Lemma 4, we obtain

$$\det \mathbb{A} = \frac{\left(-rF_{n}\right)^{n} - \left(-r\right)^{n+1}F_{n-1}^{n-1}}{1 - rL_{n-1} - rL_{n} + \left(-1\right)^{n-1}r^{2}} + \frac{\left(-r\right)^{n+1}F_{n-1}^{n-1}F_{n}\left(g_{1}^{n-1} + h_{1}^{n-1}\right)}{1 - rL_{n-1} - rL_{n} + \left(-1\right)^{n-1}r^{2}} + \frac{\left(-r\right)^{n+1}F_{n-1}^{n}\left(g_{1}^{n} + h_{1}^{n}\right)}{1 - rL_{n-1} - rL_{n} + \left(-1\right)^{n-1}r^{2}},$$
(23)

where

$$g_{1} = \frac{(rF_{n} + rF_{n-1} - 1)}{-2rF_{n-1}} + \frac{\sqrt{r^{2}(F_{n} - F_{n-1})^{2} - 2r(F_{n} + F_{n+1})}}{-2rF_{n-1}},$$

$$h_{1} = \frac{(rF_{n} + rF_{n-1} - 1)}{-2rF_{n-1}} - \frac{\sqrt{r^{2}(F_{n} - F_{n-1})^{2} - 2r(F_{n} + F_{n+1})}}{-2rF_{n-1}}.$$

$$(24)$$

Using the method in Theorem 5 similarly, we also have the following.

Theorem 6. Let $\mathbb{A}' = RFPrLRcirc_r fr(F_{n-1}, \ldots, F_0)$. Then

$$\det \mathbb{A}' = \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} - \frac{r(F_n - r)^n}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2}.$$
(25)

Theorem 7. Let $\mathbb{F} = RLPrFLcirc_r fr(F_0, \ldots, F_{n-1})$. Then

$$\det \mathbb{F} = \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right) \left(F_n - r\right)^{n-1}}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} \times (-1)^{n(n-1)/2}$$
(26)

$$-\frac{r(F_n-r)^n}{(-1)^n+rL_{n-1}-rL_n+r^2}(-1)^{n(n-1)/2}.$$

Proof. The matrix \mathbb{F} can be written as

$$\mathbb{F} = \begin{pmatrix}
F_{0} & \cdots & F_{n-2} & F_{n-1} \\
F_{1} & \cdots & F_{n-1} + rF_{0} & rF_{0} \\
F_{2} & \cdots & rF_{0} + rF_{1} & rF_{1} \\
\cdots & \cdots & \cdots & \cdots \\
F_{n-1} + rF_{0} & \cdots & rF_{n-3} + rF_{n-2} & rF_{n-2}
\end{pmatrix}$$

$$= \begin{pmatrix}
F_{n-1} & F_{n-2} & \cdots & F_{0} \\
rF_{0} & F_{n-1} + rF_{0} & \cdots & F_{1} \\
\vdots & \vdots & \ddots & \vdots \\
rF_{n-3} & rF_{n-4} + rF_{n-3} & \cdots & F_{n-2} \\
rF_{n-2} & rF_{n-3} + rF_{n-2} & \cdots & F_{n-1} + rF_{0}
\end{pmatrix}$$
(27)
$$\times \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix} = \mathbb{A}'\Gamma.$$

Hence, we have

$$\det \mathbb{F} = \det \mathbb{A}' \det \Gamma, \tag{28}$$

where $\mathbb{A}' = \operatorname{RFPrLRcirc}_r \operatorname{fr}(F_{n-1}, F_{n-2}, \dots, F_0)$ and its determinant is obtained from Theorem 6,

$$\det \mathbb{A}' = \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} - \frac{r\left(F_n - r\right)^n}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2}.$$
(29)

In addition,

$$\det \Gamma = (-1)^{n(n-1)/2},$$
(30)

so

det
$$\mathbb{F} = \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} \times (-1)^{n(n-1)/2}$$
 (31)

$$-\frac{r(F_n-r)^n}{(-1)^n+rL_{n-1}-rL_n+r^2}(-1)^{n(n-1)/2}.$$

4. Determinant of the RFMrLrR and RLMrFrL Circulant Matrices with the Lucas Numbers

Theorem 8. Let $\mathbb{B} = RFPrLRcirc_r fr(L_0, L_1, \dots, L_{n-1})$. Then

$$\det \mathbb{B} = \frac{\left(2 - rL_{n}\right)^{n}}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}} + \frac{\left(-r\right)^{n}L_{n-1}^{n-1}\left(2 - rL_{n}\right)\left(g_{2}^{n-1} + h_{2}^{n-1}\right)}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}} - \frac{\left(-r\right)^{n}L_{n-1}^{n-1}\left[rL_{n-1}\left(g_{2}^{n} + h_{2}^{n}\right) - 3r\right]}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}},$$
(32)

where

$$g_{2} = \frac{1 + rL_{n-1} + rL_{n}}{-2rL_{n-1}} + \frac{\sqrt{r^{2}(L_{n} - L_{n-1})^{2} + 10rL_{n-1} + 2rL_{n} + 1}}{-2rL_{n-1}},$$

$$h_{2} = \frac{1 + rL_{n-1} + rL_{n}}{-2rL_{n-1}} - \frac{\sqrt{r^{2}(L_{n} - L_{n-1})^{2} + 10rL_{n-1} + 2rL_{n} + 1}}{-2rL_{n-1}}.$$
(33)

Proof. The matrix $\mathbb B$ can be written as

$$\mathbb{B} = \begin{pmatrix} L_0 & L_1 & \cdots & L_{n-1} \\ rL_{n-1} & L_1 + rL_{n-1} & \cdots & L_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ rL_2 & rL_3 + rL_2 & \cdots & L_1 \\ rL_1 & rL_2 + rL_1 & \cdots & L_0 + rL_{n-1} \end{pmatrix}.$$
 (34)

Using Lemma 3, we have

$$\det \mathbb{B} = \prod_{k=1}^{n} \left(L_0 + L_1 \varepsilon_k + \dots + L_{n-1} \varepsilon_k^{n-1} \right)$$
$$= \prod_{k=1}^{n} \left[2 + (\alpha + \beta) \varepsilon_k + \dots + (\alpha^{n-1} + \beta^{n-1}) \varepsilon_k^{n-1} \right]$$
$$= \prod_{k=1}^{n} \frac{-rL_{n-1} \varepsilon_k^2 - (1 + rL_n + rL_{n-1}) \varepsilon_k - 2 + rL_n}{1 - \varepsilon_k - \varepsilon_k^2}.$$
(35)

According to Lemma 4, we obtain

$$\prod_{k=1}^{n} \left[-rL_{n-1}\varepsilon_{k}^{2} - \left(1 + rL_{n} + rL_{n-1}\right)\varepsilon_{k} + 2 - rL_{n} \right]$$

= $\left(2 - rL_{n}\right)^{n} + \left(-r\right)^{n}L_{n-1}^{n-1}\left(2 - rL_{n}\right)\left(g_{2}^{n-1} + h_{2}^{n-1}\right)^{-1}$ (36)
 $- \left(-r\right)^{n}L_{n-1}^{n-1}\left[rL_{n-1}\left(g_{2}^{n} + h_{2}^{n}\right) - 3r\right].$

Then, we get

$$\det \mathbb{B} = \frac{\left(2 - rL_{n}\right)^{n}}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}} + \frac{\left(-r\right)^{n}L_{n-1}^{n-1}\left(2 - rL_{n}\right)\left(g_{2}^{n-1} + h_{2}^{n-1}\right)}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}} - \frac{\left(-r\right)^{n}L_{n-1}^{n-1}\left[rL_{n-1}\left(g_{2}^{n} + h_{2}^{n}\right) - 3r\right]}{1 - rL_{n-1} - rL_{n} + (-1)^{n-1}r^{2}},$$
(37)

where

$$g_{2} = \frac{1 + rL_{n-1} + rL_{n}}{-2rL_{n-1}} + \frac{\sqrt{r^{2}(L_{n} - L_{n-1})^{2} + 10rL_{n-1} + 2rL_{n} + 1}}{-2rL_{n-1}},$$

$$h_{2} = \frac{1 + rL_{n-1} + rL_{n}}{-2rL_{n-1}} - \frac{\sqrt{r^{2}(L_{n} - L_{n-1})^{2} + 10rL_{n-1} + 2rL_{n} + 1}}{-2rL_{n-1}}.$$
(38)

Using the method in Theorem 8 similarly, we also have the following.

Theorem 9. Let
$$\mathbb{B}' = RFPrLRcirc_r fr(L_{n-1}, \ldots, L_0)$$
. Then

$$\det \mathbb{B}' = \frac{\left(-r - L_{n-1}\right)^n}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} + \frac{2^{n-1}r^n \left(r + L_{n-1}\right) \left(g_3^{n-1} + h_3^{n-1}\right)}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} + \frac{2^{n-1}r^n \left[-2r \left(g_3^n + h_3^n\right) + r \left(L_n - L_{n-1}\right)\right]}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2},$$
(39)

where

$$g_{3} = \frac{L_{n} - r + \sqrt{\left(r - L_{n}\right)^{2} + 8r\left(r + L_{n-1}\right)}}{4r},$$

$$h_{3} = \frac{L_{n} - r - \sqrt{\left(r - L_{n}\right)^{2} + 8r\left(r + L_{n-1}\right)}}{4r}.$$
(40)

Theorem 10. Let $\mathbb{L} = RLPrFLcirc_r fr(L_0, L_1, \dots, L_{n-1})$. Then

$$\det \mathbb{L} = \frac{\left(-r - L_{n-1}\right)^{n}}{\left(-1\right)^{n} + rL_{n-1} - rL_{n} + r^{2}} (-1)^{n(n-1)/2} \\ + \frac{2^{n-1}r^{n}\left(r + L_{n-1}\right)\left(g_{3}^{n-1} + h_{3}^{n-1}\right)}{\left(-1\right)^{n} + rL_{n-1} - rL_{n} + r^{2}} (-1)^{n(n-1)/2} \\ + \frac{2^{n-1}r^{n}\left[-2r\left(g_{3}^{n} + h_{3}^{n}\right) + r\left(L_{n} - L_{n-1}\right)\right]}{\left(-1\right)^{n} + rL_{n-1} - rL_{n} + r^{2}} \\ \times (-1)^{n(n-1)/2},$$

where

$$g_{3} = \frac{L_{n} - r + \sqrt{(r - L_{n})^{2} + 8r(r + L_{n-1})}}{4r},$$

$$h_{3} = \frac{L_{n} - r - \sqrt{(r - L_{n})^{2} + 8r(r + L_{n-1})}}{4r}.$$
(42)

Proof. The matrix \mathbb{L} can be written as

$$\mathbb{L} = \begin{pmatrix} L_{0} & \cdots & L_{n-2} & L_{n-1} \\ L_{1} & \cdots & L_{n-1} + rL_{0} & rL_{0} \\ \vdots & \ddots & \vdots & \vdots \\ L_{n-2} & \cdots & rL_{n-4} + rL_{n-3} & rL_{n-3} \\ L_{n-1} - rL_{0} & \cdots & rL_{n-3} + rL_{n-2} & rL_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} L_{n-1} & L_{n-2} & \cdots & L_{0} \\ rL_{0} & L_{n-1} + rL_{0} & \cdots & L_{1} \\ \vdots & \vdots & \ddots & \vdots \\ rL_{n-3} & rL_{n-4} + rL_{n-3} & \cdots & L_{n-2} \\ rL_{n-2} & rL_{n-3} + rL_{n-2} & \cdots & L_{n-1} + rL_{0} \end{pmatrix}$$
(43)
$$\times \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} = \mathbb{B}' \Gamma.$$

Thus, we have

$$\det \mathbb{L} = \det \mathbb{B}' \det \Gamma, \tag{44}$$

where matrix $\mathbb{B}' = \operatorname{RFPrLRcirc}_r \operatorname{fr}(L_{n-1}, \ldots, L_0)$ and its determinant can be obtained from Theorem 9,

$$\det B' = \frac{(-r - L_{n-1})^n}{(-1)^n + rL_{n-1} - rL_n + r^2} + \frac{2^{n-1}r^n \left(r + L_{n-1}\right) \left(g_3^{n-1} + h_3^{n-1}\right)}{(-1)^n + rL_{n-1} - rL_n + r^2} + \frac{2^{n-1}r^n \left[-2r \left(g_3^n + h_3^n\right) + r \left(L_n - L_{n-1}\right)\right]}{(-1)^n + rL_{n-1} - rL_n + r^2},$$
(45)

where

$$g_{3} = \frac{L_{n} - r + \sqrt{(r - L_{n})^{2} + 8r(r + L_{n-1})}}{4r},$$
(46)

$$h_{3} = \frac{L_{n} - r - \sqrt{(r - L_{n})^{2} + 8r(r + L_{n-1})}}{4r}.$$

In addition,

$$\det \Gamma = (-1)^{n(n-1)/2},$$
(47)

so the determinant of matrix $\mathbb L$ is

$$\det \mathbb{L} = \frac{\left(-r - L_{n-1}\right)^n}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} (-1)^{n(n-1)/2} + \frac{2^{n-1}r^n \left(r + L_{n-1}\right) \left(g_3^{n-1} + h_3^{n-1}\right)}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} (-1)^{n(n-1)/2} + \frac{2^{n-1}r^n \left[-2r \left(g_3^n + h_3^n\right) + r \left(L_n - L_{n-1}\right)\right]}{\left(-1\right)^n + rL_{n-1} - rL_n + r^2} \times (-1)^{n(n-1)/2},$$
(48)

where

$$g_{3} = \frac{L_{n} - r + \sqrt{(r - L_{n})^{2} + 8r(r + L_{n-1})}}{4r},$$

$$h_{3} = \frac{L_{n} - r - \sqrt{(r - L_{n})^{2} + 8r(r + L_{n-1})}}{4r}.$$
(49)

5. Determinants of the RFPrLrR and RLPrFrL Circulant Matrix with the Pell Numbers

Theorem 11. If $\mathbb{C} = RFPrLRcirc_r fr(P_0, P_1, \ldots, P_{n-1})$, then

$$\det \mathbb{C} = \frac{(-rP_n)^n}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} + \frac{\left[P_n\left(g_4^{n-1} + h_4^{n-1}\right) + P_{n-1}\left(g_4^n + h_4^n\right) - 1\right]}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2}$$
(50)

$$\times (-r)^{n+1}P_{n-1}^{n-1},$$

where

$$g_{4} = \frac{rP_{n-1} + rP_{n} - 1}{-2rP_{n-1}} + \frac{\sqrt{r^{2}(P_{n} - P_{n-1})^{2} - 2r(P_{n} + P_{n-1}) + 1}}{-2rP_{n-1}},$$

$$h_{4} = \frac{rP_{n-1} + rP_{n} - 1}{-2rP_{n-1}} - \frac{\sqrt{r^{2}(P_{n} - P_{n-1})^{2} - 2r(P_{n} + P_{n-1}) + 1}}{-2rP_{n-1}}.$$
(51)

Proof. The matrix \mathbb{C} can be written as

$$\mathbb{C} = \begin{pmatrix} P_0 & P_1 & \cdots & P_{n-1} \\ rP_{n-1} & P_0 + rP_{n-1} & P_1 & P_{n-2} \\ \vdots & rP_{n-1} + rP_{n-2} & \ddots & \vdots \\ rP_2 & \vdots & \cdots & P_1 \\ rP_1 & rP_2 + rP_1 & \cdots & P_0 + rP_{n-1} \end{pmatrix}_{n \times n}$$
(52)

Using Lemma 3, the determinant of $\mathbb C$ is

$$\det \mathbb{C} = \prod_{k=1}^{n} \left(P_0 + P_1 \varepsilon_k + \dots + P_{n-1} \varepsilon_k^{n-1} \right)$$
$$= \prod_{k=1}^{n} \left(\frac{\alpha_1 - \beta_1}{\alpha_1 - \beta_1} \varepsilon_k + \dots + \frac{\alpha_1^{n-1} - \beta_1^{n-1}}{\alpha_1 - \beta_1} \varepsilon_k^{n-1} \right)$$
(53)
$$= \prod_{k=1}^{n} \frac{-r P_{n-1} \varepsilon_k^2 + (1 - r P_{n-1} - r P_n) \varepsilon_k - r P_n}{1 - 2\varepsilon_k - \varepsilon_k^2}.$$

According to Lemma 4, we can get

$$\det \mathbb{C} = \frac{(-rP_n)^n}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} + \frac{\left[P_n\left(g_4^{n-1} + h_4^{n-1}\right) + P_{n-1}\left(g_4^n + h_4^n\right) - 1\right]}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} + (54) \times (-r)^{n+1}P_{n-1}^{n-1},$$

where

$$g_{4} = \frac{rP_{n-1} + rP_{n} - 1}{-2rP_{n-1}} + \frac{\sqrt{r^{2}(P_{n} - P_{n-1})^{2} - 2r(P_{n} + P_{n-1}) + 1}}{-2rP_{n-1}},$$

$$h_{4} = \frac{rP_{n-1} + rP_{n} - 1}{-2rP_{n-1}}$$
(55)

$$-\frac{\sqrt{r^2(P_n-P_{n-1})^2-2r(P_n+P_{n-1})+1}}{-2rP_{n-1}}.$$

Using the method in Theorem 11 similarly, we also have the following.

Theorem 12. If $\mathbb{C}' = RFPrLRcirc_r fr(P_{n-1}, P_{n-2}, \dots, P_0)$, then

$$\det \mathbb{C}' = \frac{\left(r - P_{n-1}\right)^n - \left(P_n - r\right)^{n-1} \left(rP_n - rP_{n-1}\right)}{\left(-1\right)^n + rQ_{n-1} - rQ_n + 2r^2}.$$
 (56)

Theorem 13. If $\mathbb{P} = RLPrFLcirc_r fr(P_0, P_1, \dots, P_{n-1})$, then one has

$$\det \mathbb{P} = \frac{\left(r - P_{n-1}\right)^n - \left(P_n - r\right)^{n-1} \left(rP_n - rP_{n-1}\right)}{\left(-1\right)^n + rQ_{n-1} - rQ_n + 2r^2}$$
(57)
 $\times (-1)^{n(n-1)/2}.$

Proof. The matrix \mathbb{P} can be written as

$$\mathbb{P} = \begin{pmatrix}
P_{0} & \cdots & P_{n-2} & P_{n-1} \\
P_{1} & \ddots & P_{n-1} + rP_{0} & rP_{0} \\
\vdots & \ddots & rP_{0} + rP_{1} & \vdots \\
P_{n-2} & \ddots & \vdots & rP_{n-3} \\
P_{n-1} + rP_{0} & \cdots & rP_{n-3} + rP_{n-2} & rP_{n-2}
\end{pmatrix}$$

$$= \begin{pmatrix}
P_{n-1} & P_{n-2} & \cdots & P_{0} \\
rP_{0} & P_{n-1} + rP_{0} & \cdots & P_{1} \\
\vdots & \vdots & \ddots & \vdots \\
rP_{n-2} & rP_{n-3} + rP_{n-2} & \cdots & P_{n-1} + rP_{0}
\end{pmatrix}$$
(58)
$$\times \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$

Then we can get

$$\det \mathbb{P} = \det \mathbb{C}' \det \Gamma, \tag{59}$$

where $\mathbb{C}' = \text{RFPLRcircfr}(P_{n-1}, P_{n-2}, \dots, P_0)$ and its determinant could be obtained through Theorem 12; namely,

$$\det \mathbb{C}' = \frac{\left(r - P_{n-1}\right)^n - \left(P_n - r\right)^{n-1} \left(rP_n - rP_{n-1}\right)}{\left(-1\right)^n + rQ_{n-1} - rQ_n + 2r^2}, \quad (60)$$

$$\det \Gamma = (-1)^{n(n-1)/2}.$$
 (61)

So

$$\det \mathbb{P} = \det \mathbb{C}' \det \Gamma$$

$$= \frac{(r - P_{n-1})^{n} - (P_{n} - r)^{n-1} (rP_{n} - rP_{n-1})}{(-1)^{n} + rQ_{n-1} - rQ_{n} + 2r^{2}}$$
(62)
 $\times (-1)^{n(n-1)/2}.$

6. Determinants of the RFPrLrR and RLPrFrL Circulant Matrix with the Pell-Lucas Numbers

Theorem 14. If $\mathbb{D} = RFPrLRcirc_r fr(Q_0, Q_1, \dots, Q_{n-1})$, then one has

$$\det \mathbb{D} = \frac{\left(2 - rQ_n\right)^n}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} + \frac{\left(2 - rQ_n\right)\left(g_5^{n-1} + h_5^{n-1}\right) - rQ_{n-1}\left(g_5^n + h_5^n\right) - 4r}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} \times (-r)^n Q_{n-1}^{n-1},$$
(63)

where

$$g_{5} = \frac{2 + rQ_{n-1} + rQ_{n}}{-2rQ_{n-1}} + \frac{\sqrt{r^{2}(Q_{n} - Q_{n-1})^{2} + 12rQ_{n-1} + 4rQ_{n} + 4}}{-2rQ_{n-1}},$$

$$h_{5} = \frac{2 + rQ_{n-1} + rQ_{n}}{-2rQ_{n-1}} - \frac{\sqrt{r^{2}(Q_{n} - Q_{n-1})^{2} + 12rQ_{n-1} + 4rQ_{n} + 4}}{-2rQ_{n-1}}.$$
(64)

Proof. The method is similar to Theorem 11.

Certainly, we can get the following theorem.

$$\det \mathbb{D}' = \frac{\left(-2r - rQ_{n-1}\right)^n - 2^{n-1}r^n K}{\left(-1\right)^n + rQ_{n-1} - rQ_n + 2r^2},$$
(65)

where

$$K = (-2r - Q_{n-1}) \left(g_6^{n-1} + h_6^{n-1} \right) + 2r \left(g_6^n + h_6^n \right)$$
$$- r \left(Q_n - Q_{n-1} \right),$$
$$g_6 = \frac{Q_n + \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r},$$
$$h_6 = \frac{Q_n - \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r}.$$
(66)

Theorem 16. If $\mathbb{Q} = RLPrLFcirc_r fr(Q_0, Q_1, \dots, Q_{n-1})$, then

$$\det \mathbb{Q} = \frac{\left(-2r - rQ_{n-1}\right)^n - 2^{n-1}r^n K}{\left(-1\right)^n + rQ_{n-1} - rQ_n + 2r^2} (-1)^{n(n-1)/2}, \qquad (67)$$

where

$$K = (-2r - Q_{n-1}) \left(g_6^{n-1} + h_6^{n-1} \right) + 2r \left(g_6^n + h_6^n \right)$$
$$- r \left(Q_n - Q_{n-1} \right),$$
$$g_6 = \frac{Q_n + \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r},$$
$$(68)$$
$$Q_n = \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2},$$

$$h_6 = \frac{Q_n - \sqrt{Q_n^2 + 8rQ_{n-1} + 16}}{4r}$$

7. Conclusion

The determinant problems of the RFPrLrR circulant matrices and RLPrFrL circulant matrices involving the Fibonacci, Lucas, Pell, and Pell-Lucas number are considered in this paper. The explicit determinants are presented by using some terms of these numbers.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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