Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 649270, 15 pages
http://dx.doi.org/10.1155/2014/649270

Research Article

Quasiperiodic Solutions of Completely Resonant Wave Equations with Quasiperiodic Forced Terms

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Received 28 February 2014; Revised 25 April 2014; Accepted 28 April 2014; Published 18 May 2014

Academic Editor: Yongli Song

This paper is concerned with the existence of quasiperiodic solutions with two frequencies of completely resonant, quasiperiodically forced nonlinear wave equations subject to periodic spatial boundary conditions. The solutions turn out to be, at the first order, the superposition of traveling waves, traveling in the opposite or the same directions. The proofs are based on the variational Lyapunov-Schmidt reduction and the linking theorem, while the bifurcation equations are solved by variational methods.

1. Introduction

This paper is devoted to the study of the existence of small-amplitude quasiperiodic solutions of completely resonant forced nonlinear wave equations like

$$u_{tt} - u_{xx} + f(\omega_1 t, \omega_2 t, x, u) = 0, \tag{1}$$

with periodic boundary conditions

$$u(t,x) = u(t, x + 2\pi), \quad (t,x) \in \mathbb{R}^2, \tag{2}$$

while the nonlinear forced term f is

$$f(\omega_1 t, \omega_2 t, x, u) := \sum_{k\geq 3} a_k (\omega_1 t + x, \omega_2 t + x) u^k, \tag{3}$$

when the traveling waves are in the same directions, and f is

$$f(\omega_1 t, \omega_2 t, x, u) := \sum_{k\geq 3} a_k (\omega_1 t - x, \omega_2 t + x) u^k, \tag{4}$$

when the traveling waves are in the opposite directions. Moreover, the nonlinear forced terms are all analytic in a neighborhood of $u = 0$.

Periodic or quasiperiodic solutions in nonresonant PDEs have been obtained, for instance, in [1–12] by the Lyapunov-Schmidt reduction together with Nash-Moser theory and KAM theory, while the completely resonant autonomous PDEs have been originally studied by variational methods starting from Rabinowitz [13–19]. They obtained the existence of periodic solutions with period being a rational multiple of π, and such solutions correspond to a zero-measure set of values of the amplitudes. The case with period being irrational of π, which in principle could provide a large measure of values, has been mostly studied under strong Diophantine conditions; see [20–25] and the references therein. In [26, 27], using the Lindstedt series method, Gentile and Procesi obtained the existence of periodic solutions for a large measure set of frequencies for the nonlinear wave equations and nonlinear Schrödinger equations with periodic boundary conditions. In [28], Yuan obtained the existence of quasiperiodic solution for a large measure set of at least three dimensional rotation vectors by the KAM method. In [29], under the periodic boundary condition and with the periodic forced nonlinearities $f(\omega_1 t, u) = a(\omega_1 t)u^{2d-1} + O(u^{2d})$, $d > 2$, Berti and Procesi got the existence of quasiperiodic solution of nonlinear wave equation in the form of $v(t,x) = u(\omega_1 t, \omega_2 t + x)$. In [30], Procesi firstly obtained the quasiperiodic solutions with two frequencies in the form of $v(t,x) = u(\omega_1 t + x, \omega_2 t - x)$ for the specific nonlinearities $f = u^3 + O(u^5)$, where the forced terms do not depend on the time and the bifurcation equations are solved by ODE methods. In [31], Baldi proved the existence of small-amplitude quasiperiodic solutions in the form of $v(t,x) = u(\omega_1 t + x, \omega_2 t + x)$ with the general nonlinearities $f = u^3 + O(u^4)$, which also do not depend on the time. In [32], they considered the existence of quasiperiodic solution of the forced wave equation, in which the solutions are traveling in opposite directions. However, they did not give the regularity of the solutions, and the results are the special case of our results in Section 4. Moreover, we mention the work [33] of Bambusi, where a simple proof of an infinite-dimensional extension of the Lyapunov center theorem is given.

In this paper, for the completely resonant wave equation (1) subjecting to the quasiperiodic forced terms, we will prove the existence and regularity of quasiperiodic solutions with two frequencies, ω_1, ω_2, in both of the following two cases.

Case (A1). The first case considers the wave traveling in the same directions $u(t,x) = v(\omega_1 t + x, \omega_2 t + x)$.

Case (A2). The second case considers the wave traveling in opposite directions $u(t,x) = v(\omega_1 t - x, \omega_2 t + x)$.

2. Main Results

We look for quasiperiodic solutions $u(t,x)$ of (1) of the following form:

(in the same directions)

$$\begin{gathered} u(t,x) = v(\omega_1 t + x, \omega_2 t + x) = v(\varphi_1, \varphi_2), \\ v(\varphi_1, \varphi_2) = v(\varphi_1 + 2k_1\pi, \varphi_2 + 2k_2\pi), \quad \forall k_1, k_2 \in \mathbb{Z}, \end{gathered} \tag{5}$$

with frequencies $\omega = (\omega_1, \omega_2) = (1+\epsilon, 1 + a\epsilon^2)$, $a \in \mathbb{R}^-$, or

(in opposite directions)

$$\begin{gathered} u(t,x) = v(\omega_1 t - x, \omega_2 t + x) = v(\varphi_1, \varphi_2), \\ v(\varphi_1, \varphi_2) = v(\varphi_1 + 2k_1\pi, \varphi_2 + 2k_2\pi), \quad \forall k_1, k_2 \in \mathbb{Z}, \end{gathered} \tag{6}$$

with frequencies $\omega = (\omega_1, \omega_2) = (1+\epsilon, 1 + a\epsilon)$, $a \in \mathbb{R}^-$, imposing the frequencies $\omega = (\omega_1, \omega_2)$ to be close to linear frequency 1. Therefore, finding the quasiperiodic solutions of (1) with frequencies, respectively, (ω_1, ω_2) is equivalent to finding 2π periodic solutions with respect to (φ_1, φ_2) for the following equations:

(in the same directions)

$$\begin{aligned} \left(\partial_{tt}^2 - \partial_{xx}^2\right) v &= (\partial_t - \partial_x) \circ (\partial_t + \partial_x) v \\ &= \left[\omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2} - \partial_{\varphi_1} - \partial_{\varphi_2}\right] \\ &\quad \circ \left[\omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2} + \partial_{\varphi_1} + \partial_{\varphi_2}\right] v \\ &= \left(\left(\omega_1^2 - 1\right)\partial_{\varphi_1}^2 + \left(\omega_2^2 - 1\right)\partial_{\varphi_2}^2 \right. \\ &\quad \left. + 2\left(\omega_1\omega_2 - 1\right)\partial_{\varphi_1}\partial_{\varphi_2}\right) v \\ &= -f(\varphi_1, \varphi_2, v), \end{aligned} \tag{7}$$

(in opposite directions)

$$\begin{aligned} \left(\partial_{tt}^2 - \partial_{xx}^2\right) v &= (\partial_t - \partial_x) \circ (\partial_t + \partial_x) v \\ &= \left[\omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2} + \partial_{\varphi_1} - \partial_{\varphi_2}\right] \\ &\quad \circ \left[\omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2} - \partial_{\varphi_1} + \partial_{\varphi_2}\right] v \\ &= \left(\left(\omega_1^2 - 1\right)\partial_{\varphi_1}^2 + \left(\omega_2^2 - 1\right)\partial_{\varphi_2}^2 \right. \\ &\quad \left. + 2\left(\omega_1\omega_2 + 1\right)\partial_{\varphi_1}\partial_{\varphi_2}\right) v \\ &= -f(\varphi_1, \varphi_2, v). \end{aligned} \tag{8}$$

We assume that the quasiperiodic forced term $f : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$,

$$f(\varphi_1, \varphi_2, v) = a_3(\varphi_1, \varphi_2) v^3 + O\left(v^4\right), \tag{9}$$

is analytic in v but has only finite regularity in φ_1, φ_2. More precisely,

(H) $f(\varphi_1, \varphi_2, v) := \sum_{k=3}^{\infty} a_k(\varphi_1, \varphi_2) v^k$, and the coefficients $a_k(\varphi_1, \varphi_2) \in H^1(\mathbb{T}^2)$ verify, for some $r > 0$, $\sum_{k=3}^{\infty} |a_k|_{H^1} r^k < \infty$. The function $f(\varphi_1, \varphi_2, v)$ is not identically constant in (φ_1, φ_2).

We look for solutions v of (7)-(8) in the Banach space

$$\begin{aligned} &\mathscr{H}_{\sigma,s} \\ &:= \left\{ v(\varphi) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \widehat{v}_{(l_1,l_2)} e^{il_1\varphi_1} e^{il_2\varphi_2} : \widehat{v}^*_{(l_1,l_2)} = \widehat{v}_{(-l_1,-l_2)}, \right. \\ &\qquad \left. |v|_{\sigma,s} = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \left|\widehat{v}_{(l_1,l_2)}\right| e^{|l_2|\sigma} (\max\{0, |l_1|\})^s < +\infty \right\}, \end{aligned} \tag{10}$$

where $l = (l_1, l_2) \in \mathbb{Z}^2$, $\widehat{v}^*_{(l_1,l_2)}$ denotes its complex conjugate, and $\sigma > 0$, $s \geq 0$.

The space $\mathscr{H}_{\sigma,s}$ is a Banach algebra with respect to multiplications of functions, namely,

$$v_1, v_2 \in \mathscr{H}_{\sigma,s} \Longrightarrow v_1 v_2 \in \mathscr{H}_{\sigma,s}, \qquad \left|v_1 v_2\right|_{\sigma,s} \leq C\left|v_1\right|_{\sigma,s}\left|v_2\right|_{\sigma,s}. \tag{11}$$

We will prove the following theorems.

Theorem 1. *Assume that the nonlinearity f satisfies (H) and $a_3(\varphi_1,\varphi_2)\neq 0$, $\forall(\varphi_1,\varphi_2)\in\mathbb{T}^2$. Let $\mathscr{B}_\gamma$ be the uncountable zero-measure Cantor set*

$$\begin{aligned}&\mathscr{B}_\gamma\\&:=\Big\{(a,\epsilon)\in\mathbb{R}^-\times\mathbb{R},\ 1+\epsilon\neq 0,\ 1+a\epsilon^2\neq 0,\ 2+a\epsilon^2\neq 0,\\&\qquad\left(\frac{2+\epsilon}{2+a\epsilon^2},a\epsilon\right)\in\mathscr{C}_{\gamma_1}\cap(1-\epsilon_0,1+\epsilon_0)\\&\qquad\times\mathscr{C}_{\gamma_2}\cap(-\epsilon_0,\epsilon_0),\ \frac{1+\epsilon}{1+a\epsilon^2}\notin\mathbb{Q},\ \text{for }\epsilon_0\in\left(0,\frac{1}{2}\right)\Big\},\end{aligned}\tag{12}$$

where $\mathscr{C}_{\gamma_i}$, $i=1,2$, are sets of badly approximate numbers defined as

$$\begin{aligned}\mathscr{C}_{\gamma_1}&:=\left\{\frac{2+\epsilon}{2+a\epsilon^2}:\left|l_2+\frac{2+\epsilon}{2+a\epsilon^2}l_1\right|>\frac{\gamma}{|l_1|}\right\},\\\mathscr{C}_{\gamma_2}&:=\left\{a\epsilon:|l_1+a\epsilon l_2|>\frac{\gamma}{|l_2|}\right\}\end{aligned}\tag{13}$$

for $\forall l_1,l_2\in\mathbb{Z}\setminus\{0\}$, and $0<\gamma<1/4$. There exist constants $\overline{\sigma}>0$, $\overline{s}>2$, and $\overline{\epsilon}(R)>0$, such that, for $(a,\epsilon)\in\mathscr{B}_\gamma$, $|\epsilon|/\gamma<\overline{\epsilon}(R)$, there exists a solution $v(\epsilon,\varphi_1,\varphi_2)\in\mathscr{H}_{\overline{\sigma},\overline{s}}$ of (7), having the form

$$\begin{aligned}&v(\epsilon,\varphi_1,\varphi_2)\\&\quad=|\epsilon|\left(\widehat{q}_0+\widehat{q}_-(\varphi_2)+\widehat{q}_+(\varphi_1+\varphi_2)+\widehat{p}(\epsilon,\varphi_1,\varphi_2)\right),\end{aligned}\tag{14}$$

with

$$\left|\widehat{p}(\epsilon,\varphi_1,\varphi_2)\right|_{\overline{\sigma},\overline{s}}\le\overline{C}\left(\frac{|\epsilon|}{\gamma}\right),\tag{15}$$

where $\overline{C}$ is a constant. As a consequence, (1) possesses the quasiperiodic solutions, traveling in the same directions, $u(\epsilon,t,x)=v(\epsilon,\omega_1t+x,\omega_2t+x)$, with two frequencies $(\omega_1,\omega_2)=(1+\epsilon,1+a\epsilon^2)$.

Theorem 2. *Assume that f satisfies assumption (H) and $a_3(\varphi_1,\varphi_2)\neq 0$, $\forall(\varphi_1,\varphi_2)\in\mathbb{T}^2$. Let $\mathscr{D}_\gamma\subset(-\epsilon_0,\epsilon_0)\times(-\epsilon_0,\epsilon_0)$ be the uncountable zero-measure Cantor set*

$$\begin{aligned}\mathscr{D}_\gamma:=&\Big\{(a,\epsilon)\in\mathbb{R}^-\times\mathbb{R},\ \left(\frac{a\epsilon}{2+\epsilon},\frac{\epsilon}{2+a\epsilon}\right)\in\mathscr{E}_\gamma,\\&\quad\frac{1+\epsilon}{1+a\epsilon}\notin\mathbb{Q},\ 1+\epsilon\neq 0,\ 1+a\epsilon\neq 0,\ 2+a\epsilon\neq 0\Big\},\end{aligned}\tag{16}$$

where $\mathscr{E}_\gamma$ is a set of badly approximate numbers defined as

$$\begin{aligned}&\mathscr{E}_\gamma\\&:=\Big\{\left(\frac{a\epsilon}{2+\epsilon},\frac{\epsilon}{2+a\epsilon}\right):=(\epsilon_1,\epsilon_2)\in(-\epsilon_0,\epsilon_0)\times(-\epsilon_0,\epsilon_0):\\&\qquad|l_1+\epsilon_1l_2|>\frac{\gamma}{|l_2|},\ |l_2+\epsilon_2l_1|>\frac{\gamma}{|l_1|}\Big\}\end{aligned}\tag{17}$$

for $\forall l_1,l_2\in\mathbb{Z}\setminus\{0\}$, and $0<\gamma<1/4$, $\epsilon_0\in(0,1/2)$. There exist positive numbers $\overline{\sigma},\overline{\epsilon},\overline{C},\overline{s}>2$, such that, $\forall(a,\epsilon)\in\mathscr{D}_\gamma$, (8) admits solutions in the form of

$$\begin{aligned}v_{a,\epsilon}(t,x)&=v(\epsilon,\varphi_1,\varphi_2)\\&=\sqrt{|\epsilon|}\left(\widehat{q}_0+\widehat{q}_-(\varphi_2)+\widehat{q}_+(\varphi_1)\right.\\&\qquad\left.+\widehat{p}(\epsilon,\varphi_1,\varphi_2)\right)\in\mathscr{H}_{\overline{\sigma},s},\end{aligned}\tag{18}$$

satisfying

$$\left|\widehat{p}(\epsilon,\varphi_1,\varphi_2)\right|_{\overline{\sigma},\overline{s}}\le\overline{C}\left(\frac{|\epsilon|}{\gamma}\right).\tag{19}$$

As a consequence, (1) possesses the quasiperiodic solutions, $u_{a,\epsilon}(t,x)=v_{a,\epsilon}((1+\epsilon)t+x,(1+a\epsilon)t-x)$, traveling in opposite directions.

Remark 3. The quasiperiodic solutions of traveling waves we obtained are different from the ones got by KAM methods since the quasiperiodic solutions we get depending on x and t are coupled and in the form of the traveling waves.

Remark 4. We can get the similar result with more general nonlinearity, such as $f(\omega_1t,\omega_2t,x,u)=a_d(\varphi_1,\varphi_2)u^d+O(u^{(d+1)})$, for any $d\in\mathbb{N}$, $d\ge 3$.

This paper is organized as follows: we first prove the existence of quasiperiodic solutions, at the first order, to the superposition of traveling waves, traveling in the same directions. In Section 4, we prove the existence of quasiperiodic solutions traveling in opposite directions.

3. Waves Traveling in the Same Directions

Substituting $\omega_1=1+\epsilon,\omega_2=1+a\epsilon^2$ into (7), we can obtain the equations

$$\mathscr{L}v+f(\varphi_1,\varphi_2,v)=0,\tag{20}$$

where, see (7), we have

$$\begin{aligned}\mathscr{L}&=\left(\left(\omega_1^2-1\right)\partial_{\varphi_1}^2+\left(\omega_2^2-1\right)\partial_{\varphi_2}^2+2\left(\omega_1\omega_2-1\right)\partial_{\varphi_1}\partial_{\varphi_2}\right)v\\&=\epsilon\left(2\partial_{\varphi_1}^2+2\partial_{\varphi_1}\partial_{\varphi_2}\right)\\&\quad+\epsilon^2\left(\partial_{\varphi_1}^2+\left(2a+a^2\epsilon^2\right)\partial_{\varphi_2}^2+2a(1+\epsilon)\partial_{\varphi_1}\partial_{\varphi_2}\right),\end{aligned}\tag{21}$$

and $f(\varphi_1,\varphi_2,v)=a_3(\varphi_1,\varphi_2)v^3+O(v^4)$. To prove Theorem 1, instead of looking for solutions of (7) in a shrinking neighborhood of zero, it is convenient to perform the rescaling $v(\varphi_1,\varphi_2)\to\epsilon v(\varphi_1,\varphi_2)$, enhancing the relation between the amplitude and the frequencies. Without confusion, we define

$$\begin{aligned}\mathscr{L}_{a,\epsilon}&=\left(2\partial_{\varphi_1}^2+2\partial_{\varphi_1}\partial_{\varphi_2}\right)\\&\quad+\epsilon\left(\partial_{\varphi_1}^2+\left(2a+a^2\epsilon^2\right)\partial_{\varphi_2}^2+2a(1+\epsilon)\partial_{\varphi_1}\partial_{\varphi_2}\right)\\&=\mathscr{L}_0+\epsilon\mathscr{L}_1,\end{aligned}\tag{22}$$

so the problem becomes

$$\mathscr{L}_{a,\epsilon} + \epsilon f(\varphi_1, \varphi_2, v, \epsilon) = 0. \tag{23}$$

To find the solutions of (23), we will apply the Lyapunov-Schmidt reduction method which leads to solving separately a "range equation" and a "bifurcation equation." In order to solve the range equation (avoiding small divisor problems), we restrict ϵ to the uncountable zero-measure set $\mathscr{B}_\gamma$ for Theorem 1, and we apply the Contraction Mapping Theorem; similar nonresonance conditions have been employed, for example, in [21–23, 25, 29, 30].

Equation (23) is the Euler-Lagrange equation of the action functional $\Psi_\epsilon \in C^1(\mathscr{H}_{\sigma,s}, \mathbb{R})$ defined by

$$\begin{aligned}\Psi_\epsilon(v) := &\int_{\mathbb{T}^2} \left(\partial_{\varphi_1} v\right)^2 + \left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right) \\ &+ \epsilon\left(\frac{1}{2}\left(\partial_{\varphi_1} v\right)^2 + \frac{2a + a^2\epsilon^2}{2}\left(\partial_{\varphi_2} v\right)^2 \right. \\ &\left. + a(1+\epsilon)\left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right)\right) \\ &- \epsilon F(\varphi_1, \varphi_2, v, \epsilon) \\ = &\ \Psi_0(v) + \epsilon \Psi_1(v, \epsilon),\end{aligned} \tag{24}$$

where

$$\begin{aligned}F(\varphi_1, \varphi_2, v, \epsilon) &:= \int_0^v f(\varphi_1, \varphi_2, \xi, \epsilon)\, d\xi, \\ \Psi_0(v) &= \int_{\mathbb{T}^2} \left(\partial_{\varphi_1} v\right)^2 + \left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right), \\ \Psi_1(v, \epsilon) &:= \int_{\mathbb{T}^2} \left(\frac{1}{2}\left(\partial_{\varphi_1} v\right)^2 + \frac{2a + a^2\epsilon^2}{2}\left(\partial_{\varphi_2} v\right)^2 \right. \\ &\quad \left. + a(1+\epsilon)\left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right)\right) \\ &\quad - \epsilon F(\varphi_1, \varphi_2, v, \epsilon).\end{aligned} \tag{25}$$

To find critical points of $\Psi_\epsilon(v)$, we perform a variational Lyapunov-Schmidt reduction inspired by Berti and Bolle [22, 23, 29]; see also Ambrosetti and Badiale [34].

3.1. The Variational Lyapunov-Schmidt Reduction. The operator $\mathscr{L}_{a,\epsilon}$ is diagonal defined on the Banach space $\mathscr{H}_{\sigma,s}$ under the Fourier basis $e_l = e^{il_1\varphi_1} e^{il_2\varphi_2}$ with eigenvalues

$$\begin{aligned}-D_l &= \left(2l_1^2 + 2l_1 l_2\right) + \epsilon\left(l_1^2 + \left(2a + a^2\epsilon^2\right) l_2^2 + 2a(1+\epsilon) l_1 l_2\right) \\ &= \left(l_1 + a\epsilon l_2\right)\left((2+\epsilon) l_1 + \left(2 + a\epsilon^2\right) l_2\right).\end{aligned} \tag{26}$$

So, we have

$$\mathscr{L}_{a,\epsilon}[v] = \sum_{(l_1,l_2)\in\mathbb{Z}^2} D_l \widehat{v}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2}, \quad \forall v \in \mathscr{H}_{\sigma,s}. \tag{27}$$

The critical points of the unperturbed functional $\Psi_0 : \mathscr{H}_{\sigma,s} \to \mathbb{R}$ form an infinite-dimensional linear space Q, and they are the solutions of the equation

$$\mathscr{L}_0 q = \left(2\partial_{\varphi_1}^2 + 2\partial_{\varphi_1}\partial_{\varphi_2}\right) q = 0. \tag{28}$$

The space Q can be written as

$$Q = \left\{ q = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \widehat{q}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2} \in \mathscr{H}_{\sigma,s} \mid \widehat{q}_{l_1,l_2} = 0, \right. \\ \left. \text{for } l_1(2l_1 + 2l_2) \neq 0 \right\}. \tag{29}$$

In view of the variational argument that we will use to solve the bifurcation equation, we split Q as $Q = Q_+ + Q_0 + Q_-$, where

$$\begin{aligned}Q_+ &:= \left\{q \in Q : \widehat{q}_{l_1,l_2} = 0, \text{ for } (l_1, l_2) \notin \Lambda_+\right\} \\ &= \left\{q_+ := q_+(\varphi) \in \mathscr{H}^0_{\sigma,s}\right\}, \\ Q_0 &:= \left\{q : q_{0,0} \in \mathbb{R}\right\}, \\ Q_- &:= \left\{q \in Q : \widehat{q}_{l_1,l_2} = 0, \text{ for } (l_1, l_2) \notin \Lambda_-\right\} \\ &= \left\{q_- := q_-(\varphi) \in \mathscr{H}^0_{\sigma,s}\right\},\end{aligned} \tag{30}$$

with

$$\begin{aligned}\Lambda_+ &:= \left\{(l_1, l_2) \in \mathbb{Z}^2 : l_1 = 0, l_2 \neq 0\right\}, \\ \Lambda_- &:= \left\{(l_1, l_2) \in \mathbb{Z}^2 : l_1 + l_2 = 0, \ (l_1, l_2) \neq (0, 0)\right\}, \\ \Lambda_0 &:= \left\{(l_1, l_2) \in \mathbb{Z}^2 : (l_1, l_2) \equiv (0, 0)\right\}.\end{aligned} \tag{31}$$

We will also use in Q the norm

$$\begin{aligned}|q|^2_{H^1} &= |q_+|^2_{H^1(\mathbb{T})} + q^2_{0,0} + |q_-|^2_{H^1(\mathbb{T})} \\ &\sim \sum_{(l_1,l_2)\in\Lambda_+\cup\Lambda_0\cup\Lambda_+} \widehat{q}^2_{l_1,l_2} \left(|l_1|^2 + |l_2|^2 + 1\right).\end{aligned} \tag{32}$$

So, we can decompose the space $\mathscr{H}_{\sigma,s} = Q + P$, where

$$P := \left\{ p = \sum_{l_1,l_2\in\mathbb{Z}} \widehat{p}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2} \in \mathscr{H}_{\sigma,s} \mid \widehat{p}_{l_1,l_2} = 0, \right. \\ \left. \text{for } l_1(2l_1 + 2l_2) = 0 \right\}. \tag{33}$$

Projecting (23) onto the closed subspaces Q and P, setting $v = q + p \in \mathscr{H}_{\sigma,s}$ with $q \in Q$ and $p \in P$, we obtain

$$\begin{aligned}&(Q) \ \ \mathscr{L}_1[q] + \Pi_Q f(\varphi_1, \varphi_2, q + p, \epsilon) = 0, \\ &(P) \ \ \mathscr{L}_{a,\epsilon}[p] + \epsilon \Pi_P f(\varphi_1, \varphi_2, q + p, \epsilon) = 0,\end{aligned} \tag{34}$$

where $\Pi_Q : \mathscr{H}_{\sigma,s} \to Q$, $\Pi_P : \mathscr{H}_{\sigma,s} \to P$ are the projectors, respectively, onto Q and P.

In order to prove analyticity of the solutions and to highlight the compactness of the problem, we perform a finite-dimensional Lyapunov-Schmidt reduction, introducing the decomposition $Q = Q_1 + Q_2$, where

$$\begin{aligned} Q_1 &:= \left\{ q = \sum_{|l_1|+|l_2|\le N} \widehat{q}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2} \in Q \right\}, \\ Q_2 &:= \left\{ q = \sum_{|l_1|+|l_2|\le N} \widehat{q}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2} \in Q \right\}. \end{aligned} \tag{35}$$

Setting $q = q_1 + q_2$ with $q_1 \in Q_1$ and $q_2 \in Q_2$, we finally get

$$\begin{aligned} &(Q_1)\ \mathscr{L}_1 [q_1] + \Pi_{Q_1} [f(\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon)] = 0 \\ &\quad \Longleftrightarrow d\Psi_\epsilon (v) [h] = 0, \quad \forall h \in Q_1; \end{aligned} \tag{36}$$

$$\begin{aligned} &(Q_2)\ \mathscr{L}_1 [q_2] + \Pi_{Q_2} [f(\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon)] = 0 \\ &\quad \Longleftrightarrow d\Psi_\epsilon (v) [h] = 0, \quad \forall h \in Q_2; \end{aligned} \tag{37}$$

$$\begin{aligned} &(P)\ \mathscr{L}_{a,\epsilon} [p] + \epsilon \Pi_P [f(\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon)] = 0 \\ &\quad \Longleftrightarrow d\Psi_\epsilon (v) [h] = 0, \quad \forall h \in P, \end{aligned} \tag{38}$$

where $\Pi_{Q_i} : \mathscr{H}_{\sigma,s} \to Q_i$ are the projectors onto Q_i $(i = 1, 2)$, and $\Pi_P : \mathscr{H}_{\sigma,s} \to P$ is the projector onto P. We will solve first the (Q_2)-(P)-equations for all $|q_1|_{H^1} \le 2R$, provided ϵ belongs to a suitable Cantor-like set, $|\epsilon| \le \epsilon_0(R)$ is sufficiently small, and $N \ge N_0(R)$ is large enough (see Lemma 7). Next, we will solve the Q_1-equation by means of a variational linking argument; see Section 3.4.

3.2. The (Q_2)-(P)-Equations. We first prove that $\mathscr{L}_{a,\epsilon}$ restricted to P has a bounded inverse when (a, ϵ) belongs to the uncountable zero-measure set

$$\begin{aligned} &\mathscr{B}_\gamma \\ &:= \Bigg\{ (a, \epsilon) \in \mathbb{R}^- \times \mathbb{R},\ 1 + \epsilon \ne 0,\ 1 + a\epsilon^2 \ne 0,\ 2 + a\epsilon^2 \ne 0, \\ &\quad \left(\frac{2 + \epsilon}{2 + a\epsilon^2}, a\epsilon \right) \in \mathscr{C}_{\gamma_1} \cap (1 - \epsilon_0, 1 + \epsilon_0) \\ &\quad \times \mathscr{C}_{\gamma_2} \cap (-\epsilon_0, \epsilon_0),\ \frac{1 + \epsilon}{1 + a\epsilon^2} \notin \mathbb{Q},\ \text{for } \epsilon_0 \in \left(0, \frac{1}{2}\right) \Bigg\}, \end{aligned} \tag{39}$$

where $\mathscr{C}_{\gamma_i}$, $i = 1, 2$, is a set of badly approximate numbers defined by

$$\begin{aligned} \mathscr{C}_{\gamma_1} &:= \left\{ \frac{2 + \epsilon}{2 + a\epsilon^2} : \left| l_2 + \frac{2 + \epsilon}{2 + a\epsilon^2} l_1 \right| > \frac{\gamma}{|l_1|} \right\}, \\ \mathscr{C}_{\gamma_2} &:= \left\{ a\epsilon : |l_1 + a\epsilon l_2| > \frac{\gamma}{|l_2|} \right\}, \end{aligned} \tag{40}$$

for $\forall l_1, l_2 \in \mathbb{Z} \setminus \{0\}$, and $0 < \gamma < 1/4$. $\mathscr{C}_{\gamma_i}$, $i = 1, 2$, accumulate at 1 and zero, respectively, from both the right and the left; see [21, 31, 35].

The operator $\mathscr{L}_\epsilon$ is diagonal in the Fourier basis $\{e^{il_1\varphi_1} e^{il_2\varphi_2}, (l_1, l_2) \in \mathbb{Z}^2\}$ with eigenvalues

$$\begin{aligned} D_{l_1,l_2} &= -(l_1 + a\epsilon l_2)\left((2 + \epsilon) l_1 + (2 + a\epsilon^2) l_2\right) \\ &= -(2 + a\epsilon^2)(l_1 + a\epsilon l_1)\left(l_2 + \frac{2 + \epsilon}{2 + a\epsilon^2} l_1\right). \end{aligned} \tag{41}$$

Lemma 5. *For $(a, \epsilon) \in \mathscr{B}_\gamma$, the eigenvalues D_{l_1,l_2} of $\mathscr{L}_{a,\epsilon}$ restricted to P satisfy*

$$\begin{aligned} |D_{l_1,l_2}| = \left|(l_1 + a\epsilon l_2)\left((2 + \epsilon) l_1 + (2 + a\epsilon^2) l_2\right)\right| > \gamma, \\ \forall l_1, l_2 \ne 0. \end{aligned} \tag{42}$$

As a consequence, the operator $\mathscr{L}_{a,\epsilon} : P \to P$ has a bounded inverse $\mathscr{L}_{a,\epsilon}^{-1}$ and satisfies

$$\left| \mathscr{L}_{a,\epsilon}^{-1} [h] \right|_{\sigma,s} \le \frac{|h|_{\sigma,s}}{\gamma}, \quad \forall h \in P. \tag{43}$$

Proof. Denote by $[x]$ the nearest integer close to x and by $\{x\} = x - [x]$ its fractional part. If both $l_1 \ne -[a\epsilon l_2]$ and $l_2 \ne -[((2 + \epsilon)/(2 + a\epsilon^2))l_1]$, we have

$$|D_{l_1,l_2}| = |(l_1 + a\epsilon l_2)| \cdot |(2 + a\epsilon^2)| \left| \left(l_2 + \frac{2 + \epsilon}{2 + a\epsilon^2} l_1\right) \right| > 1. \tag{44}$$

If $l_1 = -[a\epsilon l_2]$, then $|l_1| < (1/2)|l_2|$, so that $|l_2 + ((2 + \epsilon)/(2 + a\epsilon^2))l_1| > (1/2)|l_2|$. This implies that

$$\begin{aligned} &|D_{l_1,l_2}| \\ &= |l_1 + a\epsilon l_2| \cdot |2 + a\epsilon^2| \left| \frac{2 + \epsilon}{2 + a\epsilon^2} l_1 + l_2 \right| \\ &\ge \frac{\gamma}{|l_2|} \cdot \Bigg| (2 + \epsilon) \\ &\quad \times \left(\epsilon \left\{ \frac{a\epsilon}{2 + \epsilon} l_2 \right\} + 2l_2 + a\epsilon \left(1 - \frac{\epsilon}{2 + \epsilon}\right) l_2 \right) \Bigg| \ge \gamma. \end{aligned} \tag{45}$$

In the same way, if $l_2 = -[((2 + \epsilon)/(2 + a\epsilon^2))l_1]$, we have

$$|D_{l_1,l_2}| \ge \frac{\gamma}{|l_1|} \cdot \left| (2 + a\epsilon^2) \left(\epsilon \left\{ \frac{(2 + \epsilon) a}{2 + a\epsilon^2} l_1 \right\} + \frac{2 - 2a\epsilon}{2 + a\epsilon^2} l_1 \right) \right| \ge \gamma. \tag{46}$$

So, the operator $\mathscr{L}_{a,\epsilon}$ restricted to P has a bounded inverse and satisfies

$$\begin{aligned} \left| \mathscr{L}_{a,\epsilon}^{-1} [h] \right|_{\sigma,s} &= \sum_{(l_1,l_2)\in\mathbb{Z}^2} \frac{\left|\widehat{h}_{l_1,l_2}\right| e^{|l_2|\sigma} (\max\{0, |l_1|\})^s}{D_{l_1,l_2}} \\ &\le \frac{|h|_{\sigma,s}}{\gamma}, \quad \forall h \in P. \end{aligned} \tag{47}$$

$\square$

Lemma 6. *The operator $\mathscr{L}_1 : Q_2 \to Q_2$ has a bounded inverse $\mathscr{L}_1^{-1}$, satisfying*

$$\left|\mathscr{L}_1^{-1}[h]\right|_{\sigma,s} \le \frac{|h|_{\sigma,s}}{N^2}. \tag{48}$$

Proof. $\mathscr{L}_1$ is diagonal in the Fourier basis of $Q : e^{il_1\varphi_1}e^{il_2\varphi_2}$ with $(l_1,l_2) \in \Lambda_+ \cup \Lambda_0 \cup \Lambda_-$ with eigenvalues

$$d_{l_1,l_2} = \begin{cases} \left(1+\epsilon\left(-2a+a^2\epsilon\right)\right)l_2^2, & \text{if } l_1+l_2=0, \\ a\left(2+a\epsilon^2\right)l_2^2, & \text{if } l_1=0. \end{cases} \tag{49}$$

The eigenvalues of $\mathscr{L}_1$ restricted to $Q_2(N)$ verify $|d_{l_1,l_2}| \ge N^2/C$, where the constant C depends on (ϵ, a), and (48) holds. □

Fixed points of the nonlinear operator $\mathfrak{P} : Q_2 \oplus P \to Q_2 \oplus P$ defined by

$$\begin{aligned}\mathfrak{P}(q_2,p,q_1) := \Big(&-\mathscr{L}_1^{-1}\Pi_{Q_2}f(\varphi_1,\varphi_2,q_1+q_2+p,\epsilon), \\ &-\epsilon\mathscr{L}_{a,\epsilon}^{-1}\Pi_P f(\varphi_1,\varphi_2,q_1+q_2+p,\epsilon)\Big)\end{aligned} \tag{50}$$

are solutions of the (Q_2)-(P)-equations. Using the Contraction Mapping Theorem, we can prove the following lemma.

Lemma 7. *For any $R>0$, there exist an integer $N_0(R) \in \mathbb{N}^+$ and positive constants $\epsilon_0(R)>0, C_0(R)>0$ such that*

$$\begin{gathered}\forall |q_1|_{H^1} \le 2R, \quad \forall \epsilon \in \mathscr{B}_\gamma, \quad |\epsilon|\gamma^{-1} \le \epsilon_0(R), \\ \forall N \ge N_0(R) : 0 \le N\sigma \le 1,\end{gathered} \tag{51}$$

and there exists a unique solution $(q_2(q_1), p(q_1)) := (q_2(\epsilon,N,q_1), p(\epsilon,N,q_1)) \in Q_2 \oplus P$ of the (Q_2)-(P)-equations satisfying

$$\begin{gathered}\left|q_2(\epsilon,N,q_1)\right|_{\sigma,s} \le \frac{C_0(R)}{N^2}, \\ \left|p(\epsilon,N,q_1)\right|_{\sigma,s} \le C_0(R)|\epsilon|\gamma^{-1}.\end{gathered} \tag{52}$$

Moreover, the map $q_1 \to (q_2(q_1), p(q_1))$ is $C^1(B_{2R}, Q_2 \oplus P)$ and

$$\begin{gathered}\left|q_2'(q_1)[h]\right|_{\sigma,s} \le \frac{C_0(R)}{N^2}|h|_{H^1}, \\ \left|p'(q_1)[h]\right|_{\sigma,s} \le C_0(R)|\epsilon|\gamma^{-1}|h|_{H^1} \quad \forall h \in Q_1.\end{gathered} \tag{53}$$

Proof. Let us consider the ball

$$B := \left\{(q_2,p) \in Q_2 \oplus P,\ |q_2|_{\sigma,s} \le \rho_1,\ |p|_{\sigma,s} \le \rho_2\right\} \tag{54}$$

with norm $|(q_2,p)|_{\sigma,s} := |q_2|_{\sigma,s} + |p|_{\sigma,s}$. We can claim that, under assumptions (51), there exist $\rho_1, \rho_2 \in (0,1)$ such that the map $(q_2,p) \to \mathfrak{P}(q_2,p;q_1)$ is a contraction mapping in B, that is, we have to prove

(i) $(q_2,p) \in B \Rightarrow \mathfrak{P}(q_2,p;q_1) \in B$;

(ii) $|\mathfrak{P}(q_2,p;q_1) - \mathfrak{P}(\widehat{q}_2,\widehat{p};q_1)|_{\sigma,s} \le \eta|(q_2,p)-(\widehat{q}_2,\widehat{p})|_{\sigma,s}$, $\forall (q_2,p),(\widehat{q}_2,\widehat{p}) \in B$,

where the constant $\eta \in (0,1)$. In the following, κ_i $(i = 1,2,\ldots,5)$ denote different constants. By (48) and the Banach property of $\mathscr{H}_{\sigma,s}$,

$$\begin{aligned}\left|\mathfrak{P}_1(q_2,p;q_1)\right|_{\sigma,s} &= \left|\mathscr{L}_1^{-1}\Pi_{Q_2}f(\varphi_1,\varphi_2,q_1+q_2+p,\epsilon)\right| \\ &\le \frac{\kappa_1}{N^2}\left(|q_1|_{\sigma,s}^3 + |q_2|_{\sigma,s}^3 + |p|_{\sigma,s}^3\right).\end{aligned} \tag{55}$$

Similarly, for $(a,\epsilon) \in \mathscr{B}_\gamma$, by (43), we have

$$\begin{aligned}\left|\mathfrak{P}_2(q_2,p;q_1)\right|_{\sigma,s} &= \left|\epsilon\mathscr{L}_{a,\epsilon}^{-1}\Pi_P f(\varphi_1,\varphi_2,q_1+q_2+p,\epsilon)\right| \\ &\le \frac{\kappa_2|\epsilon|}{\gamma}\left(|q_1|_{\sigma,s}^3 + |q_2|_{\sigma,s}^3 + |p|_{\sigma,s}^3\right).\end{aligned} \tag{56}$$

For all $q_1 \in Q_1(N)$, setting $[x] = \max\{0, |x|\}$, we can get

$$\begin{aligned}|q_1|_{\sigma,s} &= \sum_{|l_2|\le N, l_1=0}\left|\widehat{q}_{0,l_2}\right|e^{|l_2|\sigma} + \sum_{|l_2|\le N, l_1+l_2=0}\left|\widehat{q}_{-l_2,l_2}\right|e^{|l_2|\sigma}[-l_2]^s \\ &\le e^{N\sigma}\Bigg(\sum_{|l_2|\le N, l_1=0}\left|\widehat{q}_{l_1,l_2}\right|\left(\left[-l_2-l_2^2\right]\right)^s \\ &\qquad + \sum_{|l_2|\le N, l_1+l_2=0}\left|\widehat{q}_{-l_2,l_2}\right|[-l_2]^s\Bigg) \\ &\le \kappa_3\Bigg(\Bigg(\sum_{|l_2|\le N, l_1=0}\left|\widehat{q}_{0,l_2}\right|^2[l_2]^2\Bigg)^{1/2}\Bigg(\sum_{l_2\in\mathbb{Z}}\frac{1}{[l_2]^2}\Bigg)^{1/2} \\ &\qquad + \Bigg(\sum_{|l_2|\le N, l_1+l_2=0}\left|\widehat{q}_{-l_2,l_2}\right|^2[l_2]^2\Bigg)^{1/2} \\ &\qquad \times\Bigg(\sum_{l_2\in\mathbb{Z}}\frac{1}{([l_2])^{2(1-s)}}\Bigg)^{1/2}\Bigg) \\ &\le \kappa_4|q_1|_{H_1}\end{aligned} \tag{57}$$

whenever $0 \le N\sigma \le 1$, $0 \le s < 1/2$. Thus, $\forall |q_1|_{H_1} \le 2R$, $|q_2|_{\sigma,s} \le \rho_1$, $|p|_{\sigma,s} \le \rho_2$, we get

$$\begin{gathered}\left|\mathfrak{P}_1(q_2,p;q_1)\right|_{\sigma,s} \le \frac{\kappa_5}{N^2}\left(R^3+\rho_1^3+\rho_2^3\right), \\ \left|\mathfrak{P}_2(q_2,p;q_1)\right|_{\sigma,s} \le \frac{\kappa_5|\epsilon|}{\gamma}\left(R^3+\rho_1^3+\rho_2^3\right).\end{gathered} \tag{58}$$

Now, set $C_0(R) := \kappa_5 R^3$, and we define $\rho_1 := 2C_0(R)/N^{3/2}$, $\rho_2 := 2C_0(R)(|\epsilon|/\gamma)$. By the inequality above, there exist

$N_0(R) \in \mathbb{N}^+$ and $\epsilon_0(R) > 0$ such that $\forall N \geq N_0(R)$ and $\forall |\epsilon|\gamma^{-1} \leq \epsilon_0(R)$,

$$\left|\mathfrak{P}_1 (q_2, p; q_1)\right|_{\sigma,s} \leq \rho_1, \qquad \left|\mathfrak{P}_2 (q_2, p; q_1)\right|_{\sigma,s} \leq \rho_2. \quad (59)$$

So, we get the proof of (i). Item (ii) can be obtained with the same estimates.

By the Contraction Mapping Theorem, there exists a unique fixed point $(q_2(q_1), p(q_1)) := (q_2(\epsilon, N, q_1), p(\epsilon, N, q_1))$ of $\mathfrak{P}$ in B. The bounds of (52) follow by the definition of ρ_1 and ρ_2.

Since the map $\mathfrak{P} \in C^1(Q_2 \oplus P \times Q_1; Q_2 \oplus P \times Q_1)$, the Implicit Function Theorem implies that the map $q_1 \to (q_2(\epsilon, N, q_1), p(\epsilon, N, q_1))$ is C^1. Differentiating both sides of $(q_2(q_1), p(q_1)) = \mathfrak{P}(q_2(q_1), p(q_1), q_1)$, we can get

$$\begin{aligned} & q_2' (q_1) [h] \\ & \quad = -\mathscr{L}_1^{-1} \Pi_{Q_2} (\partial_v f) (\varphi_1, \varphi_2, q_1 + q_2 (q_1) + p (q_1), \epsilon) \\ & \qquad \times \left(h + q_2' (q_1) [h] + p' (q_1) [h]\right), \\ & p' (q_1) [h] \\ & \quad = -\epsilon \mathscr{L}_\epsilon^{-1} \Pi_P (\partial_v f) (\varphi_1, \varphi_2, q_1 + q_2 (q_1) + p (q_1), \epsilon) \\ & \qquad \times \left(h + q_2' (q_1) [h] + p' (q_1) [h]\right). \end{aligned} \quad (60)$$

Using (43)–(48) and the Banach property of $\mathscr{H}_{\sigma,s}$, we get

$$\begin{aligned} & \left|q_2' (q_1) [h]\right|_{\sigma,s} \\ & \quad \leq \frac{C (R)}{N^2} \left(|h|_{\sigma,s} + \left|q_2' (q_1) [h]\right|_{\sigma,s} + \left|p' (q_1) [h]\right|_{\sigma,s}\right), \\ & \left|p' (q_1) [h]\right|_{\sigma,s} \\ & \quad \leq \frac{C (R) |\epsilon|}{\gamma} \left(|h|_{\sigma,s} + \left|q_2' (q_1) [h]\right|_{\sigma,s} + \left|p' (q_1) [h]\right|_{\sigma,s}\right), \end{aligned} \quad (61)$$

which implies the bounds (53), since when $C(R)(|\epsilon|/\gamma+1/N^2)$ is small enough, we can get

$$\det \begin{vmatrix} 1 - \dfrac{C (R)}{N^2} & -\dfrac{C (R)}{N^2} \\ -\dfrac{C (R) |\epsilon|}{\gamma} & 1 - \dfrac{C (R) |\epsilon|}{\gamma} \end{vmatrix} \geq \frac{1}{2}. \quad (62)$$

□

3.3. The (Q_1)*-Equation.* Once the (Q_2)-(P)-equations have been solved by $(q_2(q_1), p(q_1)) \in Q_2 \oplus P$, there remains the finite-dimensional Q_1-equation

$$\mathscr{L}_1 [q_1] + \Pi_{Q_1} f (\varphi_1, \varphi_2, q_1 + q_2 (q_1) + p (q_1), \epsilon) = 0. \quad (63)$$

The geometric interpretation of the construction of $(q_2(q_1), p(q_1))$ is that, on the finite-dimensional submanifold $Z := \{q_1 + q_2(q_1) + p(q_1) : |q_1| < 2R\}$, diffeomorphic to the ball

$$B_{2R} := \{q \in Q_1 : |q_1|_{H^1} < 2R\}, \quad (64)$$

the partial derivatives of the action functional Ψ_ϵ with respect to the variables (q_2, p) vanish. We claim that, at a critical point of Ψ_ϵ restricted to Z, also the partial derivative of Ψ_ϵ with respect to the variable q_1 vanishes, and therefore such a point is critical also for the nonrestricted functional $\Psi_\epsilon : \mathscr{H}_{\sigma,s} \to \mathbb{R}$.

Actually, the bifurcation equation (63) is the Euler-Lagrange equation of the reduced action functional

$$\begin{gathered} \Phi_{\epsilon,N} : B_{2R} \subset Q_1 \longrightarrow \mathbb{R}, \\ \Phi_{\epsilon,N} (q_1) := \Psi_\epsilon (q_1 + q_2 (q_1 + p (q_1))). \end{gathered} \quad (65)$$

Lemma 8. *$\Phi_{\epsilon,N} \in C^1(B_{2R}, \mathbb{R})$ and a critical point $q_1 \in B_{2R}$ of $\Phi_{\epsilon,N}$ is a solution of the bifurcation (63). Moreover, $\Phi_{\epsilon,N}$ can be written as*

$$\Phi_{\epsilon,N} (q_1) = const + \epsilon \left(\Gamma (q_1) + \mathfrak{R}_{\epsilon,N} (q_1)\right), \quad (66)$$

where

$$\begin{aligned} \Gamma (q_1) := & \int_{\mathbb{T}^2} \frac{1}{2}\left(\partial_{\varphi_1} q_1\right)^2 + \frac{\left(2a + a^2\epsilon^2\right)}{2}\left(\partial_{\varphi_2} q_1\right)^2 \\ & + a (1 + \epsilon) \left(\partial_{\varphi_1} q_1\right) \left(\partial_{\varphi_2} q_1\right) - a_3 (\varphi_1, \varphi_2) \frac{q_1^4}{4}, \\ \mathfrak{R}_{\epsilon,N} (q_1) := & \int_{\mathbb{T}^2} F (\varphi_1, \varphi_2, q_1, \epsilon = 0) \\ & - F (\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon) \\ & + \frac{1}{2} f (\varphi_1, \varphi_2, v, \epsilon) (q_2 + p), \end{aligned} \quad (67)$$

and for some positive constant $C_2(R) \geq C_1(R)$, we can get

$$\begin{aligned} & \left|\mathfrak{R}_{\epsilon,N} (q_1)\right| \leq C_2 (R) \left(\epsilon + \frac{|\epsilon|}{\gamma} + \frac{1}{N^2}\right), \\ & \left|\mathfrak{R}_{\epsilon,N}' (q_1) [h]\right| \leq C_2 (R) \left(\epsilon + \frac{|\epsilon|}{\gamma} + \frac{1}{N^2}\right) |h|_{H^1}, \\ & \qquad \forall h \in Q_1. \end{aligned} \quad (68)$$

Proof. By (37) and (38), we have that, at $v := q_1 + q_2(q_1) + p(q_1)$,

$$\begin{aligned} & d\Psi_\epsilon (v) [h] = 0, \quad \forall h \in Q_2, \\ & d\Psi_\epsilon (v) [h] = 0, \quad \forall h \in P. \end{aligned} \quad (69)$$

Since $q_2'(q_1)[h] \in Q_2$ and $p'(q_1)[h] \in P, \forall h \in Q_1$, we deduce that

$$\begin{aligned} d\Phi_{\epsilon,N} (q_1) [h] & = d\Psi_\epsilon (v) \left[[h] + q_2' (q_1) [h] + p' (q_1) [h]\right] \\ & = d\Psi_\epsilon (v) [h], \quad \forall h \in Q_1, \end{aligned} \quad (70)$$

and therefore $v := q_1 + q_2(q_1) + p(q_1)$ solves also the (Q_1)-equation (36). Write $\Psi_\epsilon(v) := \Psi_\epsilon^{(2)}(v) - \epsilon \int_{\mathbb{T}^2} F(\varphi_1, v, \epsilon)$, where

$$\Psi_\epsilon^{(2)}(v) = \int_{\mathbb{T}^2} \left(1 + \frac{1}{2}\epsilon\right)\left(\partial_{\varphi_1} v\right)^2 + \epsilon \frac{2a + a^2\epsilon^2}{2}\left(\partial_{\varphi_2} v\right)^2 + (1 + \epsilon a(1+\epsilon))\left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right) \tag{71}$$

is a homogeneous functional of degree two. By homogeneity,

$$\Psi_\epsilon(v) = \frac{1}{2} d\Psi_\epsilon^{(2)}(v)[v] - \epsilon \int_{\mathbb{T}^2} F(\varphi_1, \varphi_2, v, \epsilon), \tag{72}$$

and, according to (69), we have

$$\begin{aligned} d\Psi_\epsilon^{(2)}&(q_1 + q_2(q_1) + p(q_1))[q_2(q_1) + p(q_1)] \\ &= \epsilon \int_{\mathbb{T}^2} f(\varphi_1, \varphi_2, v, \epsilon)(q_2(q_1) + p(q_1)). \end{aligned} \tag{73}$$

Substituting the above equality into (72), we obtain, at $v := q_1 + q_2(q_1) + p(q_1)$,

$$\begin{aligned} \Phi_{\epsilon,N}(q_1) &= \Psi_\epsilon(q_1 + q_2(q_1) + p(q_1)) \\ &= \frac{1}{2} d\Psi_\epsilon^{(2)}(v)[q_1 + q_2(q_1) + p(q_1)] \\ &\quad - \epsilon \int_{\mathbb{T}^2} F(\varphi_1, \varphi_2, v, \epsilon) \\ &= \frac{1}{2} d\Psi_\epsilon^{(2)}(q_1)[q_1] \\ &\quad + \frac{1}{2}\epsilon \int_{\mathbb{T}^2} f(\varphi_1, \varphi_2, v, \epsilon)(q_2(q_1) + p(q_1)) \\ &\quad - \epsilon \int_{\mathbb{T}^2} F(\varphi_1, \varphi_2, v, \epsilon) \\ &= \int_{\mathbb{T}^2} \left(\partial_{\varphi_1} q_1\right)^2 + \left(\partial_{\varphi_1} q_1\right)\left(\partial_{\varphi_2} q_1\right) \\ &\quad + \epsilon \left(\int_{\mathbb{T}^2} \frac{1}{2}\left(\partial_{\varphi_1} q_1\right)^2 + \frac{\left(2a + a^2\epsilon^2\right)}{2}\left(\partial_{\varphi_2} q_1\right)^2 \right. \\ &\quad + a(1+\epsilon)\left(\partial_{\varphi_1} q_1\right)\left(\partial_{\varphi_2} q_1\right) \\ &\quad + \frac{1}{2} f(\varphi_1, \varphi_2, v, \epsilon)(q_2(q_1) + p(q_1)) \\ &\quad \left. - F(\varphi_1, \varphi_2, v, \epsilon) \right) \\ &= \Psi_0(q_1) \\ &\quad + \epsilon \left(\int_{\mathbb{T}^2} \frac{1}{2}\left(\partial_{\varphi_1} q_1\right)^2 + \frac{\left(2a + a^2\epsilon^2\right)}{2}\left(\partial_{\varphi_2} q_1\right)^2 \right. \\ &\quad + a(1+\epsilon)\left(\partial_{\varphi_1} q_1\right)\left(\partial_{\varphi_2} q_1\right) \\ &\quad + \frac{1}{2} f(\varphi_1, \varphi_2, v, \epsilon)(q_2(q_1) + p(q_1)) \\ &\quad \left. - F(\varphi_1, \varphi_2, v, \epsilon) \right). \end{aligned} \tag{74}$$

Because $\Psi_0(q_1) \equiv$ const, by (52), we can get the bounds of (68). □

The problem of finding nontrivial solutions of the Q_1-equation is reduced to finding nontrivial critical points of the reduced action functional $\Phi_{\epsilon,N}$ in B_{2R}. By (66), this is equivalent to finding critical points of the rescaled functional

$$\begin{aligned} \widehat{\Phi}_{\epsilon,N} &= \Gamma(q_1) + \mathfrak{R}_{\epsilon,N}(q_1) \\ &= \left(\mathfrak{Q}(q_1) - \int_{\mathbb{T}^2} a_3(\varphi_1, \varphi_2) \frac{q_1^4}{4} \right) + \mathfrak{R}_{\epsilon,N}(q_1), \end{aligned} \tag{75}$$

where the quadratic form

$$\begin{aligned} \mathfrak{Q}(q) := \int_{\mathbb{T}^2} \frac{1}{2}\left(\partial_{\varphi_1} q_1\right)^2 + \frac{\left(2a + a^2\epsilon^2\right)}{2}\left(\partial_{\varphi_2} q_1\right)^2 \\ + a(1+\epsilon)\left(\partial_{\varphi_1} q_1\right)\left(\partial_{\varphi_2} q_1\right) \end{aligned} \tag{76}$$

is positive definite on Q_+, negative definite on Q_-, and zero definite on Q_0. For $q_1 = q_+ + q_- + q_0 \in Q_1$ and $a < 0$, we have

$$\begin{aligned} \mathfrak{Q}(q_+) &= \int_{\mathbb{T}^2} \frac{1}{2}\left(\partial_{\varphi_1} q_+\right)^2 + \frac{\left(2a + a^2\epsilon^2\right)}{2}\left(\partial_{\varphi_2} q_+\right)^2 \\ &\quad + a(1+\epsilon)\left(\partial_{\varphi_1} q_+\right)\left(\partial_{\varphi_2} q_+\right) \\ &= \int_{\mathbb{T}^2} \sum_{(l_1,l_2)\in\mathbb{Z}^2} l_2^2 \left(-a - \frac{a^2\epsilon^2}{2}\right) \widehat{q}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2} \\ &= \frac{\alpha_+(a)}{2} |q_+|_{H^1}^2, \\ \mathfrak{Q}(q_0) &= 0, \\ \mathfrak{Q}(q_-) &= \int_{\mathbb{T}^2} \sum_{(l_1,l_2)\in\mathbb{Z}^2} l_2^2 \frac{(1 + a\epsilon)^2}{2} \widehat{q}_{l_1,l_2} e^{il_1\varphi_1} e^{il_2\varphi_2} \\ &= -\frac{\alpha_-(a)}{2} |q_-|_{H^1}^2, \end{aligned} \tag{77}$$

where the positive constants $\alpha_+(a), \alpha_-(a)$ are bounded away from zero and independent of ϵ. We will prove the existence of critical point of $\widehat{\Phi}_{\epsilon,N}$ in B_{2R} of linking type.

3.4. Linking Critical Points of the Reduced Action Functional $\widehat{\Phi}_{\epsilon,N}$. We cannot directly apply the linking theorem because $\widehat{\Phi}_{\epsilon,N}$ is defined only in the ball B_{2R}. Therefore, we first extend

$\widehat{\Phi}_{\epsilon,N}$ to the whole space Q_1. We define the extended action function $\widetilde{\Phi}_{\epsilon,N} \in C^1(Q_1, \mathbb{R})$ as

$$\widetilde{\Phi}_{\epsilon,N}(q_1) = \Gamma(q_1) + \widetilde{\mathfrak{R}}_{\epsilon,N}(q_1), \tag{78}$$

where $\widetilde{\mathfrak{R}}_{\epsilon,N}(q_1) : Q_1 \to \mathbb{R}$ is

$$\widetilde{\mathfrak{R}}_{\epsilon,N}(q_1) := \lambda\left(\frac{|q_1|_{H^1}^2}{R^2}\right)\mathfrak{R}_{\epsilon,N}(q_1), \tag{79}$$

and $\lambda : [0, +\infty) \to [0,1]$ is a smooth, nonincreasing, cut-off function such that

$$\begin{gathered}\lambda(x) = 1, \quad \text{if } |x| \le 1; \\ \lambda(x) = 0, \quad \text{if } |x| \ge 4, \quad \left|\lambda'(x)\right| < 1.\end{gathered} \tag{80}$$

By definition, $\widetilde{\Phi}_{\epsilon,N}(q_1) \equiv \widehat{\Phi}_{\epsilon,N}(q_1)$ on $B_R := \{q_1 \in Q_1 : |q_1|_{H^1} \le R\}$ and $\widetilde{\Phi}_{\epsilon,N}(q_1) = \Gamma(q_1)$ outside B_{2R}. Moreover, by (68), there is a constant $C_3(R) \ge C_2(R) > 0$ such that $\forall |q|_{H^1} \le 2R$, and

$$\left|\widetilde{\mathfrak{R}}_{\epsilon,N}(q_1)\right| \le C_3(R)\left(\epsilon + |\epsilon|\gamma^{-1} + \frac{1}{N^2}\right), \tag{81}$$

$$\left|\widetilde{\mathfrak{R}}'_{\epsilon,N}(q_1)[h]\right| \le C_3(R)\left(\epsilon + |\epsilon|\gamma^{-1} + \frac{1}{N^2}\right)|h|_{H^1}, \quad \forall h \in Q_1. \tag{82}$$

Second, we will verify that $\widetilde{\Phi}_{\epsilon,N}(q_1)$ satisfies the geometrical hypotheses of the linking theorem.

Lemma 9. *There exist positive constants $\rho, \beta, r_1, r_2 > \rho$, and $0 < \epsilon_1(R) \le \epsilon_0(R)$, $N_1(R) \ge N_0(R)$, which are independent of (ϵ, N, γ), such that, $\forall (|\epsilon|/\gamma) \le \epsilon_1$ and $\forall N \ge N_1(R)$,*

(i) $\inf_{q_1 \in S^+} \widetilde{\Phi}_{\epsilon,N}(q_1) \ge \beta > 0$, $\forall q_1 \in S^+ := \{q_1 \in Q_1 \cap Q_+ : |q_1|_{H^1} = \rho\}$,

(ii) $\sum_{q_1 \in \partial W^-} \widetilde{\Phi}_{\epsilon,N}(q_1) \le \beta/2, \forall q_1 \in \partial W^-$,

where W^- is the rectangle in $Q_- \oplus Q_0$,

$$\begin{gathered}W^- := \{q_1 = q_0 + q_+ + re^+,\ |q_0 + q_-| \le r_1,\ q_- \in Q_1 \cap Q_-, \\ q_0 \in \mathbb{R},\ r \in [0, r_2]\},\end{gathered} \tag{83}$$

and e_+ is the unit vector in $Q_1 \oplus Q_0$.

Proof. (i) $\forall q_+ \in Q_1 \cap Q_+$ with $|q_+|_{H^1} = \rho < R$, we have

$$\begin{aligned}\widetilde{\Phi}_{\epsilon,N}(q_+) &= \widehat{\Phi}_{\epsilon,N}(q_+) \\ &= \mathfrak{Q}(q_+) - \int_{\mathbb{T}^2} a_3(\varphi_1, \varphi_2)\frac{q_+^4}{4} + \mathfrak{R}_{\epsilon,N}(q_+) \\ &\ge \frac{\alpha_+(a)}{2}\rho^2 - \kappa_1\rho^4 - C_3(R)\left(\epsilon + \frac{\epsilon}{\gamma} + \frac{1}{N^2}\right).\end{aligned} \tag{84}$$

Now, we fix $\rho > 0$ small such that $(\alpha_+(a)/2)\rho^2 - \kappa_1\rho^4 \ge (\alpha_+(a)/4)\rho^2$. Since $C_3(R)(\epsilon + \epsilon/\gamma + 1/N^2) \le (\alpha_+(a)/8)\rho^2$, by (84), we can get

$$\widetilde{\Phi}_{\epsilon,N}(q_+) \ge \frac{\alpha_+(a)}{8}\rho^2 := \beta > 0, \quad \forall q_+ \in Q_1 \cap Q_+, \quad \text{with } |q_+|_{H^1} = \rho. \tag{85}$$

(ii) Let

$$\begin{gathered}B_1 := \{q_1 = q_0 + q_- + r_2e_+ \text{ with } |q_0 + q_-|_{H^1} \le r_1, \\ q- \in Q_1 \cap Q_-\} \subset \partial W^-,\end{gathered} \tag{86}$$

$$\begin{gathered}B_2 := \{q_1 = q_0 + q_- + re_+ \text{ with } |q_0 + q_-|_{H^1} = r_1, \\ q- \in Q_1 \cap Q_-, r \in [0, r_2]\} \subset \partial W^-,\end{gathered} \tag{87}$$

with $r_1, r_2 > 2R$. For $q_1 = q_0 + q_- + re_+ \in B_1 \cup B_2$,

$$\begin{aligned}\widetilde{\Phi}_{\epsilon,N}(q_1) &= \gamma(q_1) \\ &= \mathfrak{Q}(q_1) - \int_{\mathbb{T}^2} a_3(\varphi_1, \varphi_2)\frac{(q_0 + q_- + re_+)^4}{4} \\ &\le -\frac{\alpha_-(a)}{2}|q_-|_{H^1}^2 + r^2\mathfrak{Q}(e_+) \\ &\quad - \int_{\mathbb{T}^2} a_3(\varphi_1, \varphi_2)\frac{(q_0 + q_- + re_+)^4}{4} \\ &\le -\frac{\alpha_-(a)}{2}|q_-|_{H^1}^2 + r^2\mathfrak{Q}(e_+) \\ &\quad - \alpha\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^4,\end{aligned} \tag{88}$$

because $a_3(\varphi_1, \varphi_2)/4 \ge \alpha > 0$. Now, by Hölder inequality and orthogonality,

$$\begin{aligned}\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^4 &\ge \kappa_2\left(\int_{\mathbb{T}^2}(q_0 + q_- + re_+)^2\right)^2 \\ &= \kappa_2\left(q_0^2 + q_-^2 + r^2e_+^2\right)^2 \\ &\ge \kappa_3\left(q_0^2 + r^2\right)^2 \ge \kappa_3\left(q_0^4 + r^4\right),\end{aligned} \tag{89}$$

and by (88) we deduce that

$$\begin{aligned}\widetilde{\Phi}_{\epsilon,N}(q_0 + q_- + re_+) \le{}& \left(\kappa_4r^2 - \kappa_3r^4\right) \\ &- \left(\frac{\alpha_-(a)}{2}|q_-|_{H^1}^2 + \kappa_3q_0^4\right).\end{aligned} \tag{90}$$

Now, we fix r_2 large such that $\kappa_4r_2^2 - \kappa_3r_2^4 \le 0$, and therefore

$$\widetilde{\Phi}_{\epsilon,N}(q_1) \le \kappa_4r_2^2 - \kappa_3r_2^4 \le 0, \quad \forall q_1 \in B_1. \tag{91}$$

Next, setting $M := \max_{[0,r_2]} \kappa_4r^2 - \kappa_3r^4$, we fix r_1 large such that

$$\frac{\alpha_-(a)}{2}|q_-|_{H^1}^2 + \kappa_3q_0^4 \ge M, \quad \forall\,|q_- + q_0| = r_1, \tag{92}$$

and therefore

$$\widetilde{\Phi}_{\epsilon,N}(q_1) \le M - \left(\frac{\alpha_-(a)}{2}|q_-|_{H^1}^2 + \kappa_3 q_0^4\right) \le 0, \quad \forall q_1 \in B_2. \tag{93}$$

Finally, if $q_1 = q_- + q_0$,

$$\begin{aligned}\widetilde{\Phi}_{\epsilon,N}(q_1) &= \mathfrak{Q}(q_-) - \int_{\mathbb{T}^2} a_3(\varphi_1,\varphi_2)\frac{q_1^4}{4} + \widetilde{\mathfrak{R}}_{\epsilon,N}(q_1) \\ &\le \left|\widetilde{\mathfrak{R}}_{\epsilon,N}(q_1)\right| \le C_3(R)\left(\epsilon + |\epsilon|\gamma^{-1} + \frac{1}{N^2}\right).\end{aligned} \tag{94}$$

So, if $C_3(R)(\sqrt{\epsilon}+|\epsilon|\gamma^{-1}+(1/N^2)) \le \beta/2$, we can get $\widetilde{\Phi}_{\epsilon,N}(q_1) \le \beta/2$, $\forall q_1 \in \partial W^-$. □

We introduce the minimax class $\mathcal{S} := \{\psi \in C(\overline{W^-}, Q) \mid \psi = \mathrm{Id}$ on $\partial W_-\}$. According to Proposition 5.9 of [19], the maps of $\mathcal{S}$ have an important intersection property as follows.

Proposition 10. *(S^+ and W^- link with respect to ψ). Consider*

$$\psi \in \mathcal{S} \Longrightarrow \psi(W^-) \cap S^+ \ne \emptyset. \tag{95}$$

One defines the minimax linking level as follows:

$$\widetilde{c}_\epsilon := \inf_{\psi\in\mathcal{S}} \max_{q_1\in W^-} \widetilde{\Phi}_{\epsilon,N}(\psi(q_1)). \tag{96}$$

Obviously, by Proposition 10 and Lemma 9,

$$\max_{q_1\in W^-} \widetilde{\Phi}_{\epsilon,N}(\psi(q_1)) \ge \min_{q_1\in S^+} \widetilde{\Phi}_{\epsilon,N}(q_1) \ge \beta > 0, \quad \forall \psi \in \mathcal{S}, \tag{97}$$

and, therefore, $\widetilde{c}_\epsilon > \beta > 0$.

Since $\mathrm{Id} \in \mathcal{S}$, so

$$\begin{aligned}\widetilde{c}_\epsilon &\le \max_{q\in W^-} \widetilde{\Phi}_{\epsilon,N}((q_1)) \le \max_{q\in W^-}\left(\Gamma(q_1) + \widetilde{\mathfrak{R}}_{\epsilon,N}(q_1)\right) \\ &\le \max_{q\in W^-}\left(\frac{\alpha_+(a)}{2}|q_+|_{H^1}^2 + \frac{\alpha_-(a)}{2}|q_-|_{H^1}^2 + \int_{\mathbb{T}^2}\kappa q_1^4\right) \\ &\quad + 1 \le \widetilde{c}_\infty \le +\infty,\end{aligned} \tag{98}$$

where $\widetilde{c}_\infty$ is independent of N, ϵ, γ. By the linking theorem, we deduce the existence of a Palais-Smale sequence $(q_n) \in Q_1$ at the level $\widetilde{c}_\epsilon$, namely,

$$\widetilde{\Phi}_{\epsilon,N}(q_n) \longrightarrow \widetilde{c}_\epsilon, \quad \widetilde{\Phi}'_{\epsilon,N}(q_n) \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{99}$$

Third, we will prove that the Palais-Smale sequence (q_n) (up to subsequence) converges to some nontrivial critical point $\widehat{q}_1 \ne 0$ in some open ball of Q_1, where $\widetilde{\Phi}_{\epsilon,N}$ and $\widehat{\Phi}_{\epsilon,N}$ coincide. Because the space Q_1 is finite dimensional, we only need to prove that the sequence (q_n) is bounded.

Lemma 11. *There is a constant $M > 0$ independent of R, ϵ, N, γ, such that, for all $|\epsilon|/\gamma$ small enough and N large enough, the Palais-Smale sequence (q_n) is bounded; that is, $|q_n|_{H^1} < M$. So, there exists a subsequence of the P-S sequence that converges to some critical point $\widehat{q}_1 \ne 0$, and the functional $\widetilde{\Phi}_{\epsilon,N}$ possesses a nontrivial critical point $\widehat{q}_1 \in Q_1$ with critical value $\widetilde{\Phi}_{\epsilon,N}(\widehat{q}_1) = \widetilde{c}_\epsilon$.*

Proof. In the sequel, we will always assume that $(\epsilon + |\epsilon|\gamma^{-1} + 1/N^2) < 1$. Writing $\widetilde{\Phi}_{\epsilon,N}(q) = \Gamma(q) + \widetilde{\mathfrak{R}}_{\epsilon,N}(q)$, by (81)-(82), we can get

$$\begin{aligned}\widetilde{\Phi}_{\epsilon,N}(q_n) - \frac{1}{2}\widetilde{\Phi}'_{\epsilon,N}(q_n)[q_n] &= \Gamma(q_n) - \frac{1}{2}\Gamma'(q_n)[q_n] \\ &\quad + \left(\widetilde{\mathfrak{R}}_{\epsilon,N}(q_n) - \frac{1}{2}\widetilde{\mathfrak{R}}'_{\epsilon,N}(q_n)[q_n]\right) \\ &= \left(\frac{1}{2} - \frac{1}{4}\right)\int_{\mathbb{T}^2} a_3(\varphi_1) q_j^4 \\ &\quad + \left(\widetilde{\mathfrak{R}}_{\epsilon,N}(q_n) - \frac{1}{2}\widetilde{\mathfrak{R}}'_{\epsilon,N}(q_n)[q_n]\right) \\ &\ge \alpha\left(\frac{1}{2} - \frac{1}{4}\right)\int_{\mathbb{T}^2} q_j^4 - \left(\epsilon + |\epsilon|\gamma^{-1} + \frac{1}{N^2}\right).\end{aligned} \tag{100}$$

Then, by $\widetilde{c}_\epsilon < \widetilde{c}_\infty < +\infty$ and $\widetilde{\Phi}'_{\epsilon,N}(q_n) \to 0$ $(n \to \infty)$,

$$\widetilde{c}_\infty + 1 + |q_n|_{H^1} \ge \kappa_6 \int_{\mathbb{T}^2} q_n^4 := \kappa_6 |q_n|_{L^4}^4. \tag{101}$$

By Hölder inequality and orthogonality,

$$\begin{aligned}\widetilde{c}_\infty + 1 + |q_n|_{H^1} &\ge \kappa_7\left(\int_{\mathbb{T}^2}(q_{0,n} + q_{-,n} + q_{+,n})^2\right)^2 \\ &= \kappa_7\left(\int_{\mathbb{T}^2} q_{0,n}^2 + q_{-,n}^2 + q_{+,n}^2\right)^2 \ge \kappa_8 (q_{0,n})^4,\end{aligned} \tag{102}$$

and therefore $|q_{0,n}| \le (1 + |q_n|_{H^1})^{1/4}$. In the same way, by Hölder inequality and (82),

$$\begin{aligned}\widetilde{\Phi}'_{\epsilon,N}(q_n)[q_{+,n}] &= \alpha_+(a)|q_{+,n}|_{H_1}^2 \\ &\quad - \int_{\mathbb{T}^2} a_3(\varphi_1) q_n^3 q_{+,n} + \widetilde{\mathfrak{R}}'_{\epsilon,N}(q_n)[q_{+,n}] \\ &\ge \alpha_+(a)|q_{+,n}|_{H_1}^2 - \kappa_9 |q_{+,n}|_{H_1}\int_{\mathbb{T}^2}|q_n|^3 \\ &\quad - C_3(R)\left(\sqrt{\epsilon} + |\epsilon|\gamma^{-1} + \frac{1}{N^{3/2}}\right)|q_{+,n}|_{H_1} \\ &\ge \kappa_{10}|q_{+,n}|_{H_1}\left(|q_{+,n}|_{H_1} - |q_n|_{L^4}^3 - 1\right).\end{aligned} \tag{103}$$

By (101) and the above inequalities, using $\widetilde{\Phi}'_{\epsilon,N}(q_n) \to 0$ $(n \to \infty)$, we conclude that $|q_{+,n}|_{H_1} \le \kappa_{11}(1 + |q_n|_{H_1}^{3/4})$. Estimating analogously, we derive $|q_{-,n}|_{H_1} \le \kappa_{12}(1 + |q_n|_{H_1}^{3/4})$. Finally, we deduce that

$$\begin{aligned}|q_n|_{H_1} &= |q_{0,n}| + |q_{+,n}|_{H_1} + |q_{-,n}|_{H_1} \\ &\le \kappa_{13}\left(1 + |q_n|_{H_1}^{3/4} + |q_n|_{H_1}^{3/4}\right).\end{aligned} \tag{104}$$

So, we can conclude that $|q_n|_{H_1} \leq M$ for a suitable constant $M > 0$, which is independent of R, ϵ, N, γ. Since Q_1 is finite dimensional, $\{q_n\}$ converges, up to subsequence, to some critical point $\widehat{q}$ of $\widetilde{\Phi}_{\epsilon,N}$ with $|\widehat{q}|_{H_1} < M$. Since $\widetilde{\Phi}_{\epsilon,N}(\widehat{q}) = \widetilde{c}_\epsilon \geq \beta > 0$, we conclude that $\widehat{q} \neq 0$. □

Proof of Theorem 1. Let us fix $\overline{R} := M + 1$ and take $|\epsilon|\gamma^{-1} \leq \epsilon_2(\overline{R}) := \overline{\epsilon}$. According to Lemma 7, we can get, for $0 < \sigma N(\overline{R}) \leq 1$, $0 \leq s < 1/2$, a solution $(q_2(q_1), p(q_1)) \in (Q_2(\overline{N}) \oplus P) \cap \mathscr{H}_{\sigma,s}$ of the (Q_2)-(P)-equations with $\forall |q_1|_{H_1} \leq 2R$. By Lemma 11, the extended functional $\widetilde{\Phi}_{\epsilon,N}(q_1)$ possesses a nontrivial critical point $\widehat{q}$ with $|q_1|_{H_1} \leq M < \overline{R}$. Since $\widetilde{\Phi}_{\epsilon,N}$ coincides with $\Phi_{\epsilon,N}$ on the ball $B_{\overline{R}}$, by Lemma 8, there exists a nontrivial weak solution $\widehat{q}_1 + q_2(\widehat{q}_1) + p(\widehat{q}_1) \in \mathscr{H}_{\sigma,s}$ of (23). Finally, $u = \epsilon[\widehat{q}_1 + q_2(\widehat{q}_1) + p(\widehat{q}_1)] = \epsilon[\widehat{q}_\epsilon + p(\widehat{q}_1)]$ solves (7).

According to Lemma 6, by the regularizing property of the operator $\mathscr{L}_1$, the solution $\widehat{q}_\epsilon := \widehat{q}_1 + q_2(\widehat{q}_1)$ of the (Q)-equation belongs to $\mathscr{H}_{\sigma,s+2} \cap Q$. By the P-equation

$$\begin{aligned}&\left((2+\epsilon)\partial^2_{\varphi_1} + 2\left(1 + a\left(\epsilon + \epsilon^2\right)\right)\partial_{\varphi_1}\partial_{\varphi_2}\right)\widehat{p}\\&\quad = -\epsilon\left(\left(2a + a^2\epsilon^2\right)\partial^2_{\varphi_2} + \Pi_P f\left(\varphi_1, \varphi_2, \widehat{q}_\epsilon + p, \epsilon\right)\right)\widehat{p},\end{aligned} \tag{105}$$

where $\widehat{p} = p(\widehat{q}_1)$, we can get that

$$-\epsilon\left(\left(2a + a^2\epsilon^2\right)\partial^2_{\varphi_2} + \Pi_P f\left(\varphi_1, \varphi_2, \widehat{q}_\epsilon + p, \epsilon\right)\right)\widehat{p} \in \mathscr{H}_{\sigma\prime,s}, \tag{106}$$

for $0 < \sigma' < \sigma$, satisfying $|\epsilon(2a + a^2\epsilon^2)l_2^2|e^{|l_2|\sigma'} < e^{|l_2|\sigma}$. For $(a, \epsilon) \in \mathscr{B}_\gamma$, the eigenvalues of operator $(2+\epsilon)\partial^2_{\varphi_1} + 2(1+a(\epsilon+\epsilon^2))\partial_{\varphi_1}\partial_{\varphi_2}$ restricted to P satisfy $|(2+\epsilon)l_1^2 + 2(1+a\epsilon+a\epsilon^2)l_1 l_2| \geq |2+\epsilon|(\gamma|l_1|/|l_2|)$, $\forall l_1 \neq 0, l_1 + l_2 \neq 0$, and, thus, we deduce that $\widehat{p} \in \mathscr{H}_{\sigma'',s+1}$, for all $0 < \sigma'' < \sigma'$ (satisfying $C|l_2|e^{|l_2|\sigma''} < e^{|l_2|\sigma'}$), and $|\partial_{\varphi_1}\widehat{p}|_{\sigma'',s} = O(|\epsilon|/\gamma)$. By (105),

$$\begin{aligned}&(2+\epsilon)\partial^2_{\varphi_1}\widehat{p}\\&\quad = -2\left(1 + a\left(\epsilon + \epsilon^2\right)\right)\partial_{\varphi_1}\partial_{\varphi_2}\widehat{p}\\&\qquad - \epsilon\left(\left(2a + a^2\epsilon^2\right)\partial^2_{\varphi_2} + \Pi_P f\left(\varphi_1, \varphi_2, \widehat{q}_\epsilon + p, \epsilon\right)\right)\widehat{p},\end{aligned} \tag{107}$$

we get $\widehat{p} \in \mathscr{H}_{\overline{\sigma},s+2}$, and $|\widehat{p}|_{\overline{\sigma},s+2} = O(|\epsilon|\gamma^{-1})$, with $0 < \overline{\sigma} < \sigma''$. Thus, (15) follows with $\overline{s} = s + 2$, $0 < \overline{\sigma} < \sigma''$. By (5), $u(t,x) = \epsilon v((1+\epsilon)t + x, (1+a\epsilon^2)t + x)$ is the solution of (1), for all $(a,\epsilon) \in \mathscr{B}_\gamma$. To show that $u(t,x)$ is quasiperiodic, it remains to prove that v depends on both variables (φ_1, φ_2) independently. According to Lemma 9, $\forall |\widehat{q}_1|_{H_1} \leq \overline{R}$, $\widetilde{\Phi}_{\epsilon,N}(\widehat{q}_1) \geq \beta$. On the other hand, $\widetilde{\Phi}_{\epsilon,N}(\widehat{q}_- + \widehat{q}_0) \leq \beta/2$, $\forall |\widehat{q}_- + \widehat{q}_0|_{H_1} \leq \overline{R}$, so that $\widehat{q}_1 \notin Q_- \oplus Q_0$, and, therefore, v depends on φ_2. In fact, any solution v of (23) depending only on φ_2, that is, the solutions of

$$\left(2a + a^2\epsilon^2\right)\frac{d^2 v(\varphi_2)}{d\varphi_2^2} + f\left(\varphi_1, \varphi_2, v(\varphi_2), \epsilon\right) = 0, \tag{108}$$

is $v(\varphi_2) \equiv 0$. Indeed, by the homogeneity of $f(\varphi_1, \varphi_2, v, \epsilon)$, we have

$$\epsilon^3 f\left(\varphi_1, \varphi_2, v, \epsilon\right) = f\left(\varphi_1, \varphi_2, \epsilon v\right) = \sum_{k=3}^{\infty} a_k\left(\varphi_1, \varphi_2\right)(\epsilon v)^k. \tag{109}$$

Now consider a smooth function $h(\varphi_1)$ with zero mean, ant it satisfies $\int_{\mathbb{T}} a_k(\varphi_1, \varphi_2)h(\varphi_1)d\varphi_1 \neq 0$ for some k. Multiplying (108) by $h(\varphi_1)$ and integrating over $[0, \pi]$, we have

$$\begin{aligned}&\int_0^{2\pi}\left(2a + a^2\epsilon^2\right)\frac{d^2 v(\varphi_2)}{d\varphi_2^2}h(\varphi_1)\,d\varphi_1\\&\quad + \int_0^{2\pi} f\left(\varphi_1, \varphi_2, v(\varphi_2), \epsilon\right)h(\varphi_1)\,d\varphi_1 = 0.\end{aligned} \tag{110}$$

According to $h(\varphi_1)$ that has zero mean and multiplying above equation by ϵ^3, we get

$$\begin{aligned}&\int_0^{2\pi} f\left(\varphi_1, \varphi_2, \epsilon v(\varphi_2)\right)h(\varphi_1)\,d\varphi_1\\&\quad = \sum_{k=3}^{\infty}\left(\epsilon v(\varphi_2)\right)^k\int_0^{2\pi} a_k\left(\varphi_1, \varphi_2\right)h(\varphi_1)\,d\varphi_1 = 0.\end{aligned} \tag{111}$$

The function $G(z(\varphi_2)) = \sum_{k=3}^{\infty} b_k(\varphi_2)(z(\varphi_2))^k$, with $b_k = \int_0^{2\pi} a_k(\varphi_1, \varphi_2)h(\varphi_1)d\varphi_1$, is a nontrivial analytic function. Thus, the equation $G(\epsilon v(\varphi_2)) = 0$ cannot have a sequence of zeros accumulating to zero. So, for ϵ small enough, $v(\varphi_2) \equiv 0$. □

4. Waves Traveling in Opposite Directions

Substituting $\omega_1 = 1 + \epsilon$, $\omega_2 = 1 + a\epsilon$ into (8), we get

$$\mathscr{L}_{a,\epsilon}v + f\left(\varphi_1, \varphi_2, v\right) = 0, \tag{112}$$

where (see (8))

$$\begin{aligned}\mathscr{L}_{a,\epsilon} &:= \left[(1 + \epsilon + 1)\partial_{\varphi_1} + (a\epsilon)\partial_{\varphi_2}\right] \circ \left[\epsilon\partial_{\varphi_1} + (2 + a\epsilon)\partial_{\varphi_2}\right]\\&= 4\partial_{\varphi_1}\partial_{\varphi_2} + \epsilon\left[(2+\epsilon)\partial^2_{\varphi_1} + \left(2a + a^2\epsilon\right)\partial^2_{\varphi_2}\right.\\&\qquad\left. + 2(a + 1 + a\epsilon)\partial_{\varphi_1}\partial_{\varphi_2}\right] = \mathscr{L}_0 + \epsilon\mathscr{L}_1.\end{aligned} \tag{113}$$

We rescale (112) in order to highlight the relationship between the amplitude and the variation in frequency: $v(\varphi_1, \varphi_2) \to \sqrt{|\epsilon|}v(\varphi_1, \varphi_2)$, and, for convenience, we assume $\epsilon > 0$. In the following, we consider the scaled equation

$$\mathscr{L}_{a,\epsilon}v + \epsilon f\left(\varphi_1, \varphi_2, v, \epsilon\right) = 0, \tag{114}$$

where $f(\varphi_1, \varphi_2, v, \epsilon) = a_3(\varphi_1, \varphi_2)v^3 + \sqrt{\epsilon}O(v^4)$.

Equation (114) is the Euler-Lagrange equation of the Lagrange action functional $\Psi_\epsilon \in C^1(\mathscr{H}_{\sigma,s}, \mathbb{R})$ defined by

$$\begin{aligned}\Psi_\epsilon(v) &:= \int_{\mathbb{T}^2} 2\left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right) + \frac{\epsilon(2+\epsilon)}{2}\left(\partial_{\varphi_1} v\right)^2 \\ &\quad + \frac{\epsilon\left(2a + a^2\epsilon\right)}{2}\left(\partial_{\varphi_2} v\right)^2 \\ &\quad + \epsilon(a+1+a\epsilon)\left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right) - \epsilon F(\varphi_1, v, \delta) \\ &= \Psi_0(v) + \epsilon\Psi_1(v,\delta),\end{aligned} \tag{115}$$

where $F(\varphi_1, \varphi_2, v, \epsilon) := \int_0^v f(\varphi_1, \varphi_2, \xi, \epsilon)d\xi$ and $\Psi_0(v) := \int_{\mathbb{T}^2} 2(\partial_{\varphi_1} v)(\partial_{\varphi_2} v)$,

$$\begin{aligned}\Psi_1(v,\delta) &:= \int_{\mathbb{T}^2} \frac{(2+\epsilon)}{2}\left(\partial_{\varphi_1} v\right)^2 + \frac{2a + a^2\epsilon}{2}\left(\partial_{\varphi_2} v\right)^2 \\ &\quad + (a+1+a\epsilon)\left(\partial_{\varphi_1} v\right)\left(\partial_{\varphi_2} v\right) - F(\varphi_1, \varphi_2, v, \epsilon).\end{aligned} \tag{116}$$

In order to find critical points of $\Psi_\epsilon(v)$, we use the same method as in Section 3. The operator $\mathscr{L}_{a,\epsilon}$ is diagonal defined on the Banach space $\mathscr{H}_{\sigma,s}$ under the Fourier basis $e_{l_1,l_2} = e^{il_1\varphi_1}e^{il_2\varphi_2}$ with eigenvalue $D_{l_1,l_2} = -[(2+\epsilon)l_1 + a\epsilon l_2][\epsilon l_1 + (2+a\epsilon)l_2]$. So, we have

$$\mathscr{L}_{a,\epsilon}[v] = \sum_{(l_1,l_2)\in\mathbb{Z}^2} D_{l_1,l_2}\widehat{v}_{l_1,l_2}e^{il_1\varphi_1}e^{il_2\varphi_2}, \quad \forall v \in \mathscr{H}_{\sigma,s}. \tag{117}$$

The unperturbed functional $\Psi_0 : \mathscr{H}_{\sigma,s} \to \mathbb{R}$ possesses an infinite-dimensional linear space Q of critical points, which are the solutions of the equation

$$\mathscr{L}_0 q = 4\partial_{\varphi_1}\partial_{\varphi_2} q = 0. \tag{118}$$

The space Q can be written as

$$Q = \left\{ q = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \widehat{q}_{l_1,l_2}e^{il_1\varphi_1}e^{il_2\varphi_2} \in \mathscr{H}_{\sigma,s} \mid \widehat{q}_{l_1,l_2} = 0, \text{ for } l_1 l_2 \neq 0 \right\}. \tag{119}$$

We split Q as

$$Q = Q_+ + Q_0 + Q_-, \tag{120}$$

where

$$\begin{aligned} Q_+ &:= \left\{ q \in Q : \widehat{q}_{l_1,l_2} = 0, \text{ for } (l_1,l_2) \notin \Lambda_+ \right\} \\ &= \left\{ q_+ := q_+(\varphi) \in \mathscr{H}^0_{\sigma,s} \right\}, \\ Q_0 &:= \left\{ q : q_{0,0} \in \mathbb{R} \right\}, \\ Q_- &:= \left\{ q \in Q : \widehat{q}_{l_1,l_2} = 0, \text{ for } (l_1,l_2) \notin \Lambda_- \right\} \\ &= \left\{ q_- := q_-(\varphi) \in \mathscr{H}^0_{\sigma,s} \right\}, \\ \Lambda_+ &:= \left\{ (l_1,l_2) \in \mathbb{Z}^2 : l_1 = 0, (l_1,l_2) \neq (0,0) \right\}, \\ \Lambda_- &:= \left\{ (l_1,l_2) \in \mathbb{Z}^2 : l_2 = 0, (l_1,l_2) \neq (0,0) \right\}. \end{aligned} \tag{121}$$

We decompose the space $\mathscr{H}_{\sigma,s} = Q + P$, where

$$P := \left\{ p = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \widehat{p}_{l_1,l_2}e^{il_1\varphi_1}e^{il_2\varphi_2} \in \mathscr{H}_{\sigma,s} \mid \widehat{p}_{l_1,l_2} = 0, \text{ for } l_1 l_2 = 0 \right\}. \tag{122}$$

Projecting (114) onto the closed subspaces Q and P, setting $v = q + p \in \mathscr{H}_{\sigma,s}$ with $q \in Q$ and $p \in P$, we obtain

$$\begin{aligned} &(Q) \quad \mathscr{L}_1[q] + \Pi_Q f(\varphi_1, \varphi_2, q + p, \epsilon) = 0, \\ &(P) \quad \mathscr{L}_{a,\epsilon}[p] + \epsilon\Pi_P f(\varphi_1, \varphi_2, q + p, \epsilon) = 0, \end{aligned} \tag{123}$$

where $\Pi_Q : \mathscr{H}_{\sigma,s} \to Q$, $\Pi_P : \mathscr{H}_{\sigma,s} \to P$ are the projectors, respectively, onto Q and P; moreover, they are continuous.

In the same way, we decompose the space $Q = Q_1 + Q_2$. Setting $q = q_1 + q_2$ with $q_1 \in Q_1$ and $q_2 \in Q_2$, we finally get

$$\begin{aligned} &(Q_1) \quad \mathscr{L}_1[q_1] + \Pi_{Q_1}[f(\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon)] = 0 \\ &\qquad \Longleftrightarrow d\Psi_\epsilon(v)[h] = 0, \quad \forall h \in Q_1, \end{aligned} \tag{124}$$

$$\begin{aligned} &(Q_2) \quad \mathscr{L}_1[q_2] + \Pi_{Q_2}[f(\varphi_1, \varphi_2, q_1 + q_2 + p, \delta)] = 0 \\ &\qquad \Longleftrightarrow d\Psi_\epsilon(v)[h] = 0, \quad \forall h \in Q_2, \end{aligned} \tag{125}$$

$$\begin{aligned} &(P) \quad \mathscr{L}_{a,\epsilon}[p] + \epsilon\Pi_P[f(\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon)] = 0 \\ &\qquad \Longleftrightarrow d\Psi_\epsilon(v)[h] = 0, \quad \forall h \in P. \end{aligned} \tag{126}$$

The operator $\mathscr{L}_{a,\epsilon}$ is diagonal in the Fourier basis $\{e^{il_1\varphi_1}e^{il_2\varphi_2}, (l_1,l_2) \in \mathbb{Z}^2\}$ with eigenvalues $D_{l_1,l_2} = -[(2+\epsilon)l_1 + a\epsilon l_2][\epsilon l_1 + (2+a\epsilon)l_2]$. We first prove that $\mathscr{L}_{a,\epsilon}$ restricted to P has a bounded inverse when (a,ϵ) belongs to the uncountable zero-measure set

$$\mathscr{D}_\gamma := \left\{ (a,\epsilon) \in \mathbb{R}^- \times \mathbb{R}, \left(\frac{a\epsilon}{2+\epsilon}, \frac{\epsilon}{2+a\epsilon} \right) \in \mathscr{E}_\gamma, \frac{1+\epsilon}{1+a\epsilon} \notin \mathbb{Q}, 1+\epsilon \neq 0, 1+a\epsilon \neq 0, 2+a\epsilon \neq 0 \right\}, \tag{127}$$

where $\mathscr{E}_\gamma$ is a set of badly approximate numbers defined as

$$\mathscr{E}_\gamma := \left\{ \left(\frac{a\epsilon}{2+\epsilon}, \frac{\epsilon}{2+a\epsilon} \right) := (\epsilon_1, \epsilon_2) \in (-\epsilon_0, \epsilon_0) \times (-\epsilon_0, \epsilon_0) : \right.$$
$$\left. |l_1 + \epsilon_1 l_2| > \frac{\gamma}{|l_2|},\ |l_2 + \epsilon_2 l_1| > \frac{\gamma}{|l_1|} \right\} \tag{128}$$

for $\forall l_1, l_2 \in \mathbb{Z} \setminus \{0\}$, and $0 < \gamma < 1/4$, $\epsilon_0 \in (0, 1/2)$.

Lemma 12. *For $(a,\epsilon) \in \mathscr{D}_\gamma$, the eigenvalues D_{l_1,l_2} of $\mathscr{L}_{a,\epsilon}$ restricted to P satisfy*

$$\left|D_{l_1,l_2}\right| = \left|\left((2+\epsilon)\, l_1 + a\epsilon l_2\right)\left(\epsilon l_1 + (2+a\epsilon)\, l_2\right)\right| > \gamma, \quad \forall l_1 l_2 \neq 0. \tag{129}$$

As a consequence, the operator $\mathscr{L}_{a,\epsilon} : P \to P$ has a bounded inverse $\mathscr{L}^{-1}_{a,\epsilon}$ and satisfies

$$\left|\mathscr{L}^{-1}_{a,\epsilon}[h]\right|_{\sigma,s} \le \frac{|h|_{\sigma,s}}{\gamma}, \quad \forall h \in P. \tag{130}$$

Proof. Denote by $[x]$ the nearest integer close to x and $\{x\} = x - [x]$. If both $l_1 \neq -[(a\epsilon/(2+\epsilon)) l_2]$ and $l_2 \neq -[(\epsilon/(2+a\epsilon)) l_1]$, then we have

$$\begin{aligned} &\left|D_{l_1,l_2}\right| \\ &\quad = \left|(2+\epsilon)\left(l_1 + \frac{a\epsilon}{2+\epsilon} l_2\right)\right| \cdot \left|(2+a\epsilon)\left(l_2 + \frac{\epsilon}{2+a\epsilon} l_1\right)\right| > 1. \end{aligned} \tag{131}$$

If $l_1 = -[(a\epsilon/(2+\epsilon)) l_2]$, then

$$\begin{aligned} &\left|D_{l_1,l_2}\right| \\ &= \left|(2+\epsilon)\left(l_1 + \frac{a\epsilon}{2+\epsilon} l_2\right)\right| \\ &\quad \cdot \left|\epsilon\left(l_1 + \frac{a\epsilon}{2+\epsilon} l_2\right) + (2+a\epsilon)\, l_2 - \frac{a\epsilon}{2+\epsilon}\epsilon l_2\right| \\ &\ge \frac{\gamma}{|l_2|} \cdot \left|(2+\epsilon)\left(\epsilon\left\{\frac{a\epsilon}{2+\epsilon} l_2\right\} + 2l_2\right.\right. \\ &\qquad \left.\left. + a\epsilon\left(1 - \frac{\epsilon}{2+\epsilon}\right) l_2\right)\right| \ge \gamma. \end{aligned} \tag{132}$$

In the same way, if $l_2 = -[(\epsilon/(2+a\epsilon)) l_1]$, then we have

$$\begin{aligned} &\left|D_{l_1,l_2}\right| \\ &\ge \frac{\gamma}{|l_1|} \cdot \left|(2+a\epsilon)\left(a\epsilon\left\{\frac{\epsilon}{2+a\epsilon} l_1\right\} + 2l_1\right.\right. \\ &\qquad \left.\left. + \epsilon\left(1 - \frac{a^2\epsilon}{2+a\epsilon}\right) l_1\right)\right| \ge \gamma. \end{aligned} \tag{133}$$

□

Lemma 13. *The operator $\mathscr{L}_1 : Q_2 \to Q_2$ has a bounded inverse $\mathscr{L}_1^{-1}$ which satisfies*

$$\left|\mathscr{L}^{-1}[h]\right|_{\sigma,s} \le \frac{|h|_{\sigma,s}}{N^2}. \tag{134}$$

Proof. $\mathscr{L}_1$ is diagonal in the Fourier basis of $Q : e^{il_1\varphi_1} e^{il_2\varphi_2}$ with $(l_1, l_2) \in \Lambda_+ \cup \{(0,0)\} \cup \Lambda_-$ with eigenvalues

$$d_{l_1,l_2} = \begin{cases} -(2+\epsilon)\, l_1^2, & \text{if } l_2 = 0, \\ -a(2+a\epsilon)\, l_2^2, & \text{if } l_1 = 0. \end{cases} \tag{135}$$

The eigenvalues of $\mathscr{L}_1$ restricted to $Q_2(N)$ verify $|d_{l_1,l_2}| \ge N^2/C$, where the constant C depends on (ϵ, a), and (134) holds. □

Similarly, the solution of Q_1-equation (124) is the Euler-Lagrange equation of the reduced Lagrangian action functional:

$$\begin{aligned} &\Phi_{\epsilon,N} : B_{2R} \subset Q_1 \longrightarrow \mathbb{R}, \\ &\Phi_{\epsilon,N}(q_1) := \Psi_\epsilon\left(q_1 + q_2\left(q_1 + p\left(q_1\right)\right)\right). \end{aligned} \tag{136}$$

Lemma 14. *$\Phi_{\epsilon,N} \in C^1(B_{2R}, \mathbb{R})$ and a critical point $q_1 \in B_{2R}$ of $\Phi_{\epsilon,N}$ is a solution of the bifurcation equation* (124)*. Moreover, $\Phi_{\epsilon,N}$ can be written as*

$$\Phi_{\epsilon,N}(q_1) = const + \epsilon\left(\Gamma(q_1) + \mathfrak{R}_{\epsilon,N}(q_1)\right), \tag{137}$$

where

$$\begin{aligned} \Gamma(q_1) &:= \int_{\mathbb{T}^2} \frac{(2+\epsilon)}{2}\left(\partial_{\varphi_1} q_1\right)^2 \\ &\quad + (1 + a + a\epsilon)\left(\partial_{\varphi_1} q_1\right)\left(\partial_{\varphi_2} q_1\right) \\ &\quad + \frac{2a + a^2\epsilon}{2}\left(\partial_{\varphi_2} q_1\right)^2 - a_3(\varphi_1, \varphi_2)\frac{q_1^4}{4}, \\ \mathfrak{R}_{\epsilon,N}(q_1) &:= \int_{\mathbb{T}^2} a_3(\varphi_1, \varphi_2)\frac{q_1^4}{4} \\ &\quad - F(\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon) \\ &\quad + \frac{1}{2} f(\varphi_1, \varphi_2, v, \epsilon)(q_2 + p), \end{aligned} \tag{138}$$

and, for some positive constant $C_2(R) \ge C_1(R)$, we can get

$$\begin{aligned} &\left|\mathfrak{R}_{\epsilon,N}(q_1)\right| \le C_2(R)\left(\sqrt{\epsilon} + \epsilon\gamma^{-1} + \frac{1}{N^2}\right), \\ &\left|\mathfrak{R}'_{\epsilon,N}(q_1)[h]\right| \le C_2(R)\left(\sqrt{\epsilon} + \epsilon\gamma^{-1} + \frac{1}{N^2}\right)|h|_{H^1}, \\ &\qquad \forall h \in Q_1. \end{aligned} \tag{139}$$

The problem of finding nontrivial solutions of the Q_1-equation is reduced to finding nontrivial critical points of the reduced action functional $\Phi_{\epsilon,N}$ in B_{2R}. By (137), this is equivalent to finding critical points of the rescaled functional denoted by $\widehat{\Phi}_{\epsilon,N}$ and called the reduced action functional

$$\widehat{\Phi}_{\epsilon,N}(q_1) = \Gamma(q_1) + \mathfrak{R}_{\epsilon,N}(q_1) \equiv \left(\mathfrak{Q}(q_1) - \int_{\mathbb{T}^2} a_3(\varphi_1,\varphi_2)\frac{q_1^4}{4} + \mathfrak{R}_{\epsilon,N}(q_1) \right), \tag{140}$$

where the quadratic form

$$\mathfrak{Q}(q_1) = \int_{\mathbb{T}^2} \frac{(2+\epsilon)}{2}\left(\partial_{\varphi_1} q_1\right)^2 + (1+a+a\epsilon)\left(\partial_{\varphi_1} q_1\right)\left(\partial_{\varphi_2} q_1\right) + \frac{2a+a^2\epsilon}{2}\left(\partial_{\varphi_2} q_1\right)^2 \tag{141}$$

is positive definite on Q_+, negative definite on Q_-, and zero definite on Q_0. For $q_1 = q_+ + q_- + q_0 \in Q_1$, we have

$$\mathfrak{Q}(q_+) = \int_{\mathbb{T}^2} \frac{(2+\epsilon)}{2}\left(\partial_{\varphi_1} q_+\right)^2 = \frac{\alpha_+}{2}|q_+|^2_{H^1},$$
$$\mathfrak{Q}(q_0) = 0, \tag{142}$$
$$\mathfrak{Q}(q_-) = \int_{\mathbb{T}^2} \frac{(a+a\epsilon)}{2}\left(\partial_{\varphi_1} q_-\right)^2 = -\frac{\alpha_-}{2}|q_-|^2_{H^1},$$

where the positive constants α_+, α_- are bounded away from zero and independent of ϵ. The following steps of finding the nontrivial solutions of the Q_1-equation are similar to Lemmas 9 and 11 in Section 3.4.

Proof of Theorem 2. We can get the solution of (8) as follows:

$$v = \sqrt{|\epsilon|}\left[\widehat{q}_1 + q_2(\widehat{q}_1) + p(\widehat{q}_1)\right] = \sqrt{|\epsilon|}\left[\widehat{q}_\epsilon + p(\widehat{q}_1)\right]. \tag{143}$$

According to Lemma 13, by the regularizing property of the operator $\mathscr{L}_1$, the solution $\widehat{q}_\epsilon := \widehat{q}_1 + q_2(\widehat{q}_1)$ of the (Q)-equation belongs to $\mathscr{H}_{\sigma,s+2} \cap Q$. By the P-equation

$$\left(\epsilon(2+\epsilon)\partial^2_{\varphi_1} + \left(4+2a\epsilon+2\epsilon+2a\epsilon^2\right)\partial_{\varphi_1}\partial_{\varphi_2}\right)\widehat{p} = -\epsilon\left(\left(2a+a^2\epsilon\right)\partial^2_{\varphi_2} + \Pi_P f\left(\varphi_1,\varphi_2,\widehat{q}_\epsilon + p,\epsilon\right)\right)\widehat{p}, \tag{144}$$

where $\widehat{p} = p(\widehat{q}_1)$, we can get that

$$-\epsilon\left(\left(2a+a^2\epsilon\right)\partial^2_{\varphi_2} + \Pi_P f\left(\varphi_1,\varphi_2,\widehat{q}_\epsilon + p,\epsilon\right)\right)\widehat{p} \in \mathscr{H}_{\sigma',s}, \tag{145}$$

for $0 < \sigma' < \sigma$, satisfying $|\epsilon(2a+a^2\epsilon)l_2^2|e^{|l_2|\sigma'} < e^{|l_2|\sigma}$. For $(a,\epsilon) \in \mathscr{D}_\gamma$, the eigenvalues of operator

$$\epsilon(2+\epsilon)\partial^2_{\varphi_1} + \left(4+2a\epsilon+2\epsilon+2a\epsilon^2\right)\partial_{\varphi_1}\partial_{\varphi_2} \tag{146}$$

restricted to P satisfy

$$\left|\epsilon(2+\epsilon)l_1^2 + \left(4+2a\epsilon+\epsilon+a\epsilon^2\right)l_1 l_2\right| \ge |2+\epsilon|\frac{\gamma|l_1|}{|l_2|}, \quad \forall l_1 \neq 0,\ l_2 \neq 0, \tag{147}$$

and, thus, we deduce that $\widehat{p} \in \mathscr{H}_{\sigma'',s+1}$, for all $0 < \sigma'' < \sigma'$ (satisfying $C|l_2|e^{|l_2|\sigma''} < e^{|l_2|\sigma'}$), and $|\partial_{\varphi_1}\widehat{p}|_{\sigma'',s} = O(|\varepsilon|/\gamma)$. By (144),

$$\epsilon(2+\epsilon)\partial^2_{\varphi_1}\widehat{p} = -\left(4+2a\epsilon+2\epsilon+2a\epsilon^2\right)\partial_{\varphi_1}\partial_{\varphi_2}\widehat{p} - \epsilon\left(\left(2a+a^2\epsilon\right)\partial^2_{\varphi_2} + \Pi_P f\left(\varphi_1,\varphi_2,\widehat{q}_\epsilon + p,\epsilon\right)\right)\widehat{p}, \tag{148}$$

we get $\widehat{p} \in \mathscr{H}_{\overline{\sigma},s+2}$, and $|\widehat{p}|_{\overline{\sigma},s+2} = O(|\epsilon|/\gamma)$, with $0 < \overline{\sigma} < \sigma''$. Thus, (19) follows with $\bar{s} = s+2$, $0 < \overline{\sigma} < \sigma''$. By (6), $u(t,x) = \sqrt{|\epsilon|}v((1+\epsilon)t - x, (1+a\epsilon)t + x)$ is the solution of (1), for all $(a,\epsilon) \in \mathscr{D}_\gamma$. Obviously, v depends on both variables (φ_1,φ_2) independently. So, $u(t,x)$ is a quasiperiodic solution of (1), with frequencies $(\omega_1,\omega_2) = (1+\epsilon, 1+a\epsilon)$. □

5. Conclusion

In this paper, for the completely resonant nonlinear wave equations, under periodic boundary conditions, we obtain the existence and regularity of quasiperiodic solutions. The forced terms we consider are quasiperiodic, and, according to the linking theorem, the bifurcation equations are solved by variational method. Moreover, the solutions depending on the the spatial and time variables are coupled and in the form of traveling waves. In [28], Yuan got the existence of quasiperiodic solutions with $n \in \mathbb{N}$ $(n \ge 3)$ frequencies by KAM theory, in which the form of the solutions is $u(t,x) = v(\omega_1 t, \omega_2 t, \ldots, \omega_n t, x)$. In the future work, we will investigate the existence of quasiperiodic solutions with the traveling wave form as $u(t,x) = v(\omega_1 t + x, \omega_2 + x, \ldots, \omega_n t + x)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors express their sincere thanks to Professor Yong Li for his instructions and many invaluable suggestions. The first author was partially supported by NSFC Grant (nos. 11001042, 11171056, 11171130, and 11271062) and SRFDP Grant (no. 20100043120001). The second author is supported by NSFC Grant (nos. 11001041, 11201360, 11101170, 11202192, and 10926105), SRFDP Grant (no. 200802001008), and the State Scholarship Fund of the China Scholarship Council (nos. 2011662521 and 2011842509).

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