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Research Article

Stable Perturbed Iterative Algorithms for Solving New General Systems of Nonlinear Generalized Variational Inclusion in Banach Spaces

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We introduce and study a new general system of nonlinear variational inclusions involving generalized m-accretive mappings in Banach space. By using the resolvent operator technique associated with generalized m-accretive mappings due to Huang and Fang, we prove the existence theorem of the solution for this variational inclusion system in uniformly smooth Banach space, and discuss convergence and stability of a class of new perturbed iterative algorithms for solving the inclusion system in Banach spaces. Our results presented in this paper may be viewed as an refinement and improvement of the previously known results.

1. Introduction

Let m be a given positive integer, for any $i \in \{1, 2, \ldots, m\}$, X_i a real Banach space with dual space X_i^*. X_i, X_i^* all endowed with the norm $\|\cdot\|$, and $\langle \cdot, \cdot \rangle$ the dual pair between X_i and X_i^* (as matter of convenience). Let 2^{X_i} denote the family of all the nonempty subsets of X_i, $\eta_i : X_i \times X_i \to X_i^*$, $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ single-valued mappings, and $M_i : X_i \to 2^{X_i}$ generalized m-accretive mapping for $i = 1, 2, \ldots, m$. In this paper, we consider the following new general system for nonlinear variational inclusion involving generalized m-accretive mappings. Find $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ such that

$$0 \in N_i(x_1^*, x_2^*, \ldots, x_m^*) + M_i(x_i^*) \tag{1}$$

for all $i = 1, 2, \ldots, m$. Some special cases of the problem (1) had been studied by many authors. See, for example, [1–34] and the reference therein. Here, we mention some of them as follows.

Case 1. The problem (1) with $X_i = \mathscr{H}_i$ $(i = 1, 2, \ldots, m)$, the Hilbert spaces, was introduced and studied as general system of monotone nonlinear variational inclusions problems by Peng and Zhao [29].

If $J_q^{-1}\eta_i(x_i^1, x_i^2) = x_i^1 - x_i^2$ and $M_i = \partial\varphi_i$, $\varphi_i : X_i \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous functional on X_i, and $\partial\varphi_i$ denote the subdifferential operators of the φ_i for $i = 1, 2, \ldots, m$, then the problem (1) is equivalent to finding $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \ldots \times X_m$ such that

$$\begin{aligned} &\langle N_i(x_1^*, x_2^*, \ldots, x_m^*), j(x_i - x_i^*)\rangle \\ &\quad \geq \rho_i(\varphi_i(x_i^*) - \varphi_i(x_i)), \quad \forall x_i \in X_i. \end{aligned} \tag{2}$$

When $X_i = X$, 2-uniformly smooth Banach space with the smooth constant K, C is a nonempty closed convex subset of X, $N_i(x_1, x_2, \ldots, x_m) = \rho_i A_i(x_{i+1}) + x_i - x_{i+1}$, where $A_i : C \to X$ and $\rho_i > 0$ and $x_{m+1} = x_1$ for $i = 1, 2, \ldots, m$; the problem (2) reduces to the following system of finding $(x_1^*, x_2^*, \ldots, x_m^*) \in C \times C \times \cdots \times C$ such that

$$\begin{aligned} &\langle \rho_i A_i(x_{i+1}^*) + x_i^* - x_{i+1}^*, j(x - x_i^*)\rangle \\ &\quad \geq \rho_i(\varphi_i(x_i^*) - \varphi_i(x)), \quad \forall x \in X. \end{aligned} \tag{3}$$

Further, in the problem (3), when φ_i is the indicator function of a nonempty closed convex set C, in X defined by

$$\varphi_i(y) = \begin{cases} 0, & y \in C, \\ +\infty, & y \notin C, \end{cases} \tag{4}$$

then the system (3) reduces to finding $(x_1^*, x_2^*, \ldots, x_m^*) \in C \times C \times \cdots \times C$ such that

$$\begin{aligned} &\langle \rho_1 A_1 x_2^* + x_1^* - x_2^*, j(x - x_1^*)\rangle \geq 0, \quad \forall x \in C, \\ &\langle \rho_2 A_2 x_3^* + x_2^* - x_3^*, j(x - x_2^*)\rangle \geq 0, \quad \forall x \in C, \\ &\langle \rho_3 A_3 x_4^* + x_3^* - x_4^*, j(x - x_3^*)\rangle \geq 0, \quad \forall x \in C, \\ &\qquad \cdots \\ &\langle \rho_m A_m x_1^* + x_m^* - x_1^*, j(x - x_m^*)\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{5}$$

which was introduced and studied by Zhu et al. [34].

Case 2. If $m = 3$, then the system (3) is equivalent to finding $(x_1^*, x_2^*, x_3^*) \in C \times C \times C$ such that

$$\begin{aligned} &\langle \rho_1 A_1 x_2^* + x_1^* - x_2^*, j(x - x_1^*)\rangle \\ &\qquad \geq \rho_1 (\varphi_1(x_1^*) - \varphi_1(x)), \quad \forall x \in C, \\ &\langle \rho_2 A_2 x_3^* + x_2^* - x_3^*, j(x - x_2^*)\rangle \\ &\qquad \geq \rho_2 (\varphi_2(x_2^*) - \varphi_2(x)), \quad \forall x \in C, \\ &\langle \rho_3 A_3 x_1^* + x_3^* - x_1^*, j(x - x_3^*)\rangle \\ &\qquad \geq \rho_3 (\varphi_3(x_3^*) - \varphi_3(x)), \quad \forall x \in C. \end{aligned} \tag{6}$$

It is easy to see that the mathematical model studied by Saewan and Kumam [31] is a variant of (6).

Case 3. If $m = 2$, then the problem (1) reduces to find $(x^*, y^*) \in X_1 \times X_2$ such that

$$0 \in N_1(x^*, y^*) + M_1(x^*), \qquad 0 \in N_2(x^*, y^*) + M_2(y^*). \tag{7}$$

Problem (7) is called a system of strongly nonlinear quasi-variational inclusion involving generalized m-accretive mappings, it is considered and studied by Lan [19]. There are many special cases of the problems (7) that can be found in [3, 7, 12–14, 17, 20, 28, 30] and the references cited therein.

Case 4. If $m = 1$ and $X_1 = \mathscr{H}$, then the problem (1) reduces to finding $x^* \in \mathscr{H}$ such that

$$0 \in N(x^*) + M(x^*), \tag{8}$$

which was introduced and studied by Fang and Huang [8]. We remark that for appropriate and suitable choices of positive integer m, the mappings η_i, N_i, and M_i, and the spaces X_i for $i = 1, 2, \ldots, m$, one can know that the problem (1) includes a number of general class of variational character known problems, including minimization or maximization (whether constraint or not) of functions and minimax problems et al. as special cases. For more details, see [1–34] and the reference therein.

On the other hand, many authors discussed stability of the iterative sequence generated by the algorithm for solving the problems that they studied. Lan [19] introduced the notion of S-stable or stable with respect to S. Moreover, Agarwal et al. [1, 2], Jin [16], Kazmi and Bhat [18], and Lan and Kim [21] constructed some stability under suitable conditions, respectively.

Motivated and inspired by the above works, the main purpose of this paper is to introduce and study the new general system of nonlinear variational inclusions (1) involving generalized m-accretive mapping in uniformly smooth Banach spaces. By using the resolvent operator technique for generalized m-accretive, we prove the existence theorem of the solution for this kind of system of variational inclusions in Banach spaces and discuss the convergence and stability of a new perturbed iterative algorithm for solving this general system of nonlinear variational inclusions in Banach spaces.

2. Preliminaries

In order to get the main results of the paper, we need the following concepts and lemmas. Let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \right\}, \quad \forall x \in X, \tag{9}$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that if X^* is strictly convex or X is a uniformly smooth Banach space, then J_q is single-valued (see [33]), and if $X = \mathscr{H}$, the Hilbert space, then J_2 becomes the identity mapping on $\mathscr{H}$. We will denote the single-valued duality mapping by j_q.

In order to construct convergence and stability for researching the problem (1), we need to be using the following definition and lemma.

Definition 1. Let X_i be Banach spaces, and let $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ be single mappings for $(i = 1, 2, \ldots, m)$. Then N_i is said to be

(i) σ_j-strongly accretive with respect to jth argument if for any $(x_1, \ldots, x_{j-1}, x_j^1, x_{j+1}, \ldots, x_m)$, $(x_1, \ldots, x_{j-1}, x_j^2, x_{j+1}, \ldots, x_m) \in X_1 \times X_2 \times \cdots \times X_m$, there exists $j_{q_j}(x_j^1 - x_j^2) \in J_{q_j}(x_j^1 - x_j^2)$, such that

$$\begin{aligned} &\Big\langle N_i(x_1, \ldots, x_{j-1}, x_j^1, x_{j+1}, \ldots, x_m) \\ &\quad - N_i(x_1, \ldots, x_{j-1}, x_j^2, x_{j+1}, \ldots, x_m), \\ &\quad j_{q_j}(x_j^1 - x_j^2)\Big\rangle \geq \sigma_j \|x_j^1 - x_j^2\|^{q_j}, \end{aligned} \tag{10}$$

where $q_j > 1$ is a constant;

(ii) $(\zeta_{i1},\ldots,\zeta_{ij},\ldots,\zeta_{im})$-Lipschitz continuous if there exists constants $\zeta_{i1}>0,\ldots,\zeta_{ij}>0,\ldots,\zeta_{im}>0$, such that

$$\left\|N_i\left(x_1,\ldots,x_j,\ldots,x_m\right)-N_i\left(y_1,\ldots,y_j,\ldots,y_m\right)\right\| \leq \sum_{j=1}^{m}\zeta_{ij}\left\|x_j-y_j\right\|, \tag{11}$$

for all $x_j, y_j \in X_j$ and $j=1,2\ldots,m$.

Remark 2. When $X_i=\mathscr{H}_i$ $(i=1,2,\ldots,m)$, $\mathscr{H}_i$ is different or the same as Hilbert spaces, (i) and (ii) in Definition 1 reduce to strongly monotonicity with respect to jth argument of N_i and $(\zeta_{i1},\ldots,\zeta_{ij},\ldots,\zeta_{im})$-Lipschitz continuity of N_i, respectively (see [29]).

Definition 3. Let $\eta: X\times X\to X^*$ be single-valued mapping. Then set-valued mapping $M: X\to 2^X$ is said to be

(i) accretive if

$$\left\langle u-v, J_q\left(x-y\right)\right\rangle \geq 0, \quad \forall x,y\in X,\ u\in M(x),\ v\in M(y); \tag{12}$$

(ii) η-accretive if

$$\left\langle u-v,\eta\left(x,y\right)\right\rangle \geq 0, \quad \forall x,y\in X,\ u\in M(x),\ v\in M(y); \tag{13}$$

(iii) m-accretive if M is accretive and $(I+\rho M)(X)=X$ for all $\rho>0$, where I denotes the identity operator on X;

(iv) generalized m-accretive if M is η-accretive and $(I+\rho M)(X)=X$ for all $\rho>0$.

Remark 4. When $X=X^*=\mathscr{H}$, (i)–(iv) of Definition 3 reduce to the definitions of monotone operators, η-monotone operators, classical maximal monotone operators, and maximal η-monotone operators; if $\eta(x,y)=J_2(x-y)$, then (ii) and (iv) of Definition 3 reduce to the definitions of accretive and m-accretive of uniformly smooth Banach spaces (see [10, 11]).

Definition 5. The mapping $\eta: X\times X\to X^*$ is said to be

(i) δ-strongly monotone if there exists a constant $\delta>0$ such that

$$\left\langle x^1-x^2,\eta\left(x^1,x^2\right)\right\rangle \geq \delta\left\|x^1-x^2\right\|^2, \quad \forall x^1,x^2\in X; \tag{14}$$

(ii) τ-Lipschitz continuous if there exists a constant $\tau>0$ such that

$$\left\|\eta\left(x^1,x^2\right)\right\| \leq \tau\left\|x^1-x^2\right\|, \quad \forall x^1,x^2\in X. \tag{15}$$

In [10], Huang and Fang show that for any $\rho_i>0$, inverse mapping $(I+\rho_iM_i)^{-1}$ is single-valued, if $\eta_i: X_i\times X_i\to X_i^*$ is strict monotone and $M_i: X_i\to 2^{X_i}$ is generalized m-accretive mapping, where I is the identity mapping. Based on this fact, Huang and Fang [10] gave the following definition.

Definition 6. Let $\eta_i: X_i\times X_i\to X_i^*$ be strictly monotone mapping, and let $M_i: X_i\to 2^{X_i}$ be generalized m-accretive mapping. Then the resolvent $J_{M_i}^{\rho_i}$ for M_i is defined as follows:

$$J_{M_i}^{\rho_i}\left(x_i\right)=\left(I+\rho_iM_i\right)^{-1}\left(x_i\right), \quad \forall x_i\in X_i, \tag{16}$$

where $\rho_i>0$ is a constant and I denotes the identity mapping on X_i for $i=1,2,\ldots,m$.

Lemma 7 (see [10, 11]). *Let $\eta_i: X_i\times X_i\to X_i^*$ be τ_i-Lipschitz continuous and δ_i-strongly monotone, and let $M_i: X_i\to 2_i^X$ be generalized m-accretive mapping. Then for any $\rho_i>0$, the resolvent operator $J_{M_i}^{\rho_i}$ for M_i is τ_i/δ_i-Lipschitz continuous; that is,*

$$\left\|J_{M_i}^{\rho_i}\left(x_i\right)-J_{M_i}^{\rho_i}\left(y_i\right)\right\| \leq \frac{\tau_i}{\delta_i}\left\|x_i-y_i\right\|, \quad \forall x_i,y_i\in X_i,\ i=1,2,\ldots,m. \tag{17}$$

The modules of smoothness is a measure, it is depicted geometric structure of the underlying Banach space. The modules of smoothness of Banach space X are the function $\rho_X:[0,+\infty)\to[0,+\infty)$ defined by

$$\rho_X(t)=\sup\left\{\frac{1}{2}\left(\|x+y\|+\|x-y\|\right)-1: \|x\|\leq 1, \|y\|\leq t\right\}. \tag{18}$$

A Banach space X is called uniformly smooth if $\lim_{t\to 0}(\rho_X(t)/t)=0$. X is called q-uniformly smooth if there exists a constant $c>0$ such that $\rho_X(t)\leq ct^q$, where $q>1$ is a real number.

Remark that J_q is single-valued if X is uniformly smooth, and Hilbert space and L_p (or l_p) $(2\leq p<+\infty)$ spaces are 2-uniformly smooth Banach spaces. In what follows, we will denote the single-valued generalized duality mapping by j_q.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [35] proved the following result.

Lemma 8. *Let $q>1$ be a given real number and let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that for all $x,y\in X$, $j_q(x)\in J_q(x)$, there holds the following inequality:*

$$\|x+y\|^q\leq\|x\|^q+q\left\langle y,j_q(x)\right\rangle+c_q\|y\|^q. \tag{19}$$

Definition 9. Let S be a self-map of X, $x_0\in X$, and let $x_{n+1}=h(S,x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x\in X: Sx=x\}\neq\emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\}\subset X$ and let $\epsilon_n=\|u_{n+1}-h(S,u_n)\|$. If $\lim_{n\to\infty}\epsilon_n=0$ implies that $u_n\to x^*$, then the iteration procedure defined by $x_{n+1}=h(S,x_n)$ is said to be S-stable or stable with respect to S.

Lemma 10 (see [36]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1}\leq\left(1-t_n\right)a_n+b_nt_n+c_n, \quad \forall n\geq n_0, \tag{20}$$

where $t_n\in[0,1]$, $\sum_{n=0}^{\infty}t_n=+\infty$, $\lim_{n\to\infty}b_n=0$, and $\sum_{n=0}^{\infty}c_n<+\infty$. Then a_n converges to 0 as $n\to\infty$.

3. Existence Theorem

In this section, we will give the existence theorem of the problem (1). The solvability of the problem (1) depends on the equivalence between (1) and the problem of finding the fixed point of the associated generalized resolvent operator. It follows from the definition of generalized resolvent operator $J_{M_i}^{\rho_i}$ $(i = 1, 2, \ldots, m)$ that we can obtain the following conclusion.

Lemma 11. *Let $\eta_i : X_i \times X_i \to X_i^*$, $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ single-valued mappings, and $M_i : X_i \to 2^{X_i}$ generalized m-accretive mapping for $(i = 1, 2, \ldots, m)$. Then the following statements are mutually equivalent.*

(i) *An element $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is a solution to the problem* (1).

(ii) *There is an $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ such that*

$$x_i^* = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right], \quad (21)$$

where $J_{M_i}^{\rho_i} = (I + \rho_i M_i)^{-1}$, and $\rho_i > 0$ is constants for all $i = 1, 2, \ldots, m$.

(iii) *For any given constants $\rho_i > 0$, the map $F : X_1 \times X_2 \times \cdots \times X_m \to X_1 \times X_2 \times \cdots \times X_m$ is defined by*

$$\begin{aligned} F(u_1, u_2, \ldots, u_m) &= \big(P_{\rho_1}(u_1, u_2, \ldots, u_m), \ldots, P_{\rho_i}(u_1, u_2, \ldots, u_m), \ldots, \\ &\quad P_{\rho_m}(u_1, u_2, \ldots, u_m)\big) \end{aligned} \quad (22)$$

for all $u_i \in X_i$ and $i = 1, 2, \ldots, m$, has a fixed point $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$, where maps $P_{\rho_i} : X_1 \times X_2 \times \cdots \times X_m \to X_i$ are defined by

$$\begin{aligned} P_{\rho_i}(u_1, u_2, \ldots, u_m) &= J_{M_i}^{\rho_i}\left[u_i - \rho_i N_i\left(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m\right)\right] \end{aligned} \quad (23)$$

for $u_i \in X_i$ and $i = 1, 2, \ldots, m$.

Proof. We first prove that (i) $\Leftrightarrow$ (ii). Let $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ satisfy the relation in (ii). Then, the definition of resolvent operator $J_{M_i}^{\rho_i}$ implies that this equality holds if and only if

$$x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right) \in (I + \rho_i M_i)(x_i^*) \quad (24)$$

for $i = 1, 2, \ldots, m$; that is

$$0 \in N_i\left(x_1^*, x_2^*, \ldots, x_m^*\right) + M_i\left(x_i^*\right), \quad (25)$$

where $i = 1, 2, \ldots, m$. Thus $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is the solution of the problem (1).

Next, we show (ii) $\Leftrightarrow$ (iii). If $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ satisfy following relation:

$$x_i^* = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right], \quad (26)$$

then, for any $i = 1, 2, \ldots, m$, it follows from

$$\begin{aligned} &P_{\rho_i}(x_1^*, x_2^*, \ldots, x_m^*) \\ &\quad = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right] \end{aligned} \quad (27)$$

that

$$P_{\rho_i}(x_1^*, x_2^*, \ldots, x_m^*) = x_i^*. \quad (28)$$

Hence, $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is a fixed point of the mapping

$$\begin{aligned} F(u_1, u_2, \ldots, u_m) &= \big(P_{\rho_1}(u_1, u_2, \ldots, u_m), \ldots, P_{\rho_i}(u_1, u_2, \ldots, u_m), \ldots, \\ &\quad P_{\rho_m}(u_1, u_2, \ldots, u_m)\big). \end{aligned} \quad (29)$$

Conversely, if $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is a fixed point of the mapping $F : X_1 \times X_2 \times \cdots \times X_m \to X_1 \times X_2 \times \cdots \times X_m$, then

$$P_{\rho_i}(x_1^*, x_2^*, \ldots, x_m^*) = x_i^* \quad (30)$$

for $i = 1, 2, \ldots, m$. Hence, from

$$\begin{aligned} &P_{\rho_i}(x_1^*, x_2^*, \ldots, x_m^*) \\ &\quad = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right], \end{aligned} \quad (31)$$

we have

$$x_i^* = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right] \quad (32)$$

for $i = 1, 2, \ldots, m$. Therefore $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ satisfy the relation of (ii). □

Theorem 12. *Let X_i be a real q_i-uniformly smooth Banach space with $q_i > 1$ and let $\eta_i : X_i \times X_i \to X_i^*$ be τ_i-Lipschitz continuous and δ_i-strongly monotone for any $i = 1, 2, \ldots, m$. Suppose that $M_i : X_i \to 2^{X_i}$ is generalized m-accretive mapping, and $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ is σ_i-strongly accretive in the ith argument and $(\zeta_{i1}, \ldots, \zeta_{ii}, \ldots, \zeta_{im})$-Lipschitz continuous for $i = 1, 2, \ldots, m$. If*

$$\frac{\tau_j}{\delta_j}\sqrt[q_j]{1 - q_j\rho_j\sigma_j + c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}} + \sum_{i=1, i\neq j}^{m}\frac{\zeta_{ij}\rho_i\tau_i}{\delta_i} < 1, \quad (33)$$

where c_{q_j} is the constants as in Lemma 8 for $j = 1, 2, \ldots, m$, then problem (1) *has a unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$.*

Proof. For any given $\rho_i > 0$ and $i = 1, 2, \ldots, m$, we first define $P_{\rho_i} : X_1 \times X_2 \times \cdots \times X_m \to X_i$ as follows:

$$P_{\rho_i}(u_1, u_2, \ldots, u_m) = J_{M_i}^{\rho_i}[u_i - \rho_i N_i(u_1, u_2, \ldots, u_m)], \quad (34)$$

for all $u_i \in X_i$. Now define $\|\cdot\|_*$ on $X_1 \times X_2 \times \cdots \times X_m$ by

$$\|(u_1, u_2, \ldots, u_m)\|_* = \sum_{i=1}^{m} \|u_i\|, \quad \forall (u_1, u_2, \ldots, u_m) \in X_1 \times X_2 \times \cdots \times X_m. \quad (35)$$

It is easy to see that $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$ is a Banach space. In fact

(i) $\|(u_1, u_2, \ldots, u_m)\|_* = \sum_{i=1}^{m} \|u_i\| \geq 0$, the negative being satisfied;

(ii) for all real number α,

$$\begin{aligned}\|\alpha(u_1, u_2, \ldots, u_m)\|_* &= \|(\alpha u_1, \alpha u_2, \ldots, \alpha u_m)\|_* \\ &= \sum_{i=1}^{m} \|\alpha u_i\| = \sum_{i=1}^{m} |\alpha| \|u_i\| \\ &= |\alpha| \sum_{i=1}^{m} \|u_i\| = |\alpha| \|(u_1, u_2, \ldots, u_m)\|_*\end{aligned} \quad (36)$$

homogeneity being satisfied;

(iii) for all $(u_1, u_2, \ldots, u_m), (v_1, v_2, \ldots, v_m) \in X_1 \times X_2 \times \cdots \times X_m$,

$$\begin{aligned}&\|(u_1, u_2, \ldots, u_m) + (v_1, v_2, \ldots, v_m)\|_* \\ &= \|(u_1 + v_1, u_2 + v_2, \ldots, u_m + v_m)\|_* \\ &= \sum_{i=1}^{m} \|u_i + v_i\| \leq \sum_{i=1}^{m} (\|u_i\| + \|v_i\|) \\ &= \sum_{i=1}^{m} \|u_i\| + \sum_{i=1}^{m} \|v_i\| \\ &= \|(u_1, u_2, \ldots, u_m)\|_* + \|(v_1, v_2, \ldots, v_m)\|_*,\end{aligned} \quad (37)$$

the triangle inequality being satisfied;

(iv) let $\|(u_1, u_2, \ldots, u_m)\|_* = 0$; that is, $\sum_{i=1}^{m} \|u_i\| = 0$; this implies that $\|u_i\| = 0$ $(i = 1, 2, \ldots, m)$; thus $u_i = 0$ $(i = 1, 2, \ldots, m)$; we get $\|\cdot\|_*$ is a norm on the $X_1 \times X_2 \times \cdots \times X_m$;

(v) let $(u_1^n, u_2^n, \ldots, u_m^n) \in X_1 \times X_2 \times \cdots \times X_m$ is Cauchy sequence; that is, for $\forall \epsilon > 0$, there exists a positive integer N; let $n > N$; we have

$$\begin{aligned}&\|(u_1^{n+p}, u_2^{n+p}, \ldots, u_m^{n+p}) - (u_1^n, u_2^n, \ldots, u_m^n)\|_* \\ &\quad = \sum_{i=1}^{m} \|u_i^{n+p} - u_i^n\| < \epsilon.\end{aligned} \quad (38)$$

Thus, for all $i \in \{1, 2, 3, \ldots, m\}$, we have $\|u_i^{n+p} - u_i^n\| < \epsilon$ $(n > N, p = 1, 2, 3, \ldots,)$; that is, $\{u_i^n\} \subset X_i$ is also Cauchy sequence; thus $\lim_{n\to\infty} u_i^n = u_i \in X_i$ for $i = 1, 2, \ldots, m$; we get $(u_1, u_2, \ldots, u_m) \in X_1 \times X_2 \times \cdots \times X_m$ and $(u_1, u_2, \ldots, u_m)$ is a cluster point on the $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$; we claim $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$ is a Banach space.

Now, by (34), for any given $\rho_i > 0$, define mapping $F : X_1 \times X_2 \times \cdots \times X_m \to X_1 \times X_2 \times \cdots \times X_m$ by

$$\begin{aligned}&F(u_1, u_2, \ldots, u_m) \\ &\quad = \big(P_{\rho_1}(u_1, u_2, \ldots, u_m), \ldots, P_{\rho_i}(u_1, u_2, \ldots, u_m), \ldots, \\ &\qquad P_{\rho_m}(u_1, u_2, \ldots, u_m)\big),\end{aligned} \quad (39)$$

where $u_i \in X_i$ for $i = 1, 2, \ldots, m$.

In the sequel, we prove that F is a contractive mapping on the $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$. In fact, for any $u_i, v_i \in X_i$ and $i = 1, 2, \ldots, m$, it follows from (34) and Lemma 7 that

$$\begin{aligned}&\|P_{\rho_i}(u_1, u_2, \ldots, u_m) - P_{\rho_i}(v_1, v_2, \ldots, v_m)\| \\ &= \|J_{M_i}^{\rho_i}[u_i - \rho_i N_i(u_1, u_2, \ldots, u_m)] \\ &\qquad - J_{M_i}^{\rho_i}[v_i - \rho_i N_i(v_1, v_2, \ldots, v_m)]\| \\ &\leq \frac{\tau_i}{\delta_i} \|u_i - v_i - \rho_i (N_i(u_1, u_2, \ldots, u_m) - N_i(v_1, v_2, \ldots, v_m))\| \\ &\leq \frac{\tau_i}{\delta_i} \|u_i - v_i \\ &\qquad - \rho_i (N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m))\| \\ &\quad + \frac{\rho_i \tau_i}{\delta_i} \|N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_m)\|.\end{aligned} \quad (40)$$

By assumptions and Lemma 8, we have

$$\begin{aligned}&\|u_i - v_i - \rho_i (N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m))\|^{q_i} \\ &\leq \|u_i - v_i\|^{q_i} \\ &\quad + c_{q_i} \rho_i^{q_i} \|N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m)\|^{q_i} \\ &\quad - q_i \rho_i \langle N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m), j_{q_i}(u_i - v_i)\rangle \\ &\leq (1 - q_i \rho_i \sigma_i + c_{q_i} \rho_i^{q_i} \zeta_{ii}^{q_i}) \|u_i - v_i\|^{q_i},\end{aligned}$$

$$\begin{aligned}&\left\|N_i\left(u_1,\ldots,u_{i-1},v_i,u_{i+1},\ldots,u_m\right)\right.\\&\quad\left.-N_i\left(v_1,\ldots,v_{i-1},v_i,v_{i+1},\ldots,v_m\right)\right\|\\&\le\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|.\end{aligned}\tag{41}$$

From (40)-(41), we obtain

$$\begin{aligned}&\left\|P_{\rho_i}\left(u_1,u_2,\ldots,u_m\right)-P_{\rho_i}\left(v_1,v_2,\ldots,v_m\right)\right\|\\&\quad\le\frac{\rho_i\tau_i}{\delta_i}\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|\\&\qquad+\frac{\tau_i}{\delta_i}\sqrt[q_i]{1-q_i\rho_i\sigma_i+c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}}\left\|u_i-v_i\right\|\end{aligned}\tag{42}$$

for $i=1,2,\ldots,m$. Equation (42) implies that

$$\begin{aligned}&\sum_{j=1}^{m}\left\|P_{\rho_j}\left(u_1,u_2,\ldots,u_m\right)-P_{\rho_j}\left(v_1,v_2,\ldots,v_m\right)\right\|\\&\quad=\sum_{i=1}^{m}\left\|P_{\rho_i}\left(u_1,u_2,\ldots,u_m\right)-P_{\rho_i}\left(v_1,v_2,\ldots,v_m\right)\right\|\\&\quad\le\sum_{i=1}^{m}\left(\frac{\tau_i}{\delta_i}\sqrt[q_i]{1-q_i\rho_i\sigma_i+c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}}\left\|u_i-v_i\right\|\right.\\&\qquad\left.+\frac{\rho_i\tau_i}{\delta_i}\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|\right)\\&\quad\le\sum_{i=1}^{m}\frac{\tau_i}{\delta_i}\sqrt[q_i]{1-q_i\rho_i\sigma_i+c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}}\left\|u_i-v_i\right\|\\&\qquad+\sum_{i=1}^{m}\frac{\rho_i\tau_i}{\delta_i}\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|\\&\quad=\sum_{j=1}^{m}\frac{\tau_j}{\delta_j}\sqrt[q_j]{1-q_j\rho_j\sigma_j+c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}}\left\|u_j-v_j\right\|\\&\qquad+\sum_{j=1}^{m}\left(\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}}{\delta_i}\right)\left\|u_j-v_j\right\|\\&\quad=\sum_{j=1}^{m}\left[\frac{\tau_j}{\delta_j}\sqrt[q_j]{1-q_j\rho_j\sigma_j+c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}}+\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}}{\delta_i}\right]\\&\qquad\times\left\|u_j-v_j\right\|\\&\quad\le k\sum_{j=1}^{m}\left\|u_j-v_j\right\|,\end{aligned}\tag{43}$$

where $k=\max_{1\le j\le m}\{(\tau_j/\delta_j)\sqrt[q_j]{1-q_j\rho_j\sigma_j+c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}}+\sum_{i=1,i\neq j}^{m}(\zeta_{ij}\rho_i\tau_i/\delta_i)\}$. By (33), we know that $0\le k<1$. It follows from (43) that

$$\begin{aligned}&\left\|F(u_1,u_2,\ldots,u_m)-F(v_1,v_2,\ldots,v_m)\right\|_*\\&\quad\le k\left\|(u_1,u_2,\ldots,u_m)-(v_1,v_2,\ldots,v_m)\right\|_*.\end{aligned}\tag{44}$$

This proves that $F:X_1\times X_2\times\cdots\times X_m\to X_1\times X_2\times\cdots\times X_m$ is a contraction mapping. Hence, there exists a unique $(x_1^*,x_2^*,\ldots,x_m^*)\in X_1\times X_2\times\cdots\times X_m$ such that

$$F\left(\left(x_1^*,x_2^*,\ldots,x_m^*\right)\right)=\left(x_1^*,x_2^*,\ldots,x_m^*\right);\tag{45}$$

that is, $P_{\rho_i}(x_1^*,x_2^*,\ldots,x_m^*)=x_i^*$ for $i=1,2,\ldots,m$; that is,

$$x_i^*=J_{M_i}^{\rho_i}\left[x_i^*-\rho_iN_i\left(x_1^*,x_2^*,\ldots,x_m^*\right)\right].\tag{46}$$

By Lemma 11, $(x_1^*,x_2^*,\ldots,x_m^*)$ is the unique solution of problem (1). This completes the proof. □

Remark 13. If $m=2$, then Theorem 12 reduces to Theorem 3.2 of Lan [19].

Corollary 14. *Let $\mathscr{H}_i$ be real Hilbert space and $\eta_i:\mathscr{H}_i\times\mathscr{H}_i\to\mathscr{H}_i$ be τ_i-Lipschitz continuous and δ_i-strongly monotone for any $i=1,2,\ldots,m$. Suppose that $M_i:\mathscr{H}_i\to 2^{\mathscr{H}_i}$ is maximal η_i-monotone mapping, $N_i:\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m\to\mathscr{H}_i$ is σ_i-strongly monotone in the ith argument, and $(\zeta_{i1},\ldots,\zeta_{ii},\ldots,\zeta_{im})$-Lipschitz continuous for $i=1,2,\ldots,m$. If*

$$\frac{\tau_j}{\delta_j}\sqrt[2]{1-2\rho_j\sigma_j+\rho_j^2\zeta_{jj}^2}+\sum_{i=1,i\neq j}^{m}\frac{\zeta_{ij}\rho_i\tau_i}{\delta_i}<1,\tag{47}$$

then problem (1) has a unique solution $(x_1^*,x_2^*,\ldots,x_m^*)\in\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m$.

Corollary 15. *Let $\mathscr{H}_i$ be real Hilbert space for any $i=1,2,\ldots,m$. Suppose that $\varphi_i:\mathscr{H}_i\to(-\infty,+\infty]$ is proper, convex, and lower semicontinuous functional on $\mathscr{H}_i$ and $N_i:\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m\to\mathscr{H}_i$ is σ_i-strongly monotone in the ith argument and $(\zeta_{i1},\ldots,\zeta_{ii},\ldots,\zeta_{im})$-Lipschitz continuous for $i=1,2,\ldots,m$. If*

$$\sqrt[2]{1-2\rho_j\sigma_j+\rho_j^2\zeta_{jj}^2}+\sum_{i=1,i\neq j}^{m}\rho_i\zeta_{ij}<1,\tag{48}$$

then problem (2) has a unique solution $(x_1^*,x_2^*,\ldots,x_m^*)\in\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m$.

4. Perturbed Iterative Algorithms

In this section, by using Definition 9 and Lemma 10, we construct a new perturbed iterative algorithm with mixed errors for solving problem (1) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Algorithm 16. Let $\eta_i:X_i\times X_i\to X_i^*$ and $N_i:X_1\times X_2\times\cdots\times X_m\to X_i$ be single-valued mappings and let $M_i:X_i\to 2^{X_i}$

be generalized m-accretive mapping for $i = 1, 2, \ldots, m$. For any given initial point $(x_1^0, x_2^0, \ldots, x_m^0) \in X_1 \times X_2 \times \cdots \times X_m$, the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ for problem (1) is defined by

$$x_i^{n+1} = (1-\alpha_n) x_i^n + \alpha_n J_{M_i}^{\rho_i} [x_i^n - \rho_i N_i (x_1^n, x_2^n, \ldots, x_m^n)] + \alpha_n u_i^n + w_i^n, \tag{49}$$

where $n \geq 0$, $i = 1, 2, \ldots, m$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{u_i^n\}, \{w_i^n\} \subset X_i$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

(i) $u_i^n = u_i'^n + u_i''^n$;

(ii) $\lim_{n\to\infty} \|u_i'^n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|u_i''^n\| < +\infty$, $\sum_{n=0}^{\infty} \|w_i^n\| < +\infty$ for $i = 1, 2, \ldots, m$.

Let $\{(z_1^n, z_2^n, \ldots, z_m^n)\}$ be any sequence in $X_1 \times X_2 \times \cdots \times X_m$ and define $\{(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)\}$ by

$$\epsilon_i^n = \left\| z_i^{n+1} - \left\{ (1-\alpha_n) z_i^n + \alpha_n J_{M_i}^{\rho_i} [z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)] + \alpha_n u_i^n + w_i^n \right\} \right\| \tag{50}$$

for $i = 1, 2, \ldots, m$.

Algorithm 17. Let $\eta_i : \mathscr{H}_i \times \mathscr{H}_i \to \mathscr{H}_i$ and $N_i : \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m \to \mathscr{H}_i$ be single-valued mappings and let $M_i : \mathscr{H}_i \to 2^{\mathscr{H}_i}$ be maximal η_i-monotone mapping for $i = 1, 2, \ldots, m$. For any given initial point $(x_1^0, x_2^0, \ldots, x_m^0) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$, the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ for problem (1) is defined by

$$x_i^{n+1} = (1-\alpha_n) x_i^n + \alpha_n J_{M_i}^{\rho_i} [x_i^n - \rho_i N_i (x_1^n, x_2^n, \ldots, x_m^n)] + \alpha_n u_i^n + w_i^n, \tag{51}$$

where $n \geq 0$, $i = 1, 2, \ldots, m$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{u_i^n\}, \{w_i^n\} \subset X_i$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

(i) $\lim_{n\to\infty} \|u_i^n\| = 0$;

(ii) $\sum_{n=0}^{\infty} \|w_i^n\| < +\infty$ for $i = 1, 2, \ldots, m$.

Let $\{(z_1^n, z_2^n, \ldots, z_m^n)\}$ be any sequence in $\mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ and define $\{(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)\}$ by

$$\epsilon_i^n = \left\| z_i^{n+1} - \left\{ (1-\alpha_n) z_i^n + \alpha_n J_{M_i}^{\rho_i} [z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)] + \alpha_n u_i^n + w_i^n \right\} \right\| \tag{52}$$

for $i = 1, 2, \ldots, m$.

Algorithm 18. Let $N_i : \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m \to \mathscr{H}_i$ be single-valued mappings and $\varphi_i : \mathscr{H}_i \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous functional on $\mathscr{H}_i$ for $i = 1, 2, \ldots, m$. For any given initial point $(x_1^0, x_2^0, \ldots, x_m^0) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$, the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ for problem (2) is defined by

$$x_i^{n+1} = (1-\alpha_n) x_i^n + \alpha_n J_{\varphi_i} [x_i^n - \rho_i N_i (x_1^n, x_2^n, \ldots, x_m^n)] + w_i^n, \tag{53}$$

where $n \geq 0$, $i = 1, 2, \ldots, m$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{w_i^n\} \subset X_i$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the condition $\sum_{n=0}^{\infty} \|w_i^n\| < +\infty$ for $i = 1, 2, \ldots, m$. Let $\{(z_1^n, z_2^n, \ldots, z_m^n)\}$ be any sequence in $\mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ and define $\{(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)\}$ by

$$\epsilon_i^n = \left\| z_i^{n+1} - \left\{ (1-\alpha_n) z_i^n + \alpha_n J_{\varphi_i} [z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)] + w_i^n \right\} \right\| \tag{54}$$

for $i = 1, 2, \ldots, m$.

Remark 19. If $m = 2$, then Algorithm 16 reduces to Algorithm 4.3 of Lan [19].

Next we will show the convergence and stability of Algorithm 16.

Theorem 20. *Suppose that X_i, η_i, N_i, and M_i $(i = 1, 2, \ldots, m)$ are the same as in Theorem 12. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and condition (33) holds, then the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ defined by Algorithm 16 converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ of the problem (1). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then*

$$\lim_{n\to\infty} (z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*) \tag{55}$$

if and only if

$$\lim_{n\to\infty} (\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}, \tag{56}$$

where $(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)$ is defined by (50).

Proof. From Theorem 12, we know that problem (1) has a unique solution

$$(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m. \tag{57}$$

It follows from (49) and the proof of (42) in Theorem 12 that, for $i = 1, 2, \ldots, m$,

$$\begin{aligned}
&\left\| x_i^{n+1} - x_i^* \right\| \\
&\le (1 - \alpha_n) \left\| x_i^n - x_i^* \right\| \\
&\quad + \alpha_n \left\{ \frac{\tau_i}{\delta_i} \sqrt[q_i]{1 - q_i \rho_i \sigma_i + c_{q_i} \rho_i^{q_i} \zeta_{ii}^{q_i}} \left\| x_i^n - x_i^* \right\| \right. \\
&\qquad \left. + \frac{\rho_i \tau_i}{\delta_i} \sum_{j=1, j \ne i}^{m} \zeta_{ij} \left\| x_j^n - x_j^* \right\| \right\} \\
&\quad + \alpha_n \left\| u_i'^n \right\| + \left(\left\| u_i''^n \right\| + \left\| w_i^n \right\| \right).
\end{aligned} \tag{58}$$

It follows from (58), we have

$$\begin{aligned}
&\sum_{i=1}^{m} \left\| x_i^{n+1} - x_i^* \right\| \\
&\le (1 - \alpha_n) \sum_{i=1}^{m} \left\| x_i^n - x_i^* \right\| \\
&\quad + \alpha_n \sum_{j=1}^{m} \left[\frac{\tau_j}{\delta_j} \sqrt[q_j]{1 - q_j \rho_j \sigma_j + c_{q_j} \rho_j^{q_j} \zeta_{jj}^{q_j}} + \sum_{i=1, i \ne j}^{m} \frac{\rho_i \tau_i \zeta_{ij}}{\delta_i} \right] \\
&\quad \times \left\| x_j^n - x_j^* \right\| + \alpha_n \sum_{i=1}^{m} \left\| u_i'^n \right\| + \sum_{i=1}^{m} \left\| u_i''^n \right\| + \sum_{i=1}^{m} \left\| w_i^n \right\| \\
&= (1 - \alpha_n) \sum_{j=1}^{m} \left\| x_j^n - x_j^* \right\| \\
&\quad + \alpha_n \sum_{j=1}^{m} \left[\frac{\tau_j}{\delta_j} \sqrt[q_j]{1 - q_j \rho_j \sigma_j + c_{q_j} \rho_j^{q_j} \zeta_{jj}^{q_j}} + \sum_{i=1, i \ne j}^{m} \frac{\rho_i \tau_i \zeta_{ij}}{\delta_i} \right] \\
&\quad \times \left\| x_j^n - x_j^* \right\| + \alpha_n \sum_{j=1}^{m} \left\| u_j'^n \right\| + \sum_{j=1}^{m} \left\| u_j''^n \right\| + \sum_{j=1}^{m} \left\| w_j^n \right\| \\
&\le \left[1 - \alpha_n (1 - k) \right] \sum_{j=1}^{m} \left\| x_j^n - x_j^* \right\| \\
&\quad + \alpha_n (1 - k) \cdot \frac{1}{1 - k} \sum_{j=1}^{m} \left\| u_j'^n \right\| \\
&\quad + \left(\sum_{j=1}^{m} \left\| u_j''^n \right\| + \sum_{j=1}^{m} \left\| w_j^n \right\| \right),
\end{aligned} \tag{59}$$

where k is the same as in (43). Letting $t_n = \alpha_n(1 - k) \in [0, 1]$, $b_n = (1/(1 - k)) \sum_{j=1}^{m} \| u_j'^n \|$, and $c_n = \sum_{j=1}^{m} \| u_j''^n \| + \sum_{j=1}^{m} \| w_j^n \|$ $(n \ge 0)$, then it follows from $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and (i)–(iii) of Algorithm 16 that

$$\begin{gathered}
\sum_{n=0}^{\infty} t_n = +\infty, \qquad \lim_{n \to \infty} b_n = \frac{1}{1 - k} \sum_{j=1}^{m} \left(\lim_{n \to \infty} \left\| u_j'^n \right\| \right) = 0, \\
\sum_{n=0}^{\infty} c_n = \sum_{j=1}^{m} \sum_{n=0}^{\infty} \left\| u_j''^n \right\| + \sum_{j=1}^{m} \sum_{n=0}^{\infty} \left\| w_j^n \right\| < +\infty.
\end{gathered} \tag{60}$$

Setting $a_n = \sum_{j=1}^{m} \| x_j^n - x_j^* \|$, then (59) can be rewritten as

$$a_{n+1} \le (1 - t_n) a_n + b_n t_n + c_n, \quad n = 0, 1, 2, \ldots. \tag{61}$$

It follows from Lemma 10 that $\lim_{n \to \infty} a_n = 0$; that is,

$$\lim_{n \to \infty} \sum_{j=1}^{m} \left\| x_j^n - x_j^* \right\| = 0; \tag{62}$$

thus

$$x_j^n \longrightarrow x_j^* \quad (n \longrightarrow \infty), \ (j = 1, 2, \ldots, m). \tag{63}$$

Hence, we know that the sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*)$ of the problem (1).

Now we prove the second conclusion. By (50), now we know

$$\begin{aligned}
&\left\| z_i^{n+1} - x_i^* \right\| \\
&\quad \le \left\| (1 - \alpha_n) z_i^n + \alpha_n J_{M_i}^{\rho_i} \left[z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n) \right] \right. \\
&\qquad \left. + \alpha_n u_i^n + w_i^n - x_i^* \right\| + \epsilon_i^n,
\end{aligned} \tag{64}$$

where $i = 1, 2, \ldots, m$. As the proof of inequality (59), we have

$$\begin{aligned}
&\sum_{j=1}^{m} \left\| (1 - \alpha_n) z_j^n + \alpha_n J_{M_j}^{\rho_j} \left[z_j^n - \rho_j N_j (z_1^n, z_2^n, \ldots, z_m^n) \right] \right. \\
&\qquad \left. + \alpha_n u_j^n + w_j^n - x_j^* \right\| \\
&\le \left[1 - \alpha_n (1 - k) \right] \sum_{j=1}^{m} \left\| z_j^n - x_j^* \right\| \\
&\quad + \alpha_n (1 - k) \cdot \frac{1}{1 - k} \sum_{j=1}^{m} \left\| u_j'^n \right\| \\
&\quad + \left(\sum_{j=1}^{m} \left\| u_j''^n \right\| + \sum_{j=1}^{m} \left\| w_j^n \right\| \right).
\end{aligned} \tag{65}$$

Since $0 < a \le \alpha_n$ $(n = 0, 1, 2, \ldots,)$, it follows from (64) and (65) that

$$\begin{aligned}\sum_{j=1}^{m} \left\| z_j^{n+1} - x_j^* \right\| &\le \left[1 - \alpha_n (1-k)\right] \sum_{j=1}^{m} \left\| z_j^n - x_j^* \right\| \\ &\quad + \alpha_n (1-k) \cdot \frac{1}{1-k} \left(\sum_{j=1}^{m} \left\| u_j'^n \right\| + \frac{1}{a} \sum_{j=1}^{m} \epsilon_j^n \right) \\ &\quad + \left(\sum_{j=1}^{m} \left\| u_j''^n \right\| + \sum_{j=1}^{m} \left\| w_j^n \right\| \right).\end{aligned} \tag{66}$$

Suppose that $\lim_{n\to\infty}(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}$. Letting $b_n' = (1/(1-k))(\sum_{j=1}^{m} \|u_j'^n\| + (1/a)\sum_{j=1}^{m} \epsilon_j^n)$ and $a_n' = \sum_{j=1}^{m} \|z_j^n - x_j^*\|$, then (66) implies that

$$a_{n+1}' \le (1 - t_n) a_n' + b_n' t_n + c_n, \quad n = 0, 1, 2, \ldots, \tag{67}$$

where t_n, c_n are the same as previously. Since $\lim_{n\to\infty}\|u_j'^n\| = 0$ and $\lim_{n\to\infty}\epsilon_j^n = 0$ $(j = 1, 2, \ldots, m)$,

$$\lim_{n\to\infty} b_n' = \frac{1}{1-k} \left[\sum_{j=1}^{m} \left(\lim_{n\to\infty} \left\| u_j'^n \right\| \right) + \frac{1}{a} \sum_{j=1}^{m} \left(\lim_{n\to\infty} \epsilon_j^n \right) \right] = 0. \tag{68}$$

It again follows from Lemma 10, we have $\lim_{n\to\infty} a_n' = 0$ and so

$$\lim_{n\to\infty} (z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*). \tag{69}$$

Conversely, if $\lim_{n\to\infty}(z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*)$, it follows from (50), then we get

$$\begin{aligned}\epsilon_i^n &\le \left\| z_i^{n+1} - x_i^* \right\| \\ &\quad + \left\| (1 - \alpha_n) z_i^n + \alpha_n J_{M_j}^{\rho_i} \left[z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n) \right] \right. \\ &\quad \left. + \alpha_n u_i^n + w_i^n - x_i^* \right\|, \quad \forall i = 1, 2, \ldots, m.\end{aligned} \tag{70}$$

Combining (65) with (70), we have

$$\begin{aligned}\sum_{i=1}^{m} \epsilon_i^n &\le \sum_{i=1}^{m} \left\| z_i^{n+1} - x_i^* \right\| + \left[1 - \alpha_n (1-k)\right] \sum_{j=1}^{m} \left\| z_j^n - x_j^* \right\| \\ &\quad + \alpha_n (1-k) \cdot \frac{1}{1-k} \sum_{j=1}^{m} \left\| u_j'^n \right\| \\ &\quad + \left(\sum_{j=1}^{m} \left\| u_j''^n \right\| + \sum_{j=1}^{m} \left\| w_j^n \right\| \right) \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{71}$$

This completes the proof. □

Corollary 21. *Suppose that $\mathscr{H}_i$, η_i, N_i, and M_i $(i = 1, 2, \ldots, m)$ are the same as in Corollary 14. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and condition (47) holds, then the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ defined by Algorithm 17 converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ of the problem (1). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \ge 0$, then $\lim_{n\to\infty}(z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*)$ if and only if*

$$\lim_{n\to\infty} (\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}, \tag{72}$$

where $(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)$ is defined by (50).

Corollary 22. *Assume that $\mathscr{H}_i$, N_i, and φ_i $(i = 1, 2, \ldots, m)$ are the same as in Corollary 15. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and condition (48) holds, then the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ defined by Algorithm 18 converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ of the problem (2). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \ge 0$, then $\lim_{n\to\infty}(z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*)$ if and only if*

$$\lim_{n\to\infty} (\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}, \tag{73}$$

where $(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)$ is defined by (54).

Remark 23. If $m = 2$, then Theorem 20 reduces to Theorem 4.3 of Lan [19]. Further, one can easily see that our results presented in this paper may be viewed as an refinement and improvement of the previously known results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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