Research Article

Convexity Properties for Certain Classes of Analytic Functions Associated with an Integral Operator

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We introduce a new form of generalized integral operator defined on the class of analytic functions $\mathcal{A}_0$. By making use of this novel integral operator, we give the convexity of other integral operators. We also briefly indicate the relevant connections of our presented results to the formerly reported results. Furthermore, other interesting properties are also discussed.

1. Introduction

In the field of geometric function theory, the class of univalent functions [1, 2] has been mainly studied. There are many distinguished geometric properties that played important role in the theory of univalent functions, such as starlikeness, convexity, and close-to-convexity (see, e.g., [3–5]). One of the generalizations of univalent functions is the theory of multivalent functions or p -valent functions. Also, the geometric properties for the subclasses of p -valent functions are investigated by many authors (see, e.g., [6–9]).

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disc in the complex plane. For a positive integer p , $\mathcal{A}_p$ denotes the class of p -valent functions of the following form:

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}, \quad (1)$$

which is analytic in $\mathbb{D}$. In particular, we set $\mathcal{A}_1 \equiv \mathcal{A}$. Furthermore, let $\mathcal{A}_0$ be the class of analytic functions f in $\mathbb{D}$ of the following form:

$$f(z) = 1 + \sum_{j=1}^{\infty} b_j z^j. \quad (2)$$

A function $f \in \mathcal{A}_p$ is said to be p -valently starlike of order γ ($0 \leq \gamma < p$) in $\mathbb{D}$ if f satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma, \quad z \in \mathbb{D}. \quad (3)$$

We denote this class by $\mathcal{S}_p^*(\gamma)$ which was introduced by Patil and Thakare [10]. In particular, we set $\mathcal{S}_p^*(0) \equiv \mathcal{S}_p^*$ for a class of p -valent starlike functions in $\mathbb{D}$. Denoted by $\mathcal{S}^*(p, \gamma)$, the subclass of $\mathcal{S}_p^*(\gamma)$ consists of functions $f \in \mathcal{A}_p$ for which

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \gamma, \quad z \in \mathbb{D}. \quad (4)$$

On the other hand, a function $f \in \mathcal{A}_p$ is said to be p -valently convex of order γ ($0 \leq \gamma < p$) in $\mathbb{D}$ if f satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \gamma, \quad z \in \mathbb{D}. \quad (5)$$

We denote this class by $\mathcal{K}_p(\gamma)$ which was introduced by Owa [11]. In particular, we set $\mathcal{K}_p(0) \equiv \mathcal{K}_p$ for a class of p -valent convex functions in $\mathbb{D}$. Analogously, we denote that the subclass $\mathcal{K}(p, \gamma)$ of $\mathcal{K}_p(\gamma)$ consists of functions $f \in \mathcal{A}_p$ for which

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| < p - \gamma, \quad z \in \mathbb{D}. \quad (6)$$

By using the Alexander-type criterion, it follows that

$$f(z) \in \mathcal{K}_p(\gamma) \iff \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\gamma). \quad (7)$$

The statement is also true if we replace $\mathcal{S}_p^*(\gamma)$ and $\mathcal{K}_p(\gamma)$ by $\mathcal{S}^*(p, \gamma)$ and $\mathcal{K}(p, \gamma)$, respectively. Moreover, we note that

$$\mathcal{S}^*(p, \gamma) \subset \mathcal{S}_p^*(\gamma) \subset \mathcal{S}_p^*, \quad \mathcal{K}(p, \gamma) \subset \mathcal{K}_p(\gamma) \subset \mathcal{K}_p. \quad (8)$$

A function $f \in \mathcal{US}_p(\delta, \gamma)$ is said to be δ -uniformly p -valent starlike of order γ ($-1 \leq \gamma < p, \delta \geq 0$) in $\mathbb{D}$ if f satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta \left| \frac{zf'(z)}{f(z)} - p \right| + \gamma, \quad z \in \mathbb{D}. \quad (9)$$

Furthermore, a function $f \in \mathcal{UK}_p(\delta, \gamma)$ is said to be δ -uniformly p -valent starlike of order γ ($-1 \leq \gamma < p, \delta \geq 0$) in $\mathbb{D}$ if f satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| + \gamma, \quad z \in \mathbb{D}. \quad (10)$$

Both $\mathcal{US}_p(\delta, \gamma)$ and $\mathcal{UK}_p(\delta, \gamma)$ are comprehensive classes of analytic functions that include various classes of analytic univalent functions as well as some very well-known ones. For example, in the case $p = 1$, we have $\mathcal{US}_1(\delta, \gamma) \equiv \mathcal{US}(\delta, \gamma)$ and $\mathcal{UK}_1(\delta, \gamma) \equiv \mathcal{UK}(\delta, \gamma)$ which are introduced by Bharati et al. [12]. For $\gamma = 0$, $\mathcal{US}_1(\delta, 0) \equiv \mathcal{US}(\delta)$ is the class of δ -uniformly convex function [13]. In the special case, $p = \delta = 1$ and $\gamma = 0$, the class $\mathcal{US}_1(1, 0) \equiv \mathcal{US}$ of uniformly starlike functions and $\mathcal{UK}_1(1, 0) \equiv \mathcal{UK}$ of uniformly convex functions were introduced by Goodman [14, 15]. Using the Alexander type relation, statement (7) holds for the $\mathcal{UK}_p(\beta, \gamma)$ and $\mathcal{US}_p(\beta, \gamma)$; that is,

$$f(z) \in \mathcal{UK}_p(\beta, \gamma) \iff \frac{zf'(z)}{p} \in \mathcal{US}_p(\beta, \gamma). \quad (11)$$

Many researchers have studied the geometric properties of integral operators. The common investigation is finding sufficient conditions of integral operators in order to transform analytic functions into classes with each of those mentioned properties. The well-known integral transformation defining a subclass of univalent functions was introduced by Alexander in [16]. It is of the following form:

$$F[f](z) = \int_0^z \frac{f(t)}{t} dt. \quad (12)$$

In [17], Kim and Merkes extended the integral operator (12) by introducing a complex parameter α as

$$F^\alpha[f](z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt. \quad (13)$$

Another object of investigation for the studies of the integral operator by Pfaltzgraff [18] is G^α defined by

$$G^\alpha[f](z) = \int_0^z (f'(t))^\alpha dt. \quad (14)$$

Until now, the various generalized form of the integral operators F^α in (13) and G^α in (14) has been investigated. However, Breaz and Stanciu [19] introduced and studied the more general form of integral operator $J_n^{\alpha, \mu} : \mathcal{A}^n \times \mathcal{A}^n \rightarrow \mathcal{A}$, which is

$$J_n^{\alpha, \mu}[f, g](z) = \int_0^z \prod_{k=1}^n \left(\frac{f_k(t)}{t} \right)^{\alpha_k} (g'_k(t))^{\mu_k} dt. \quad (15)$$

By setting appropriate values for the parameters n , α , and μ , integral operators that have been previously introduced can be obtained. In particular, if $\mu_k = 0$, then the integral operator $J_n^{\alpha, 0}$ becomes the integral operator F_n^α introduced by D. Breaz and N. Breaz [20]. Also, when $\alpha_k = 0$, the integral operator $J_n^{0, \mu}$ is exactly the integral operator G_n^α defined by Breaz et al. [21]. The specialized form of F_n^α and G_n^α involving the Bessel functions was introduced and studied in [22–24]. In addition, the specific case $n = 1$ for $J_n^{\alpha, \mu}$ in (15), $J_1^{\alpha, \mu} = J^{\alpha, \mu}$ was investigated by Pescar in [25]. The univalence and their properties of the integral operators are reported in [26–30].

In [31], Bulut developed the integral operator $I_{p,n}^{\alpha, \mu} : \mathcal{A}_p^n \times \mathcal{A}_p^n \rightarrow \mathcal{A}_p$ which extends the class of analytic functions $\mathcal{A}$ to the class of p -valent functions $\mathcal{A}_p$; that is,

$$I_{p,n}^{\alpha, \mu}[f, g](z) = \int_0^z pt^{p-1} \prod_{k=1}^n \left(\frac{f_k(t)}{t^p} \right)^{\alpha_k} \left(\frac{g'_k(t)}{pt^{p-1}} \right)^{\mu_k} dt. \quad (16)$$

By setting $\mu_k = 0$ and $\alpha_k = 0$, we obtain the integral operators $I_{p,n}^{\alpha, 0} = F_{p,n}^\alpha$ and $I_{p,n}^{0, \mu} = G_{p,n}^\mu$, respectively, which were introduced by Frasin [32]. Also, some properties of these integral operators have been studied in [32–34].

Recently, many authors modified integral operators associated with differential operator such as Salagean operator [35], Ruscheweyh operator [36], Al-Ouboudi operator [37], and Carlson-Shaffer operator [38]. In [39], Frasin investigated one of the generalized integral operators by using the Hadamard product to demonstrate most of the previously defined integral operators. Frasin [39] defined the integral operator $H_n : \mathcal{A}^n \times \mathcal{A}^n \rightarrow \mathcal{A}$ by

$$H_n[f, g](z) = \int_0^z \prod_{k=1}^n \left(\frac{(f_k * g_k)(t)}{t} \right)^{\alpha_k} dt, \quad (17)$$

where $(f * g)(z)/z \neq 0$, $z \in \mathbb{D}$ and the Hadamard product is defined by

$$(f * g)(z) \equiv z + \sum_{j=2}^{\infty} a_j b_j z^j, \quad z \in \mathbb{D}, \quad (18)$$

where $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$. It was reported that, for appropriate functions $g_k \in \mathcal{A}$, the integral operator H_n generalizes many integral operators introduced and studied by several authors [40–44]. Moreover, the integral operator H_n is generalized integral operators of those in (12)–(15). In a similar idea, H_n can be extended to more generalized on $\mathcal{A}_p^n \times \mathcal{A}_p^n$ to $\mathcal{A}_p$ by

$$H_{p,n}[f, g](z) = \int_0^z pt^{p-1} \prod_{k=1}^n \left(\frac{(f_k * g_k)(t)}{t^p} \right)^{\alpha_k} dt, \quad (19)$$

where $(f * g)(z)/z^p \neq 0$, $f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}$, and $g(z) = z^p + \sum_{j=1}^{\infty} b_{p+j} z^{p+j}$, and the Hadamard product is defined by

$$(f * g)(z) \equiv z^p + \sum_{j=1}^{\infty} a_{p+j} b_{p+j} z^{p+j}, \quad z \in \mathbb{D}. \quad (20)$$

Certainly, the integral operator $H_{p,n}$ generalizes many operators when we choose suitable functions $g_k \in \mathcal{A}_p$ (see, e.g., [45–47]).

In fact, we can write the Hadamard product in the form $(f * g)(z) = z^p h(z)$ where h is analytic in $\mathbb{D}$ and $h(0) = 1$. Indeed,

$$h(z) = 1 + \sum_{j=1}^{\infty} a_{p+j} b_{p+j} z^j, \quad z \in \mathbb{D}, \quad (21)$$

which usually appears in most of integral operators and always belongs to the class $\mathcal{A}_0$. That is why we are interested in replacing the term $(f_k * g_k)(z)/z^p$ in the integral operator (19) with general function in $\mathcal{A}_0$. Additionally, replacing the term z^{p-1} with a function $\phi(z) \in \mathcal{A}_{p-1}$ yields the integral operator, which is still contained in $\mathcal{A}_p$.

We now define the following general integral operator $\mathcal{J}_\phi^\alpha : \mathcal{A}_0^n \rightarrow \mathcal{A}_p$, for $p \geq 1$, $\phi(z) \in \mathcal{A}_{p-1}$, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_+^n$ by

$$\mathcal{J}_\phi^\alpha[h](z) = \int_0^z p\phi(t) \prod_{k=1}^n (h_k(t))^{\alpha_k} dt, \quad (22)$$

where $h_k \in \mathcal{A}_0$ for all $k = 1, 2, \dots, n$.

The main purpose of the paper is to investigate the sufficient conditions on convexity of the integral operator $\mathcal{J}_\phi^\alpha[h]$ on classes $\mathcal{S}_p^*(\gamma)$, $\mathcal{S}^*(p, \gamma)$, and $\mathcal{US}(\delta, \gamma)$ of analytic functions. Our main results will be applied to reinstate the results of former researches with related integral operators.

2. Main Results

In this section, we investigate sufficient conditions for the convexity of the integral operator $\mathcal{J}_\phi^\alpha[h]$ which is defined by (22). For the convenience, we introduce the transformation operator $\mathcal{T}_p : \mathcal{A}_q \rightarrow \mathcal{A}_{p+q}$ by

$$\mathcal{T}_p(f)(z) = z^p f(z), \quad z \in \mathbb{D}, \quad (23)$$

where $f \in \mathcal{A}_q$ and q is a nonnegative integer. In particular, we set $\mathcal{T}_1 = \mathcal{I}$.

We now prove a general property which guarantees the convexity of the proposed integral operator on the class $\mathcal{S}_p^*(\gamma)$.

Theorem 1. Let $0 \leq \lambda$, $\gamma_k < p$, $\mathcal{T}(\phi) \in \mathcal{S}_p^*(\lambda)$, and $\mathcal{T}_p(h_k) \in \mathcal{S}_p^*(\gamma_k)$ for $k = 1, 2, \dots, n$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_+^n$ satisfies

$$\sum_{k=1}^n \alpha_k (p - \gamma_k) \leq \lambda, \quad (24)$$

then the integral operator $\mathcal{J}_\phi^\alpha[h]$ defined by (22) is p -valently convex of order $\lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k)$.

Proof. From the definition of integral operator in (22), we observe that $\mathcal{J}_\phi^\alpha[h] \in \mathcal{A}_p$. By calculating the first derivative of $\mathcal{J}_\phi^\alpha[h]$, we obtain

$$(\mathcal{J}_\phi^\alpha[h])'(z) = p\phi(z) \prod_{k=1}^n (h_k(z))^{\alpha_k}. \quad (25)$$

Differentiating on both sides of (25) logarithmically and multiplying by z give

$$\frac{z(\mathcal{J}_\phi^\alpha[h])''(z)}{(\mathcal{J}_\phi^\alpha[h])'(z)} = \frac{z\phi'(z)}{\phi(z)} + \sum_{k=1}^n \alpha_k \frac{zh'_k(z)}{h_k(z)}. \quad (26)$$

Since $\mathcal{T}(\phi) \in \mathcal{S}_p^*(\lambda)$, it follows that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z\mathcal{T}(\phi(z))'(z)}{\mathcal{T}(\phi)(z)} \right\} &= \operatorname{Re} \left\{ \frac{z(z\phi(z))'(z)}{z\phi(z)} \right\} \\ &= \operatorname{Re} \left\{ 1 + \frac{z\phi'(z)}{\phi(z)} \right\} > \lambda. \end{aligned} \quad (27)$$

Also, since $\mathcal{T}_p(h_k) \in \mathcal{S}_p^*(\gamma_k)$, we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z(\mathcal{T}_p(h_k))'(z)}{\mathcal{T}_p(h_k)(z)} \right\} &= \operatorname{Re} \left\{ \frac{z(z^p h_k(z))'(z)}{z^p h_k(z)} \right\} \\ &= \operatorname{Re} \left\{ p + \frac{zh'_k(z)}{h_k(z)} \right\} > \gamma_k. \end{aligned} \quad (28)$$

From (27) and (28), by taking the real part of (26), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ 1 + \frac{z(\mathcal{J}_\phi^\alpha[h])''(z)}{(\mathcal{J}_\phi^\alpha[h])'(z)} \right\} &= \operatorname{Re} \left\{ 1 + \frac{z\phi'(z)}{\phi(z)} \right\} + \sum_{k=1}^n \alpha_k \operatorname{Re} \left\{ \frac{zh'_k(z)}{h_k(z)} \right\} \\ &> \lambda + \sum_{k=1}^n \alpha_k \operatorname{Re} \left\{ p + \frac{zh'_k(z)}{h_k(z)} \right\} - p \sum_{k=1}^n \alpha_k \\ &> \lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k). \end{aligned} \quad (29)$$

Therefore, $\mathcal{J}_\phi^\alpha[h]$ is p -valently convex of order $\lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k)$. $\square$

We note that by suitable functions $\phi \in \mathcal{A}_{p-1}$, $h_k \in \mathcal{A}_0$, and $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_+^n$ in Theorem 1, we obtain the earlier result. For example, if $p = 1$ and $\phi(z) = 1$, we obtain Theorem 2.1 in [44] and Theorems 2.1 and 2.2 in [48]. If $p \geq 1$ and $\phi(z) = z^{p-1}$, by using the Alexander type relation (7), Theorem 1 in [19] is obtained.

Using the same method and technique as that in Theorem 1 with the nonnegativity of modulus of complex numbers, we are led easily to Theorem 2. The proof is omitted.

Theorem 2. Let $\mathcal{T}(\phi) \in \mathcal{US}_p(\delta, \lambda)$ and $\mathcal{T}_p(h_k) \in \mathcal{US}_p(\delta_k, \gamma_k)$ for $k = 1, 2, \dots, n$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_+^n$ satisfies

$$\sum_{k=1}^n \alpha_k (p - \gamma_k) \leq \lambda, \quad (30)$$

then the integral operator $\mathcal{J}_\phi^\alpha[h]$ defined by (22) is p -valently convex of order $\lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k)$.

Theorem 2 generalizes many results proposed by several authors. For $p = 1$ and $\phi(z) = 1$, we obtain Theorem 2.1 in [26] and Theorem 2 in [49]. For $p \geq 1$ and $\phi(z) = z^{p-1}$ with the Alexander type relation (11), we obtain Theorem 2 in [46] and Theorem 2.1 in [31]. Also, Theorems 2.1 and 3.1 in [33] are obtained.

The following is a result on the transformation property of $\mathcal{J}_\phi^\alpha[h]$ on the class $\mathcal{S}^*(p, \gamma)$.

Theorem 3. Let $\mathcal{T}(\phi) \in \mathcal{S}^*(p, \lambda)$ and $\mathcal{T}_p(h_k) \in \mathcal{S}^*(p, \gamma_k)$ for $k = 1, 2, \dots, n$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}_+^n$ satisfies

$$\sum_{k=1}^n \alpha_k (p - \gamma_k) \leq \lambda, \quad (31)$$

then the integral operator $\mathcal{J}_\phi^\alpha[h]$ defined by (22) is in the class $\mathcal{K}(p, \lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k))$. Furthermore, $\mathcal{J}_\phi^\alpha[h]$ is p -valently convex of order $\lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k)$.

Proof. From (26), we obtain

$$\begin{aligned} \left| 1 + \frac{z(\mathcal{J}_\phi^\alpha[h])''(z)}{(\mathcal{J}_\phi^\alpha[h])'(z)} - p \right| &\leq \left| 1 + \frac{z\phi'(z)}{\phi(z)} - p \right| \\ &\quad + \sum_{k=1}^n \alpha_k \left| \frac{zh_k'(z)}{h_k(z)} \right|. \end{aligned} \quad (32)$$

Since $\mathcal{T}(\phi) \in \mathcal{S}^*(p, \lambda)$, it follows that

$$\begin{aligned} \left| \frac{z(\mathcal{T}(\phi))'(z)}{\mathcal{T}(\phi)(z)} - p \right| &= \left| \frac{z(z\phi)'(z)}{z\phi(z)} - p \right| \\ &= \left| 1 + \frac{z\phi'(z)}{\phi(z)} - p \right| < p - \lambda. \end{aligned} \quad (33)$$

Also, since $\mathcal{T}_p(h_k) \in \mathcal{S}^*(p, \gamma_k)$, we have

$$\begin{aligned} \left| \frac{z(\mathcal{T}_p(h_k))'(z)}{\mathcal{T}_p(h_k)(z)} - p \right| &= \left| \frac{z(z^p h_k)'(z)}{z^p h_k(z)} - p \right| \\ &= \left| \frac{zh_k'(z)}{h_k(z)} \right| < p - \gamma_k. \end{aligned} \quad (34)$$

By substitution of (33) and (34) into (32), we get

$$\left| 1 + \frac{z(\mathcal{J}_\phi^\alpha[h])''(z)}{(\mathcal{J}_\phi^\alpha[h])'(z)} - p \right| \leq p - \lambda + \sum_{k=1}^n \alpha_k (p - \gamma_k). \quad (35)$$

Therefore, $\mathcal{J}_\phi^\alpha[h] \in \mathcal{K}(p, \gamma - \sum_{k=1}^n \alpha_k (p - \beta_k))$. It follows that $\mathcal{J}_\phi^\alpha[h]$ is p -valently convex of order $\lambda - \sum_{k=1}^n \alpha_k (p - \gamma_k)$. $\square$

Remark 4. The parameter $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ in Theorem 3 can be extended to the complex number and assumption (31) becomes

$$\sum_{k=1}^n |\alpha_k| (p - \gamma_k) \leq \lambda. \quad (36)$$

That is, Theorem 3 can be applied to the integral operator $\mathcal{J}_\phi^\alpha[h]$ in case $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$.

Over the past few decades, there are many studies on the sufficient conditions that make the integral operators univalent. In fact, the class of convex functions is a subclass of the class of all univalent functions in $\mathbb{D}$. Thus, it is interesting to observe that many results on the univalence property of integral operators follow the convexity property according to main results, especially Theorem 3 or Remark 4.

We now consider the integral operator $H_n : \mathcal{A}^n \times \mathcal{A}^n \rightarrow \mathcal{A}$ defined in (17). In order to obtain the convexity of the integral operator H_n by Theorem 3 or Remark 4, we set $\phi(z) = 1$, $p = 1$, $\gamma_k = 0$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$, and

$$h_k(z) = \frac{(f_k * g_k)(z)}{z} \in \mathcal{A}_0, \quad k = 1, 2, \dots, n, \quad (37)$$

where $f_k, g_k \in \mathcal{A}$. Other than that, the univalent property of H_n is also obtained. This implies Theorem 3.1 of Frasin in [39]. Moreover, Frasin [39] noticed that for suitable functions $g_k \in \mathcal{A}$, the integral operator H_n generalizes many operators introduced by several authors, for instance, Theorem 1 in [20], Theorem 2.1 in [41], Theorem 2.3 in [42], and Theorem 2.3 in [43]. It is noteworthy to say that, under same assumptions, the former researches obtain only the univalence, while we obtain the stronger result, which is the convexity.

Our results can be used to explain the convexity of the other integral operators that are related to the Hadamad product as described next.

Remark 5. For $p \geq 1$, we set $\phi(z) = z^{p-1}$ and take

$$h_k(z) = \frac{(f_k * g_k)(z)}{z^p} \in \mathcal{A}_0, \quad k = 1, 2, \dots, n, \quad (38)$$

where $f_k, g_k \in \mathcal{A}_p$. Then, we can apply the main results to discuss the convexity of the integral operator $H_{p,n}$ defined by (19).

Remark 6. The main results are also applicable to the integral operator $G_{p,n} : \mathcal{A}_p^n \times \mathcal{A}_p^n \rightarrow \mathcal{A}_p$ of the following form:

$$G_{p,n}[f, g](z) = \int_0^z pt^{p-1} \prod_{k=1}^n \left(\frac{(f_k * g_k)'(t)}{pt^{p-1}} \right)^{\alpha_k} dt, \quad (39)$$

where $f_k, g_k \in \mathcal{A}_p$. In order to apply the main results, by the Alexander-type criterion, we note that

$$\begin{aligned} \mathcal{T}_p \left(\frac{(f_k * g_k)'(z)}{pz^{p-1}} \right) &\in \mathcal{S}_p^*(\gamma) \\ \iff \frac{z(f_k * g_k)'(z)}{p} &\in \mathcal{S}_p^*(\gamma) \\ \iff (f_k * g_k)(z) &\in \mathcal{K}_p(\gamma). \end{aligned} \quad (40)$$

The above statement also holds for the pairs of classes $\mathcal{US}_p(\beta, \gamma) - \mathcal{UK}_p(\beta, \gamma)$ and $\mathcal{S}(p, \gamma) - \mathcal{K}(p, \gamma)$.

Remark 7. For suitable functions $g_k \in \mathcal{A}_p$, by Remarks 5 and 6, we obtain new results for the convexity of other integral operators, for example, $\mathcal{F}_{p,\delta,l}^{n,m}$ and $\mathcal{G}_{p,\delta,l}^{n,m}$ in [45], $\mathcal{F}_{p,n,l,\delta}$ and $\mathcal{G}_{p,n,l,\delta}$ in [46], $\mathcal{F}_{p,m,l,\mu}$ and $\mathcal{G}_{p,m,l,\mu}$ in [50], and so forth.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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