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Research Article

Krasnosel'skii Type Hybrid Fixed Point Theorems and Their Applications to Fractional Integral Equations

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Some hybrid fixed point theorems of Krasnosel'skii type, which involve product of two operators, are proved in partially ordered normed linear spaces. These hybrid fixed point theorems are then applied to fractional integral equations for proving the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

1. Introduction

Recently, Nieto and Rodríguez-López [1] proved the following hybrid fixed point theorem for the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra and geometry.

Theorem 1 (Nieto and Rodríguez-López [1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping such that there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leqq kd(x, y) \tag{1}$$

for all comparable elements $x, y \in X$. Assume that either T is continuous or X is such that if $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$ in X, then

$$x_n \leqq x \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{2}$$

Further, if there is an element $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Another fixed point theorem in the above direction can be stated as follows.

Theorem 2 (Nieto and Rodríguez-López [1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping such that there exists a constant $k \in (0, 1)$ such that (1) satisfies for all comparable elements $x, y \in X$. Assume that either T is continuous or X is such that if $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$ in X, then*

$$x_n \geqq x \quad (n \in \mathbb{N}). \tag{3}$$

Further, if there is an element $x_0 \in X$ satisfying $x_0 \succeq Tx_0$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Remark 3. If we suppose that $d(a, c) \geqq d(b, c)$ $(a \leqq b \leqq c)$ and $\{x_n\} \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x implies the conditions (2) and (3), since (in the monotone case) the existence of a subsequence whose terms are comparable with the limit is equivalent to saying that all the terms in the sequence are also comparable with the limit.

Taking Remark 3 into account, the results discussed by Nieto and Rodríguez-López and the fact that, in conditions

$\{x_n\} \to x$, there is a sequence in X whose consecutive terms are comparable, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x implies the validity of the conditions (2) and (3). Here the key is that the terms in the sequence (starting at a certain term) are comparable to the limit. Nieto and Rodríguez-López [2] obtained the following results, which improve Theorems 1 and 2.

Theorem 4 (Nieto and Rodríguez-López [2]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone function (non-decreasing or non-increasing) such that there exists $k \in [0, 1)$ with*

$$d(T(x), T(y)) \leqq kd(x, y) \quad (x \geqq y). \qquad (1')$$

Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

After the publication of the above fixed point theorems, there is a huge upsurge in the development of the metric fixed point theory in partially ordered metric spaces. A good number of fixed and common fixed point theorems have been proved in the literature for two, three, and four mappings in metric spaces by suitably modifying the contraction condition (1) as per the requirement of the results. We claim that almost all the results proved so far along this line, though not mentioned here, have their origin in a paper due to Heikkilä and Lakshmikantham [3]. The main difference is the convergence criteria of the sequence of iterations of the monotone mappings under consideration. The convergence of the sequence in Heikkilä and Lakshmikantham [3] is straightforward, whereas the convergence of the sequence in Nieto and Rodríguez-López [1, 2] is due mainly to the metric condition of contraction. The hybrid fixed point theorem of Heikkilä and Lakshmikantham [3] for the monotone mappings in ordered metric spaces is as follows.

Theorem 5 (Heikkillä and Lakshmikantham [3]). *Let $[a, b]$ be an order interval in a subset Y of the ordered metric space X and let $G : [a, b] \to [a, b]$ be a non-decreasing mapping. If the sequence $\{Gx_n\}$ converges in Y whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the well-ordered chain of G-iterations of a has the maximum x^* which is a fixed point of G. Moreover,*

$$x^* = \max\{y \in [a, b] \mid y \leqq Gy\}. \qquad (4)$$

The above hybrid fixed point theorem is applicable in the study of discontinuous nonlinear equations and has been used throughout the research monograph of Heikkillä and Lakshmikantham [3]. We also claim that the convergence of the monotone sequence in Theorem 5 is replaced in Theorem 4 by the Cauchy sequence $\{x_n\}$ and completeness of X. Further, the Cauchy non-decreasing sequence is replaced by the equivalent contraction condition for comparable elements in X. Theorem 4 is the best hybrid fixed point theorem because it is derived for the mixed arguments from algebra and geometry. The main advantage of Theorem 4 is that the uniqueness of the fixed point of the monotone mappings is obtained under certain additional conditions on the domain space such as lattice structure of the partially ordered space under consideration and these fixed point results are useful in establishing the uniqueness of the solution of nonlinear differential and integral equations. Again, some hybrid fixed point theorems of Krasnosel'skii type for monotone mappings are proved in Dhage [4, 5] along the lines of Heikkilä and Lakshmikantham [3].

The main object of this paper is first to establish some hybrid fixed point theorems of Krasnosel'skii type in partially ordered normed linear spaces, which involve product of two operators. We then apply these hybrid fixed point theorems to fractional integral equations for proving the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

2. Hybrid Fixed Point Theorems

Let X be a linear space or vector space. We introduce a partial order $\preceq$ in X as follows. A relation $\preceq$ in X is said to be a partial order if it satisfies the following properties:

(1) reflexivity: $a \preceq a$ for all $a \in X$;

(2) antisymmetry: $a \preceq b$ and $b \preceq a$ implies $a = b$;

(3) transitivity: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$;

(4) order linearity: $x_1 \preceq y_1$ and $x_2 \preceq y_2 \Rightarrow x_1 + x_2 \preceq y_1 + y_2$; and $x \preceq y \Rightarrow tx \preceq ty$ for $t \geqq 0$.

The linear space X together with a partial order $\preceq$ becomes a partially ordered linear or vector space. Two elements x and y in a partially ordered linear space X are called comparable if the relation either $x \preceq y$ or $y \preceq x$ holds true. We introduce a norm $\|\cdot\|$ in partially ordered linear space X so that X becomes now a partially ordered normed linear space. If X is complete with respect to the metric d defined through the above norm, then it is called a partially ordered complete normed linear space.

The following definitions are frequently used in our present investigation.

Definition 6. A mapping $T : X \to X$ is called monotone non-decreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

Definition 7. A mapping $T : X \to X$ is called monotone non-increasing if $x \preceq y$ implies $Tx \succeq Ty$ for all $x, y \in X$.

Definition 8. A mapping $T : X \to X$ is called monotone if it is either monotone non-increasing or monotone non-decreasing.

Definition 9 (see [6, 7]). A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a monotone dominating function or, in short, an M-function if it is an upper or lower semicontinuous and monotonic

non-decreasing or non-increasing function satisfying the condition: $\varphi(0) = 0$.

Definition 10 (see [6, 7]). Given a partially ordered normed linear space E, a mapping $Q : E \to E$ is called partially M-Lipschitz or partially nonlinear M-Lipschitz if there is an M-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Qx - Qy\| \leqq \varphi(\|x - y\|) \tag{5}$$

for all comparable elements $x, y \in E$. The function is called an M-function of Q on E. If $\varphi(r) = kr$ $(k > 0)$, then Q is called partially M-Lipschitz with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a partially M-contraction on X with the contraction constant k. Further, if $\varphi(r) < r$, for $r > 0$, then Q is called a partially nonlinear M-contraction with an M-function φ of Q on X.

There do exist M-functions and the commonly used M-functions are $\varphi(r) = kr$ and $\varphi(r) = r/1+r$, *et cetera*. These M-functions can be used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods.

Definition 11 (see [8]). An operator Q on a normed linear space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if, for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Definition 12 (see [8]). An operator Q on a normed linear space E into itself is called partially compact if $Q(C)$ is a relatively compact subset of E for all totally ordered set or chain C in E. The operator Q is called partially totally bounded if, for any totally ordered and bounded subset C of E, $Q(C)$ is a relatively compact subset of E. If the operator Q is continuous and partially totally bounded, then it is called partially completely continuous on E.

Remark 13. We note that every compact mapping in a partially metric space is partially compact and every partially compact mapping is partially totally bounded. However, the reverse implication does not hold true. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is continuous and partially totally bounded, but the converse may not be true.

We now state and prove the basic hybrid fixed point results of this paper by using the argument from algebra, analysis, and geometry. The slight generalization of Theorem 4 and Dhage [8] using M-contraction is stated as follows.

Theorem 14. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone function (non-decreasing or non-increasing) such that there exists an M-function φ_T such that*

$$d(T(x), T(y)) \leqq \varphi_T(d(x, y)) \tag{6}$$

for all comparable elements $x, y \in X$ and satisfying $\varphi_T(r) < r$ $(r > 0)$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Proof. The proof is standard. Nevertheless, for the sake of completeness, we give the details involved. Define a sequence $\{x_n\}$ of successive iterations of T by

$$x_{n+1} = Tx_n \quad (n \in \mathbb{N}). \tag{7}$$

By the monotonicity property of T, we obtain

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \cdots \tag{8}$$

or

$$x_0 \succeq x_1 \succeq \cdots \succeq x_n \cdots. \tag{9}$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $u = x_n$ is a fixed point of T. Therefore, we assume that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. If $x = x_{n-1}$ and $y = x_n$, then, by the condition (6), we obtain

$$d(x_n, x_{n+1}) \leqq \varphi(d(x_{n-1}, x_n)) \tag{10}$$

for each $n \in \mathbb{N}$.

Let us write $r_n = d(x_n, x_{n+1})$. Since φ is an M-function, $\{r_n\}$ is a monotonic sequence of real numbers which is bounded. Hence $\{r_n\}$ is convergent and there exists a real number r such that

$$\lim_{n\to\infty} r_n = d(x_n, x_{n+1}) = r. \tag{11}$$

We show that $r = 0$. If $r \neq 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} r_n = \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &\leqq \lim_{n\to\infty} \varphi(d(x_{n-1}, x_n)) \leqq \varphi(r) < r, \end{aligned} \tag{12}$$

which is a contradiction. Hence $r = 0$.

We now show that $\{x_n\}$ is a Cauchy sequence in X. If not, then, for $\epsilon > 0$, there exists a positive integer k such that

$$d(x_{m(k)}, x_{n(k)}) \geqq \epsilon \tag{13}$$

for all positive integers $m(k) \geqq n(k) \geqq k$.

If we write $r_k = d(x_{m(k)}, x_{n(k)})$, then

$$\begin{aligned} \epsilon \leqq r_k &= d(x_{m(k)}, x_{n(k)}) \\ &\leqq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &= r_{m(k)-1} + \epsilon, \end{aligned} \tag{14}$$

so that we have

$$\lim_{k\to\infty} r_k = \epsilon. \tag{15}$$

Again, we have

$$\begin{aligned}\epsilon \leqq r_k &= d\left(x_{m(k)}, x_{n(k)}\right)\\ &\leqq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right)\\ &\quad + d\left(x_{n(k)+1}, x_{n(k)}\right) = r_{m(k)} + \varphi\left(r_k\right) + r_{n(k)}.\end{aligned} \tag{16}$$

Taking the limit as $k \to \infty$, we obtain

$$\epsilon \leqq \varphi(\epsilon) < \epsilon, \tag{17}$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in X. By the metric space (X, d) being complete, there is a point $x^* \in X$ such that $\lim_{n\to 0} x_n = x^*$. The rest of the proof is similar to above fixed point Theorem 4 given in Nieto and Rodríguez-López [2]. Hence we omit the details involved. □

Corollary 15. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone function (non-decreasing or non-increasing) such that there exists an M-function φ and a positive integer p such that*

$$d\left(T^p(x), T^p(y)\right) \leqq \varphi_T\left(d(x, y)\right) \tag{18}$$

for all comparable elements $x, y \in X$ and satisfying $\varphi_T(r) < r$ $(r > 0)$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Proof. Let us first set $Q = T^p$. Then $Q : X \to X$ is a continuous monotonic mapping. Also there exists the element $x_0 \in X$ such that $x_0 \preceq Qx_0$. Now, an application of Theorem 14 yields that Q has an unique fixed point; that is, it is a point $u \in X$ such that $Q(u) = T^p(u) = u$. Now $T(u) = T(T^p u) = Q(Tu)$, showing that Tu is again a fixed point of Q. By the uniqueness of u, we get $Tu = u$. The proof is complete. □

Fixed point Theorem 14 and Corollary 15 have some nice applications to various nonlinear problems modelled on nonlinear equations for proving existence as well as uniqueness of the solutions under generalized Lipschitz condition. The following fixed point theorem is presumably new in the literature. The basic principle in formulating this theorem is the same as that of Dhage [5, 8] and Nieto and Rodríguez-López [2]. Before stating these results, we give an useful definition.

Definition 16. The order relation $\preceq$ and the norm $\|\cdot\|$ in a nonempty set X are said to be compatible if $\{x_n\}$ is a monotone sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x_0 impling that the whole sequence $\{x_n\}$ converges to x_0. Similaraly, given a partially ordered normed linear space $(X, \preceq, \|\cdot\|)$, the ordered relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm are compatible.

Clearly, the set $\mathbb{R}$ with the usual order relation $\leqq$ and the norm defined by absolute value function has this property. Similarly, the space $C(J, \mathbb{R})$ with usual order relation defined by $x \leqq y$ if and only if $x(t) \leqq y(t)$ for all $t \in J$ or $x \leqq y$ if and only if $x(t) \geqq y(t)$ for all $t \in J$ and the usual standard supremum norm $\|\cdot\|$ are compatible.

We now state a more basic hybrid fixed point theorem. Since the proof is straightforword, we omit the details involved.

Theorem 17. *Let X be a partially ordered linear space and suppose that there is a norm in X such that X is a normed linear space. Let $T : X \to X$ be a monotonic (non-decreasing or non-increasing), partially compact and continuous mapping. Further, if the order relation $\preceq$ or $\succeq$ and the norm $\|\cdot\|$ in X are compatible and if there is an element $x_0 \in X$ satisfying $x_0 \leqq Tx_0$ or $x_0 \geqq Tx_0$, then T has a fixed point.*

In this paper, we combine Theorems 14 and 17 and Corollary 15 to derive some Krasnosel'skii type fixed point theorems in partially ordered complete normed linear spaces and discuss some of their applications to fractional integral equations of mixed type. We freely use the conventions and notations for fractional integrals as in (for example) [9–11].

3. Krasnosel'skii Type Fixed Point Theorems

We first state the following result.

Theorem 18 (see Krasnosel'skii [12]). *Let S be a closed convex and bounded subset of a Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators satisfying the following conditions:*

(a) *A is nonlinear contraction;*

(b) *B is completely continuous;*

(c) *$Ax + By = x$ for all $y \in S$ implies $x \in S$.*

Then the following operator equation

$$Ax + Bx = x \tag{19}$$

has a solution.

Theorem 18 is very much useful and applied to linear perturbations of differential and integral equations by several authors in the literature for proving the existence of the solutions. The theory of Krasnosel'skii type fixed point theorem is initiated by Dhage [5]. The following Krasosel'skii type fixed point theorem is proved in Dhage [5].

Theorem 19 (see Dhage [5]). *Let S be a nonempty, closed, convex, and bounded subset of the Banach algebra X. Also let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is D-Lipschitz with the D-function ψ;*

(b) *B is completely continuous;*

(c) $x = AxBy \Rightarrow x \in S$ *for all* $y \in S$;

$M\psi(r) < r$, $r > 0$ *where*

$$M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}. \tag{20}$$

Then the operator equation $AxBx = x$ *has a solution in* S.

Remark 20. $(I/A)^{-1}B$ is monotone (non-decreasing or non-increasing) if A and B are monotone (non-decreasing or non-increasing), but the converse may not be true.

We now obtain another version of Krasnosel'skii type fixed point theorems in partially ordered complete normed linear spaces under weaker conditions, which improve Theorem 19, and discuss some of their applications to fractional integral equations of mixed type.

Theorem 21. *Let* $(X, \preceq, \|\cdot\|)$ *be a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A, B : X \to X$ *be two monotone operators (non-decreasing or non-increasing) such that*

(a) *A is continuous and partially nonlinear M-contraction;*

(b) *B is continuous and partially compact;*

(c) *there exists an element* $x_0 \in X$ *such that* $x_0 \preceq Ax_0By$ *or* $x_0 \succeq Ax_0By$ *for all* $y \in X$;

(d) *every pair of elements* $x, y \in X$ *has a lower and an upper bound in* X;

(e) $K\varphi(r) < r$, $r > 0$ *where*

$$K = \|B(X)\| = \sup\{\|Bx\| : x \in X\}. \tag{21}$$

Then the operator equation $AxBx = x$ *has a solution.*

Proof. Define an operator $T : X \to X$ by

$$T(x) = \left(\frac{I}{A}\right)^{-1} B. \tag{22}$$

Clearly, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy. \tag{23}$$

Now, for any two comparable elements $x_1, x_2 \in X$, we have

$$\begin{aligned} &\|A_y(x_1) - A_y(x_2)\| \\ &\quad = \|Ax_1By - Ax_2By\| \leqq \|Ax_1 - Ax_2\| \cdot \|By\| \\ &\quad \leqq K\varphi_A(\|x_1 - x_2\|), \end{aligned} \tag{24}$$

where A is an M-function of T on X. Hence, by an application of fixed point Theorem 14, A_y has an unique fixed point; say $x^* \in X$. Therefore, we have an unique element $x^* \in X$ such that

$$A_y(x^*) = Ax^*By = x^*, \tag{25}$$

which implies that

$$\left(\frac{I}{A}\right)^{-1} By = x^* \tag{26}$$

or, equivalently, that

$$Ty = x^*. \tag{27}$$

Thus the mapping $T : X \to X$ is well defined.

We now define a sequence $\{x_n\}$ of iterates of T; that is, $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. It follows from the hypothesis (c) that $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$. Again, by Remark 20, we find that the mapping T is monotonic (non-decreasing or non-increasing) on X. So we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq \cdots \tag{28}$$

or

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots x_n \succeq \cdots. \tag{29}$$

Since B is partially compact and $(I/A)^{-1}$ is continuous, the composition mapping $T = (I/A)^{-1}B$ is partially compact and continuous on X into X. Therefore, the sequence $\{x_n\}$ has a convergent subsequence and, from the compatibility of the order relation and the norm, it follows that the whole sequence converges to a point in X. Hence, an application of Theorem 17 implies that T has a fixed point. This further implies that

$$\left(\frac{I}{A}\right)^{-1} Bx^* = x^* \quad \text{or} \quad Ax^*Bx^* = x^*, \tag{30}$$

which evidently completes the proof of Theorem 21. □

Theorem 22. *Let* $(X, \preceq, \|\cdot\|)$ *be a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A, B : X \to X$ *be two monotone mappings (non-decreasing or non-increasing) satisfying the following conditions:*

(a) *A is linear and bounded and A^p is partially nonlinear M-contraction for some positive integer p;*

(b) *B is continuous and partially compact;*

(c) *there exists an element* $x_0 \in X$ *such that* $x_0 \preceq Ax_0By$ *or* $x_0 \succeq Ax_0By$ *for all* $y \in X$;

(d) *every pair of elements* $x, y \in X$ *has a lower and an upper bound in* X;

(e) $K\varphi(r) < r$, $r > 0$ *where*

$$K = \|B(X)\| = \sup\{\|Bx\| : x \in X\}. \tag{31}$$

Then the operator equation $AxBx = x$ *has a solution.*

Proof. Define an operator T on X by

$$T(x) = \left(\frac{I}{A}\right)^{-1} B. \tag{32}$$

Now the mapping $(I/A)^{-1}$ exists in view of the relation

$$\left(\frac{I}{A}\right)^{-1} = \left(\frac{I}{A^p}\right)^{-1} \prod_{j=1}^{p-1} A^j, \tag{33}$$

where $\prod_{j=1}^{p-1} A^j$ is bounded and $(I/A^p)^{-1}$ exists in view of Corollary 15. Hence, $(I/A)^{-1}$ exists and is continuous on X. Next, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy. \tag{34}$$

Then, for any two comparable elements $x, y \in X$, we have

$$\begin{aligned} &\left\|A_y^p(x_1) - A_y^p(x_2)\right\| \\ &\quad = \left\|A^p x_1 By - A^p x_2 By\right\| \leqq \left\|A^p x_1 - A^p x_2\right\| \cdot \|By\| \\ &\quad \leqq K\varphi_A\left(\|x_1 - x_2\|\right). \end{aligned} \tag{35}$$

Hence, by Corollary 15 again, there exists an unique element x^* such that

$$A_y^p(x^*) = A_p(x^*)By = x^*. \tag{36}$$

This further implies that $A_y(x^*) = x^*$ and x^* is an unique fixed point of A_y. Thus we have

$$A_y(x^*) = x^* = Ax^*By \quad \text{or} \quad \left(\frac{I}{A}\right)^{-1} By = x^*. \tag{37}$$

Consequently, $Ty = x^*$ and so T is well defined. The rest of the proof is similar to that of Theorem 21 and we omit the details. The proof is complete. □

Remark 23. The hypothesis (d) of Theorems 21 and 22 holds true if the partially ordered set X is a lattice. Furthermore, the space $C(J, \mathbb{R})$ of continuous real-valued functions on the closed and bounded interval $J = [a, b]$ is a lattice, where the order relation $\leqq$ is defined as follows. For any $x, y \in C(J, \mathbb{R})$, $x \leqq y$ if and only if $x(t) \leqq y(t)$ for all $t \in J$. The real-variable operations show that $\min(x, y)$ and $\max(x, y)$ are, respectively, the lower and upper bounds for the pair of elements x and y in X.

4. Fractional Integral Equations of Mixed Type

In this section we apply the hybrid fixed point theorems proved in the preceding sections to some nonlinear fractional integral equations of mixed type.

Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$, $\mathbb{R}$ being the set of real numbers or some real numbers $t_0 \in \mathbb{R}$ and $a \in \mathbb{R}$ with $a > 0$ and given a real number $0 < q < 1$, consider the following nonlinear hybrid fractional integral equation (in short HFIE):

$$x(t) = [f(t, x(t))]\left(\frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))\, ds\right), \tag{38}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : J \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous.

We seek the solutions of HFIE (38) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We consider the following set of hypotheses in what follows.

(H_1) g is bounded on $J \times \mathbb{R}$ with bound C_g.

(H_2) $g(t, x)$ is non-decreasing in x for each $t \in J$.

(H_3) There exist constants $L > 0$ and $K > 0$ such that

$$0 \leqq \left(f(t, x) - f(t, y)\right) \leqq \frac{L(x-y)}{K + (x-y)} \tag{39}$$

for all $x, y \in \mathbb{R}$ with $x \geqq y$. Moreover, $L \leqq K$.

(H_4) There exists an element $u_0 \in X = C(J, \mathbb{R})$ such that

$$u_0(t) \leqq [f(t, u_0(t))] \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, y(s))\, ds \tag{40}$$

for all $t \in J$ and $y \in X$ or

$$u_0(t) \geqq [f(t, u_0(t))] \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, y(s))\, ds \tag{41}$$

for all $t \in J$ and $y \in X$.

Remark 24. The condition given in the hypothesis (H_4) is a little more restrictive than that of a lower solution of the HFIE (38). It is clear that u_0 is a lower solution of the HFIE (38); however, the converse is not true.

Theorem 25. *Assume that the hypotheses* (H_1) *through* (H_4) *hold true. Then the HFIE* (38) *admits a solution.*

Proof. Define two operators A and B on $X = C(J, \mathbb{R})$, the Banach space of continuous real-valued functions on J with the usual supremum norm $\|\cdot\|$ given by

$$\|x\| = \sup_{t \in J} |x(t)|. \tag{42}$$

We define an order relation $\leqq$ in X with help of a cone $\mathscr{K}$ defined by

$$\mathscr{K} = \{x : x \in C(J, \mathbb{R}), x(t) \geqq 0 \ (\forall t \in J)\}. \tag{43}$$

Clearly, the Banach space X together with this order relation becomes an ordered Banach space. Furthermore, the order relation $\leqq$ and the norm $\|\cdot\|$ in X are compatible. Define two operators $A, B : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$\begin{aligned} Ax(t) &= f(t, x(t)) \quad (t \in J), \\ Bx(t) &= \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))\, ds. \end{aligned} \tag{44}$$

Then the given Hybrid fractional integral equation (38) is transformed into an equivalent operator equation as follows:

$$Ax(t) \cdot Bx(t) = x(t) \quad (t \in J). \tag{45}$$

We show that the operators A and B satisfy all the conditions of Theorem 21 on $C(J,\mathbb{R})$. First of all, we show that A is a nonlinear M-contraction on $C(J,\mathbb{R})$. Let $x, y \in X$. Then, by the hypothesis (H_3), we obtain

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leqq \frac{L|x(t)-y(t)|}{K+|x(t)-y(t)|} \\ &\leqq \frac{L\|x-y\|}{K+\|x-y\|}. \end{aligned} \tag{46}$$

Taking the supremum over t, we get

$$\|Ax - Ay\| \leqq \frac{L\|x-y\|}{K+\|x-y\|} = \varphi(\|x-y\|), \tag{47}$$

where

$$\varphi(r) = \frac{Lr}{K+r} < r \quad (r > 0). \tag{48}$$

Clearly, φ is an M-function for the operator A on X and so A is a partially nonlinear M-contraction on X.

Next, we show that B is a compact continuous operator on X. To this end, we show that $B(X)$ is a uniformly bounded and equicontinuous set in X. Now, for any $x \in X$, we have

$$\begin{aligned} |Bx(t)| &\leqq \frac{1}{\Gamma(q)}\int_{t_0}^{t} |t-s|^{q-1} |g(s,x(s))|\, ds \\ &\leqq \frac{C_g}{\Gamma(q)}\int_{t_0}^{t} |t-s|^{q-1} ds \leqq \frac{a^q C_g}{\Gamma(q+1)}, \end{aligned} \tag{49}$$

which shows that B is a uniformly bounded set in X. We now let $t_1, t_2 \in J$. Then

$$\begin{aligned} &|Bx(t_1) - Bx(t_2)| \\ &\leqq \frac{C_g}{\Gamma(q)}\int_{t_0}^{t_2} \left|(t_1-s)^{q-1} - (t_2-s)^{q-1}\right| ds + \frac{C_g}{\Gamma(q+1)}|t_1-t_2|^q \\ &\leqq \frac{C_g}{\Gamma(q)}\int_{t_0}^{t_0+a} \left|(t_1-s)^{q-1} - (t_2-s)^{q-1}\right| ds + \frac{C_g}{\Gamma(q+1)}|t_1-t_2|^q \\ &\longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2 \end{aligned} \tag{50}$$

uniformly for all $x \in X$. Hence $B(X)$ is an equicontinuous set in X. Now we apply the Arzela-Ascoli theorem to show that $B(X)$ is a compact set in X. The continuity of B follows from the continuity of the function g on $J \times \mathbb{R}$.

Finally, since $f(t,x)$ and $g(t,x)$ are non-decreasing in x for each $t \in J$, the operators A and B are non-decreasing on X. Also the hypothesis (H_3) yields $u_0 \leqq Au_0 \cdot Bu_0$. Thus, all of the conditions of Theorem 22 are satisfied and we conclude that the fractional integral equation (38) admits a solution. This completes the proof. □

We now consider the following fractional integral equation of mixed type:

$$\begin{aligned} x(t) = &\left[\int_{t_0}^{t} v(t,s) f(s,x(s))\, ds\right] \\ &\times \left(q(t) + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\, ds\right) \end{aligned} \tag{51}$$

for all $t \in J$ and $0 < q < 1$, where the functions $v : J \times J \to \mathbb{R}^+$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

We consider the following set of hypotheses in what follows.

(H_5) The function $v : J \times J \to \mathbb{R}^+$ is continuous. Moreover, $v = \sup_{t,s\in J}|v(t,s)|$.

(H_6) $f(t,x)$ is linear in x for each $t \in J$.

(H_7) f is bounded on $J \times \mathbb{R}$ and there exists a constant $L > 0$ such that $f(t,x) < L|x|$ for all $t \in J$ and $x \in \mathbb{R}$.

(H_8) There exists an element $u_0 \in X = C(J,\mathbb{R})$ such that

$$\begin{aligned} u_0(t) \leqq &\left[\int_{t_0}^{t} v(t,s) f(s,u_0(s))\, ds\right] \\ &\times \left(q(t) + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} g(s,y(s))\, ds\right) \end{aligned} \tag{52}$$

or

$$\begin{aligned} u_0(t) \geqq &\left[\int_{t_0}^{t} v(t,s) f(s,u_0(s))\, ds\right] \\ &\times \left(q(t) + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} g(s,y(s))\, ds\right) \end{aligned} \tag{53}$$

for all $t \in J$ and $0 < q < 1$, where the functions $v : J \times J \to \mathbb{R}^+$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

Remark 26. The condition given in hypothesis (H_7) is a little more restrictive than that of a lower solution for the HFIE (51) defined on J.

Theorem 27. *Assume that the hypotheses (H_1), (H_2), and (H_5) through (H_8) hold true. Then the HFIE (51) admits a solution.*

Proof. Set $X = C(J,\mathbb{R})$ and define an order relation $\leqq$ with the help of the cone $\mathscr{K}$ defined by (43). Clearly, $C(J,\mathbb{R})$ is a lattice with respect to the above order relation $\leqq$ in it. Define two operators A and B on X by

$$\begin{aligned} Ax(t) &= \int_{t_0}^{t} v(t,s) f(s,x(s))\, ds \quad (t \in J), \\ Bx(t) &= q(t) + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\, ds \quad (t \in J). \end{aligned} \tag{54}$$

Clearly, the operator A is linear and bounded in view of the hypotheses (H_1), (H_6), and (H_7). We only show that the operator A^n is partially M-contraction on X for every positive integer n. Let $x, y \in X$ be such that $x \geqq y$. Then, by (H_6) and (H_7), we have

$$\begin{aligned} &|Ax(t) - Ay(t)| \\ &\quad \leqq \int_{t_0}^{t} |V| \cdot |f(s, x(s)) - f(s, y(s))| \, ds \\ &\quad \leqq V \int_{t_0}^{t_0+a} L |x(s) - y(s)| \, ds \leqq LVa \|x - y\|, \end{aligned} \tag{55}$$

where $|V|$ is the supremum of $v(t, s)$ over t. Thus, by taking the supremum over t, we obtain

$$\|Ax - Ay\| \leqq LVa \|x - y\|. \tag{56}$$

Similarly, it can be proved that

$$\begin{aligned} \|A^2x - A^2y\| &= |A(Ax(t)) - A(Ay(t))| \\ &\leqq LV \int_{t_0}^{t_0+a} \left(\int_{t_0}^{t} |Ax(s) - Ay(s)| \, ds \right) ds \\ &\leqq \frac{L^2V^2a^2}{2!} \|x - y\|. \end{aligned} \tag{57}$$

In general, proceeding in the same way, for any positive integer n, we have

$$\|A^n x - A^n y\| \leqq \frac{L^n V^n a^n}{n!} \|x - y\|. \tag{58}$$

Therefore, for large n, A^n is partially a nonlinear M-contraction mapping on X. The rest of the proof is similar to that of Theorem 25. The desired result now follows by an application of Theorem 22. This completes the proof. □

5. An Illustrative Example

Example 1. Consider a distributed-order fractional hybrid differential equation (DOFHDES) involving the Reimann-Liouville derivative operator of order $0 < q < 1$ with respect to the negative density function $b(q) > 0$ as follows:

$$\begin{gathered} \int_0^1 b(q) D^q \left[\frac{x(t)}{f(t, x(t))} \right] dq = g(t, x(t)) \quad (t \in J), \\ \int_0^1 b(q) \, dq = 1, \\ x(0) = 0. \end{gathered} \tag{59}$$

Moreover, the function $t \to x/f(t, x)$ is continuous for each $x \in \mathbb{R}$, where $J = [0, T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J, \mathbb{R})$. It is well known that the DOFHDES (59) is equivalent to the following integral equation:

$$\begin{aligned} &x(t) \\ &\quad = \frac{f(t, x(t))}{\pi} \int_0^t L \left\{ S \left\{ \frac{1}{B(re^{-i\pi})} \right\}; t - \tau \right\} g(\tau, x(\tau)) \, d\tau \end{aligned} \tag{60}$$

such that $0 \leqq \tau \leqq t \leqq T$ and

$$B(s) = \int_0^1 b(q) s^q dq. \tag{61}$$

The integral equation (60) is valid for all $x \in C(J, \mathbb{R})$. Hence, if Theorem 25 holds true then we further have

$$\frac{LM|h|_{L'}}{\pi} < 1 \quad (M > 0), \tag{62}$$

then the above-mentioned DOFHDES (59) has a solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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