

## Research Article

# Existence and Uniqueness of Solution for Perturbed Nonautonomous Systems with Nonuniform Exponential Dichotomy

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Nonuniform exponential dichotomy has been investigated extensively. The essential condition of these previous results is based on the assumption that the nonlinear term satisfies  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$ . However, this condition is very restricted. There are few functions satisfying  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$ . In some sense, this assumption is not reasonable enough. More suitable assumption should be  $|f(t, x)| \leq \mu$ . To the best of the authors' knowledge, there is no paper considering the existence and uniqueness of solution to the perturbed nonautonomous system with a relatively conservative assumption  $|f(t, x)| \leq \mu$ . In this paper, we prove that if the nonlinear term is bounded, the perturbed nonautonomous system with nonuniform exponential dichotomy has a unique solution. The technique employed to prove Theorem 4 is the highlight of this paper.

## 1. Introduction

The notion of exponential dichotomy, introduced by Perron in [1], plays an important role in the theory of differential equations and dynamical systems (also see [2–5]). It is well known that if linear system  $\dot{x}(t) = A(t)x(t)$  admits an (uniform) exponential dichotomy, the nonlinear term  $f(t, x)$  is bounded and has a small Lipschitz constant, then the nonlinear system  $\dot{x}(t) = A(t)x(t) + f(t, x)$  has a unique bounded solution (see [6]). However, many scholars argued that (uniform) exponential dichotomy restricted the behavior of dynamical systems. For this reason, we need a more general concept of hyperbolicity. Recently, Barreira and Valls [7, 8] have introduced the notion of nonuniform exponential dichotomy. General nonuniform exponential dichotomy has also been proposed (see [9–11]). Many properties of nonuniform exponential dichotomy have been extensively studied. For example, the topological conjugacies between linear

and nonlinear perturbations were explored and some new Grobman-Hartman type theorems for nonuniform exponential dichotomy were established ([12, 13]). However, the essential condition of these results is based on the assumption that the nonlinear term satisfies  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$ . Under the same condition, Zhang et al. studied nonlinear perturbations of nonuniform exponential dichotomy on measure chains ([14]).

However, the condition  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$  is very restricted. There are few functions satisfying  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$ . Thus, it is necessary to find a more conservative condition for the nonlinear term  $f(t, x)$ . In this paper, our main objective is to explore the existence and uniqueness of solution to the perturbed nonautonomous system with a relatively conservative assumption  $|f(t, x)| \leq \mu$ . Finally, we prove that if  $|f(t, x)| \leq \mu$ , the perturbed nonautonomous system with nonuniform exponential dichotomy has a unique solution  $x(t)$  satisfying  $|x(t)| = O(e^{\varepsilon|t|})$ .

The outline of this paper is arranged as follows. Next section is to state our main results. In Section 3, we prove the main results.

### 2. Main Results

In this section, we will state our main theorems. First, we introduce the definition of nonuniform exponential dichotomy.

Consider systems

$$\dot{x}(t) = A(t)x(t), \tag{1}$$

$$\dot{x}(t) = A(t)x(t) + f(t, x), \tag{2}$$

where  $t \in \mathbb{R}, x \in \mathbb{R}^n, A(t)$  is a continuous matrix function, and  $f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function.

Let  $T_A(t, s)$  be the evolution operator satisfying  $x(t) = T_A(t, s)x(s), t, s \in \mathbb{R}$ , where  $x(t)$  is a solution of (1).

*Definition 1.* Linear system (1) is said to admit a nonuniform exponential dichotomy if there exists a projection  $P(t)$  ( $P^2 = P$ ) and constants  $\alpha > 0, K > 0, \varepsilon \geq 0$ , such that

$$\begin{aligned} |T_A(t, s)P(s)| &\leq Ke^{-\alpha(t-s)} \cdot e^{\varepsilon|s|}, & t \geq s, \\ |T_A(t, s)Q(s)| &\leq Ke^{\alpha(t-s)} \cdot e^{\varepsilon|s|}, & t \leq s, \end{aligned} \tag{3}$$

where  $P(t) + Q(t) = \text{Id}$  (identity),  $T_A(t, s)P(s) = P(t)T_A(t, s), t, s \in \mathbb{R}$ .

*Remark 2.* When  $\varepsilon \equiv 0$ , system (1) is said to have an exponential dichotomy; and when  $\varepsilon \equiv 0, \alpha \equiv 0$ , system (1) is said to have a uniform dichotomy.

To present our main results, we give a theorem under the trivial condition  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$ .

**Theorem 3.** *Suppose that linear system (1) admits a nonuniform exponential dichotomy. For  $t \in \mathbb{R}, x, x_1, x_2 \in \mathbb{R}^n$ , if the nonlinear term  $f(t, x)$  satisfies*

$$\begin{aligned} (\widetilde{H}_1) \quad &|f(t, x)| \leq \mu e^{-\varepsilon|t|}, \\ (\widetilde{H}_2) \quad &|f(t, x_1) - f(t, x_2)| \leq re^{-\varepsilon|t|}|x_1 - x_2|, \\ (\widetilde{H}_3) \quad &4Kr < \alpha, \end{aligned}$$

where  $\mu, r, \varepsilon, \alpha$  are all positive constants, then nonlinear system (2) has a unique bounded solution  $\tilde{x}(t)$  satisfying

$$\begin{aligned} \tilde{x}(t) &= \int_{-\infty}^t T_A(t, s)P(s)f(s, \tilde{x}(s))ds \\ &\quad - \int_t^{+\infty} T_A(t, s)Q(s)f(s, \tilde{x}(s))ds. \end{aligned} \tag{4}$$

*Discussion.* One of the essential conditions of Theorem 3 is  $(\widetilde{H}_1)$ . However, this condition is very restricted. There are few functions satisfying  $|f(t, x)| \leq \mu e^{-\varepsilon|t|}$ . Thus, it is necessary to find a more conservative condition for the nonlinear term

$f(t, x)$ . The main objective of this paper is to prove that the perturbed system has a unique solution under  $|f(t, x)| \leq \mu$ . But Theorem 3 cannot be valid yet. For this case, we have the following.

**Theorem 4.** *Suppose that linear system (1) admits a nonuniform exponential dichotomy with the estimates (3). For  $t \in \mathbb{R}, x, x_1, x_2 \in \mathbb{R}^n$ , if  $f(t, x)$  satisfies*

$$\begin{aligned} (H_1) \quad &|f(t, x)| \leq \mu, \\ (H_2) \quad &|f(t, x_1) - f(t, x_2)| \leq re^{-\varepsilon|t|}|x_1 - x_2|, \\ (H_3) \quad &4Kr < \alpha - \varepsilon, \end{aligned}$$

where  $\alpha - \varepsilon$  is a positive constant, then system (2) has a unique solution  $x(t)$  satisfying

$$|x(t)| = O(e^{\varepsilon|t|}). \tag{5}$$

*Remark 5.* The method used to prove Theorem 3 cannot be applied to this case. To see how to overcome the difficulty, one can refer to the main proof of Theorem 4. The technique employed to prove Theorem 4 is very skillful and interesting, which is the highlight of this paper.

### 3. Proofs of Main Results

In what follows, to prove Theorem 3, a preliminary lemma is needed.

**Lemma 6** (see [15], Lemma 4). *If system (1) admits a nonuniform exponential dichotomy, then system (1) has no nontrivial bounded solutions; that is,  $x(t) = 0$  is the unique bounded solution of (1).*

*3.1. Proof of Theorem 3.* Let  $\mathbf{B} = \{\varphi(t) \mid \varphi(t) \text{ be continuous and } |\varphi(t)| \leq 2K\mu\alpha^{-1}\}$ , for  $\forall \varphi \in \mathbf{B}$ , define a mapping  $\mathcal{T}_1$ :

$$\begin{aligned} \mathcal{T}_1\varphi(t) &= \int_{-\infty}^t T_A(t, s)P(s)f(s, \varphi(s))ds \\ &\quad - \int_t^{+\infty} T_A(t, s)Q(s)f(s, \varphi(s))ds. \end{aligned} \tag{6}$$

From (3) and  $(\widetilde{H}_1)$  and  $(\widetilde{H}_2)$ , we have

$$\begin{aligned} |\mathcal{T}_1\varphi(t)| &\leq \int_{-\infty}^t Ke^{-\alpha(t-s)}e^{\varepsilon|s|} \cdot \mu e^{-\varepsilon|s|} ds \\ &\quad + \int_t^{+\infty} Ke^{\alpha(t-s)}e^{\varepsilon|s|} \cdot \mu e^{-\varepsilon|s|} ds \\ &= K\mu\alpha^{-1} + K\mu\alpha^{-1} \\ &= 2K\mu\alpha^{-1}. \end{aligned} \tag{7}$$

Therefore,  $\mathcal{T}_1\varphi(t) \in \mathbf{B}$ , which implies  $\mathcal{T}_1$  maps  $\mathbf{B}$  onto itself. On the other hand,

$$\begin{aligned}
 & |\mathcal{T}_1\varphi_1(t) - \mathcal{T}_1\varphi_2(t)| \\
 & \leq \int_{-\infty}^t T_A(t,s)P(s)re^{-\varepsilon|s|}|\varphi_1(s) - \varphi_2(s)|ds \\
 & \quad + \int_t^{+\infty} T_A(t,s)Q(s)re^{-\varepsilon|s|}|\varphi_1(s) - \varphi_2(s)|ds \\
 & \leq \int_{-\infty}^t Kre^{-\alpha(t-s)}|\varphi_1(s) - \varphi_2(s)|ds \\
 & \quad + \int_t^{+\infty} Kre^{\alpha(t-s)}|\varphi_1(s) - \varphi_2(s)|ds \\
 & \leq 2K\alpha^{-1}\sup_{s \geq 0}|\varphi_1(s) - \varphi_2(s)| \\
 & \leq \frac{1}{2}\sup_{s \geq 0}|\varphi_1(s) - \varphi_2(s)|.
 \end{aligned} \tag{8}$$

Then  $\mathcal{T}_1$  is a contraction mapping. Therefore, in  $\mathbf{B}$ , there exists a unique fixed point  $\varphi_0(t)$ , such that

$$\begin{aligned}
 \varphi_0(t) = \mathcal{T}_1\varphi_0(t) = & \int_{-\infty}^t T_A(t,s)P(s)f(s,\varphi_0(s))ds \\
 & - \int_t^{+\infty} T_A(t,s)Q(s)f(s,\varphi_0(s))ds.
 \end{aligned} \tag{9}$$

Differentiating the above equality, we see that  $\varphi_0(t)$  satisfies (2). Now we are going to show that the solution of system (2) satisfying  $(\bar{H}_1)$ ,  $(\bar{H}_2)$ , and  $(\bar{H}_3)$  is unique. Assume that system (2) has another bounded solution  $\varphi^*(t)$  satisfying  $(\bar{H}_1)$ ,  $(\bar{H}_2)$ , and  $(\bar{H}_3)$ ; we have

$$\begin{aligned}
 \varphi^*(t) = T_A(t,0)\varphi^*(0) & + \int_0^t T_A(t,s)T^{-1}(s,s)f(s,\varphi^*(s))ds \\
 = T_A(t,0)\varphi^*(0) & + \int_0^t T_A(t,s)P(s)f(s,\varphi^*(s))ds \\
 & + \int_0^t T_A(t,s)Q(s)f(s,\varphi^*(s))ds \\
 = T_A(t,0)\varphi^*(0) & + \int_{-\infty}^t T_A(t,s)P(s)f(s,\varphi^*(s))ds
 \end{aligned}$$

$$\begin{aligned}
 & - \int_{-\infty}^0 T_A(t,s)P(s)f(s,\varphi^*(s))ds \\
 & + \int_0^{+\infty} T_A(t,s)Q(s)f(s,\varphi^*(s))ds \\
 & - \int_t^{+\infty} T_A(t,s)Q(s)f(s,\varphi^*(s))ds \\
 = T_A(t,0) & \left[ \varphi^*(0) \right. \\
 & - \left( \int_{-\infty}^0 T_A(0,s)P(s)f(s,\varphi^*(s))ds \right. \\
 & \quad \left. - \int_0^{+\infty} T_A(0,s)Q(s)f(s,\varphi^*(s))ds \right) \\
 & + \left( \int_{-\infty}^t T_A(t,s)P(s)f(s,\varphi^*(s))ds \right. \\
 & \quad \left. - \int_t^{+\infty} T_A(t,s)Q(s)f(s,\varphi^*(s))ds \right).
 \end{aligned} \tag{10}$$

By calculating, we get

$$\begin{aligned}
 & \int_{-\infty}^t T_A(t,s)P(s)f(s,\varphi^*(s))ds \\
 & - \int_t^{+\infty} T_A(t,s)Q(s)f(s,\varphi^*(s))ds \leq 2K\mu\alpha^{-1}.
 \end{aligned} \tag{11}$$

As  $\varphi^*(t)$  is bounded, we obtain

$$\begin{aligned}
 T_A(t,0) & \left[ \varphi^*(0) - \left( \int_{-\infty}^0 T_A(0,s)P(s)f(s,\varphi^*(s))ds \right. \right. \\
 & \quad \left. \left. - \int_0^{+\infty} T_A(0,s)Q(s)f(s,\varphi^*(s))ds \right) \right],
 \end{aligned} \tag{12}$$

is bounded. In addition, the formula above is the solution of system (1), so it is a bounded solution. From Lemma 6, we have

$$\begin{aligned}
 & T_A(t,0) \left[ \varphi^*(0) \right. \\
 & - \left( \int_{-\infty}^0 T_A(0,s)P(s)f(s,\varphi^*(s))ds \right. \\
 & \quad \left. - \int_0^{+\infty} T_A(0,s)Q(s)f(s,\varphi^*(s))ds \right) \Big] = 0.
 \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
 \varphi^*(t) = & \int_{-\infty}^t T_A(t,s)P(s)f(s,\varphi^*(s))ds \\
 & - \int_t^{+\infty} T_A(t,s)Q(s)f(s,\varphi^*(s))ds.
 \end{aligned} \tag{14}$$

From (3),  $(\widetilde{H}_2)$ , and  $(\widetilde{H}_3)$ , we have

$$\begin{aligned} & |\varphi_0(t) - \varphi^*(t)| \\ & \leq \int_{-\infty}^t K e^{-\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |\varphi_0(s) - \varphi^*(s)| ds \\ & \quad + \int_t^{+\infty} K e^{\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |\varphi_0(s) - \varphi^*(s)| ds \quad (15) \\ & = 2Kr\alpha^{-1} |\varphi_0(s) - \varphi^*(s)| \\ & = \frac{1}{2} \sup_{t \in \mathbb{R}} |\varphi_0(t) - \varphi^*(t)|. \end{aligned}$$

That is,  $\sup_{t \in \mathbb{R}} |\varphi_0(t) - \varphi^*(t)| \leq (1/2) \sup_{t \in \mathbb{R}} |\varphi_0(t) - \varphi^*(t)|$ , which implies  $\varphi_0(t) = \varphi^*(t)$ . Then the uniqueness is proved. The proof of Theorem 3 is complete.

**3.2. Proof of Theorem 4.** To prove Theorem 4, a standard method is to employ a linear transformation  $x = e^{\varepsilon|t|}y$ . However,  $x = e^{\varepsilon|t|}y$  is not differentiable at  $t = 0$ . Thus, we cannot use such transformation directly. We have to discuss by dividing into two pieces  $t \geq 0$  and  $t \leq 0$ .

Consider system

$$\dot{x}(t) = B(t)v(t) + F(t, v), \quad (16)$$

where  $u \in \mathbb{R}^n$ ,  $B(t)$  is a continuous matrix function, and  $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function.

**Lemma 7.** Suppose that system  $\dot{v}(t) = B(t)v(t)$  admits a nonuniform exponential dichotomy; that is, its evolution operator  $T_B(t, s)$  satisfies

$$\begin{aligned} |T_B(t, s)P(s)| & \leq K e^{-\lambda(t-s)} \cdot e^{\varepsilon|s|}, \quad t \geq s, \\ |T_B(t, s)Q(s)| & \leq K e^{\lambda(t-s)} \cdot e^{\varepsilon|s|}, \quad t \leq s, \end{aligned} \quad (17)$$

where  $\lambda$  is a positive constant. In addition,

$$\begin{aligned} |F(t, v_1) - F(t, v_2)| & \leq r e^{-\varepsilon|t|} |v_1 - v_2|, \\ 4Kr & < \lambda. \end{aligned} \quad (18)$$

If  $|F(t, v)| \leq \mu e^{-\varepsilon|t|}$  for  $t \geq 0$ , then for any  $a \in \mathbb{R}^n$ , system (16) has a unique solution  $v^+(t)$  satisfying the following:

- (i)  $|v^+(t)| < +\infty$ , for  $t \geq 0$ ;
- (ii)  $P(0)v^+(0) = P(0)a$ ;
- (iii) in  $\mathbb{R}^+$ ,  $v^+(t)$  satisfies integral equation

$$\begin{aligned} v^+(t) & = T_B(t, 0)P(0)a + \int_0^t T_B(t, s)P(s)F(s, v^+(s)) ds \\ & \quad - \int_t^{+\infty} T_B(t, s)Q(s)F(s, v^+(s)) ds. \end{aligned} \quad (19)$$

If  $|F(t, v)| \leq \mu e^{-\varepsilon|t|}$  for  $t \leq 0$ , then for any  $a \in \mathbb{R}^n$ , system (16) has a unique solution  $v^-(t)$  satisfying the following:

- (i)  $|v^-(t)| < +\infty$ , for  $t \leq 0$ ;
- (ii)  $Q(0)v^-(0) = Q(0)a$ ;
- (iii) in  $\mathbb{R}^-$ ,  $v^-(t)$  satisfies integral equation

$$\begin{aligned} v^-(t) & = T_B(t, 0)Q(0)a + \int_{-\infty}^t T_B(t, s)P(s)F(s, v^-(s)) ds \\ & \quad - \int_t^0 T_B(t, s)Q(s)F(s, v^-(s)) ds. \end{aligned} \quad (20)$$

*Proof.* We prove the existence of  $v^+(t)$  by successive approximation method. For any  $a \in \mathbb{R}^n$ , let  $v_0^+(t) = T_B(t, 0)P(0)a$ . We define  $v_m^+(t), v_{m+1}^+(t)$  recursively as follows:

$$\begin{aligned} v_{m+1}^+(t) & = T_B(t, 0)P(0)a + \int_0^t T_B(t, s)P(s)F(s, v_m^+(s)) ds \\ & \quad - \int_t^{+\infty} T_B(t, s)Q(s)F(s, v_m^+(s)) ds. \end{aligned} \quad (21)$$

From (17) and  $|F(t, v)| \leq \mu e^{-\varepsilon|t|}$ , for  $t \geq 0$ , we have

$$\begin{aligned} |v_{m+1}^+(t)| & \leq K e^{-\lambda t} |a| + \int_0^t K e^{-\lambda(t-s)} e^{\varepsilon s} \cdot \mu e^{-\varepsilon s} ds \\ & \quad + \int_t^{+\infty} K e^{\lambda(t-s)} e^{\varepsilon s} \cdot \mu e^{-\varepsilon s} ds \\ & = K e^{-\lambda t} |a| + K\mu\lambda^{-1} (1 - e^{-\lambda t}) + K\mu\lambda^{-1} \\ & \leq K |a| + 2K\mu\lambda^{-1}. \end{aligned} \quad (22)$$

For any bounded function  $v(t)$  defined on  $\mathbb{R}^+$ , denote  $\|v\| = \sup_{t \in \mathbb{R}^+} |v(t)|$ ; then it follows from (18) and (21) that

$$\begin{aligned} & |v_{m+1}^+(t) - v_m^+(t)| \\ & \leq \int_0^t K e^{-\lambda(t-s)} e^{\varepsilon s} \cdot r e^{-\varepsilon s} |v_m^+(s) - v_{m-1}^+(s)| ds \\ & \quad + \int_t^{+\infty} K e^{\lambda(t-s)} e^{\varepsilon s} \cdot r e^{-\varepsilon s} |v_m^+(s) - v_{m-1}^+(s)| ds \\ & \leq 2Kr\lambda^{-1} |v_m^+(s) - v_{m-1}^+(s)| \\ & \leq 2Kr\lambda^{-1} \|v_m^+ - v_{m-1}^+\| \\ & \leq \frac{1}{2} \|v_m^+ - v_{m-1}^+\|, \end{aligned} \quad (23)$$

which implies that  $\|v_{m+1}^+ - v_m^+\| \leq (1/2)\|v_m^+ - v_{m-1}^+\|$ . Hence, the series  $\sum_{m=0}^{\infty} (v_{m+1}^+(t) - v_m^+(t))$  converges uniformly on  $\mathbb{R}^+$ . It means that the series  $\{v_m^+(t)\}$  converges uniformly to a limit  $v^+(t)$  on  $\mathbb{R}^+$ .

From (21), for any fixed  $t$ , let  $m \rightarrow \infty$ , we have

$$\begin{aligned} v^+(t) & = T_B(t, 0)P(0)a + \int_0^t T_B(t, s)P(s)F(s, v^+(s)) ds \\ & \quad - \int_t^{+\infty} T_B(t, s)Q(s)F(s, v^+(s)) ds. \end{aligned} \quad (24)$$

Differentiating the above equality, we see that  $v^+(t)$  satisfies the system (16).

From (22) and (24), we know that  $|v^+(t)| < \infty$  for  $t \geq 0$ , and it is easy to demonstrate that  $P(0)v^+(0) = P(0)a$ . Now we are going to show the uniqueness of  $v^+(t)$ . If there is another bounded function  $\tilde{v}^+(t)$  satisfying (i), (ii), and (iii) on  $\mathbb{R}^+$ , in view of (iii) and (18) we have

$$\begin{aligned} & |\tilde{v}^+(t) - v^+(t)| \\ & \leq \int_0^t Ke^{-\lambda(t-s)}e^{\varepsilon s} \cdot re^{-\varepsilon s} |v^+(s) - \tilde{v}^+(s)| ds \\ & \quad + \int_t^{+\infty} Ke^{\lambda(t-s)}e^{\varepsilon s} \cdot re^{-\varepsilon s} |\tilde{v}^+(s) - v^+(s)| ds \quad (25) \\ & \leq 2Kr\lambda^{-1} \|\tilde{v}^+ - v^+\| \\ & \leq \frac{1}{2} \|\tilde{v}^+ - v^+\|, \end{aligned}$$

which implies  $\|\tilde{v}^+ - v^+\| \leq (1/2)\|\tilde{v}^+ - v^+\|$ . Hence,  $\tilde{v}^+(t) \equiv v^+(t)$ .

The proof of the existence and uniqueness of  $v^-(t)$  is similar to that of  $v^+(t)$ . The proof of Lemma 7 is complete.  $\square$

**Lemma 8.** Suppose that  $\alpha > 0, \delta > 0, C, L$ , and  $M$  are nonnegative constants and that  $v(t)$  is a nonnegative bounded continuous function which satisfies two of the following inequalities:

$$\begin{aligned} v(t) & \leq Ce^{-\alpha t} + L \int_0^t e^{-\alpha(t-s)} v(s) ds \\ & \quad + M \int_t^{+\infty} e^{\delta(t-s)} v(s) ds, \quad (t \geq 0), \\ v(t) & \leq Ce^{\alpha t} + L \int_t^0 e^{\alpha(t-s)} v(s) ds \\ & \quad + M \int_{-\infty}^t e^{-\delta(t-s)} v(s) ds, \quad (t \leq 0). \end{aligned} \quad (26)$$

In addition, if  $\gamma = L/\alpha + M/\delta < 1$ , then for  $t \geq 0$  or  $t \leq 0$ , one has

$$v(t) \leq (1 - \gamma)^{-1} Ce^{-[\alpha - (1-\gamma)^{-1}L]|t|}. \quad (27)$$

*Proof.* The proof is straightforward by Lemma 6.2 of Chapter 3 in [6].  $\square$

**Lemma 9.** For any  $a \in \mathbb{R}^n$ , system (2) has a unique solution  $x^+(t)$  with the following properties:

- (i)  $|x^+(t)e^{-\varepsilon t}| < +\infty$ , for  $t \geq 0$ ;
- (ii)  $P(0)x^+(0) = P(0)a$ ;
- (iii)  $x^+(t)$  on  $\mathbb{R}^+$  satisfies integral equation

$$\begin{aligned} x^+(t) & = T_A(t, 0)P(0)a + \int_0^t T_A(t, s)P(s)f(s, x^+(s)) ds \\ & \quad - \int_t^{+\infty} T_A(t, s)Q(s)f(s, x^+(s)) ds. \end{aligned} \quad (28)$$

Similarly, for any  $a \in \mathbb{R}^n$ , system (2) also has a unique solution  $x^-(t)$  with the following properties:

- (i)  $|x^-(t)e^{\varepsilon t}| < +\infty$ , for  $t \leq 0$ ;
- (ii)  $Q(0)x^-(0) = Q(0)a$ ;
- (iii)  $x^-(t)$  on  $\mathbb{R}^-$  satisfies integral equation

$$\begin{aligned} x^-(t) & = T_A(t, 0)Q(0)a + \int_{-\infty}^t T_A(t, s)P(s)f(s, x^-(s)) ds \\ & \quad - \int_t^0 T_A(t, s)Q(s)f(s, x^-(s)) ds. \end{aligned} \quad (29)$$

*Proof.* We firstly prove the existence and uniqueness of  $x^+(t)$ . Let  $v = xe^{-\varepsilon t}$ , then system (2) becomes

$$\dot{v}(t) = (A(t) - \varepsilon I)v(t) + e^{-\varepsilon t}f(t, ve^{\varepsilon t}). \quad (30)$$

Let  $F(t, v) = e^{-\varepsilon t}f(t, ve^{\varepsilon t})$ . From (H<sub>1</sub>) and (H<sub>2</sub>), we have

$$\begin{aligned} |F(t, v)| & \leq e^{-\varepsilon t}\mu = \mu e^{-\varepsilon|t|}, \quad \text{for } t \geq 0; \\ |F(t, v_1) - F(t, v_2)| & = e^{-\varepsilon t} |f(t, v_1e^{\varepsilon t}) - f(t, v_2e^{\varepsilon t})| \\ & \leq e^{-\varepsilon t}re^{-\varepsilon|t|} |v_1e^{\varepsilon t} - v_2e^{\varepsilon t}| \\ & \leq re^{-\varepsilon|t|} |v_1 - v_2|. \end{aligned} \quad (31)$$

Let  $T_C(t, s)$  be the evolution operator of the linear system  $\dot{v}(t) = (A(t) - \varepsilon t)v(t)$ . Since  $x(t) = T_A(t, s)x(s)$ ,  $v = e^{-\varepsilon t}x$ , we have  $T_C(t, s) = e^{-\varepsilon(t-s)}T_A(t, s)$ . Hence, from (3), we obtain

$$\begin{aligned} |T_C(t, s)P(s)| & \leq Ke^{-(\alpha-\varepsilon)(t-s)} \cdot e^{\varepsilon|s|}, \quad \text{for } t \geq s, \\ |T_C(t, s)Q(s)| & \leq Ke^{(\alpha-\varepsilon)(t-s)} \cdot e^{\varepsilon|s|}, \quad \text{for } t \leq s. \end{aligned} \quad (32)$$

Since  $4Kr < \alpha - \varepsilon$ , system (30) satisfies all conditions of Lemma 7. Therefore, for any  $a \in \mathbb{R}^n$ , system (30) has a unique solution  $v^+(t)$  with the following properties:

- (i)  $|v^+(t)| < +\infty$ , for  $t \geq 0$ ;
- (ii)  $P(0)v^+(0) = P(0)a$ ;
- (iii)  $v^+(t)$  on  $\mathbb{R}^+$  satisfies integral equation

$$\begin{aligned} v^+(t) & = T_C(t, 0)P(0)a + \int_0^t T_C(t, s)P(s)F(s, v^+(s)) ds \\ & \quad - \int_t^{+\infty} T_C(t, s)Q(s)F(s, v^+(s)) ds \\ & = e^{-\varepsilon t}T_A(t, 0)P(0)a \\ & \quad + \int_0^t e^{-\varepsilon(t-s)}T_A(t, s)P(s) \cdot e^{-\varepsilon s}f(s, v^+(s)e^{\varepsilon s}) ds \\ & \quad - \int_t^{+\infty} e^{-\varepsilon(t-s)}T_A(t, s)Q(s) \\ & \quad \cdot e^{-\varepsilon s}f(s, v^+(s)e^{\varepsilon s}) ds. \end{aligned} \quad (33)$$

Hence,

$$\begin{aligned} v^+(t) e^{et} &= T_A(t, 0) P(0) a \\ &+ \int_0^t T_A(t, s) P(s) f(s, v^+(s) e^{es}) ds \\ &- \int_t^{+\infty} T_A(t, s) Q(s) f(s, v^+(s) e^{es}) ds. \end{aligned} \quad (34)$$

Let  $x^+(t) = v^+(t)e^{et}$ , we have

$$\begin{aligned} x^+(t) &= T_A(t, 0) P(0) a + \int_0^t T_A(t, s) P(s) f(s, x^+(s)) ds \\ &- \int_t^{+\infty} T_A(t, s) Q(s) f(s, x^+(s)) ds. \end{aligned} \quad (35)$$

Then  $x^+(t)$  is the solution of system (2) and it satisfies all conditions of Lemma 9.

The proof for the existence and uniqueness of  $x^-(t)$  is similar to that of  $x^+(t)$ , so we omit it. This completes the proof of Lemma 9.  $\square$

**Lemma 10.** *If  $|T_A(t, 0)P(0)a| \leq Me^{\varepsilon|t|}$ , then  $a = 0$ .*

*Proof.* If  $a \neq 0$ , then  $P(0)a \neq 0$  or  $Q(0)a \neq 0$ . Without loss of generality, we assume  $P(0)a \neq 0$ , then we have

$$\begin{aligned} |T_A(t, 0) P(0) a| &= |T_A(t, 0) P(0) T_A^{-1}(s, 0) T_A(s, 0) P(0) a| \\ &= |T_A(t, 0) T_A(0, s) P(s) T_A(s, 0) P(0) a| \\ &= |T_A(t, s) P(s) T_A(s, 0) P(0) a| \\ &\leq |T_A(t, s) P(s)| \cdot |T_A(s, 0) P(0) a|. \end{aligned} \quad (36)$$

Since  $|T_A(t, s)P(s)| \leq Ke^{-\alpha(t-s)}e^{\varepsilon|s|}$  for  $t \geq s$ ,

$$|T_A(t, 0) P(0) a| \leq Ke^{-\alpha(t-s)}e^{\varepsilon|s|} \cdot |T_A(s, 0) P(0) a|, \quad (37)$$

which implies

$$|T_A(s, 0) P(0) a| \geq \frac{|T_A(t, 0) P(0) a|}{Ke^{-\alpha(t-s)}e^{\varepsilon|s|}}. \quad (38)$$

Taking  $t = 0$ , we obtain

$$\begin{aligned} |T_A(s, 0) P(0) a| &\geq \frac{|P(0) a|}{Ke^{\alpha s}e^{-\varepsilon s}} \\ &= K^{-1}e^{-(\alpha-\varepsilon)s} |P(0) a| \quad (s \leq 0). \end{aligned} \quad (39)$$

Therefore, when  $s \leq 0$ ,

$$\frac{|T_A(s, 0) P(0) a|}{e^{\varepsilon|s|}} \geq K^{-1}e^{-\alpha s} |P(0) a| \longrightarrow +\infty \quad \text{as } s \longrightarrow -\infty. \quad (40)$$

On the other hand, when  $s \leq 0$ , from (3), we have

$$|T_A(s, 0) Q(0) a| \leq Ke^{\alpha s} e^{\varepsilon|0|} = Ke^{\alpha s}, \quad (41)$$

hence,

$$\frac{|T_A(s, 0) Q(0) a|}{e^{\varepsilon|s|}} \leq Ke^{(\alpha+\varepsilon)s}. \quad (42)$$

It follows from (40) and (42) that

$$\begin{aligned} \frac{|T_A(s, 0) a|}{e^{\varepsilon|s|}} &= \frac{|T_A(s, 0) (P(s) + Q(s)) a|}{e^{\varepsilon|s|}} \\ &\geq \frac{|T_A(s, 0) P(s) a|}{e^{\varepsilon|s|}} - \frac{|T_A(s, 0) Q(s) a|}{e^{\varepsilon|s|}} \\ &\geq \frac{|T_A(s, 0) P(s) a|}{e^{\varepsilon|s|}} - Ke^{(\alpha+\varepsilon)s}. \end{aligned} \quad (43)$$

From the above inequality, we know that  $|T_A(s, 0)a|/e^{\varepsilon|s|} \rightarrow +\infty$  as  $s \rightarrow -\infty$ , which contradicts the original condition  $|T_A(s, 0)a|/e^{\varepsilon|s|} \leq M$  and it implies  $a = 0$ . This ends the proof of Lemma 10.  $\square$

*Proof of Theorem 4.* For any solution  $x(t)$  of system (2), it can be written as follows:

$$\begin{aligned} x(t) &= T_A(t, 0) x(0) + \int_0^t T_A(t, s) T_A^{-1}(s, 0) f(s, x(s)) ds \\ &= T_A(t, 0) x(0) + \int_0^t T_A(t, s) P(s) f(s, x(s)) ds \\ &\quad + \int_0^t T_A(t, s) Q(s) f(s, x(s)) ds \\ &= T_A(t, 0) x(0) + \int_{-\infty}^t T_A(t, s) P(s) f(s, x(s)) ds \\ &\quad - \int_{-\infty}^0 T_A(t, s) P(s) f(s, x(s)) ds \\ &\quad + \int_0^{+\infty} T_A(t, s) Q(s) f(s, x(s)) ds \\ &\quad - \int_t^{+\infty} T_A(t, s) Q(s) f(s, x(s)) ds \\ &= T_A(t, 0) \left[ x(0) \right. \\ &\quad \left. - \left( \int_{-\infty}^0 T_A(0, s) P(s) f(s, x(s)) ds \right. \right. \\ &\quad \left. \left. - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x(s)) ds \right) \right] \\ &\quad + \left( \int_{-\infty}^t T_A(t, s) P(s) f(s, x(s)) ds \right. \\ &\quad \left. - \int_t^{+\infty} T_A(t, s) Q(s) f(s, x(s)) ds \right). \end{aligned} \quad (44)$$

Let  $\xi(t)$  be any  $n$ -variable continuous function defined on  $\mathbb{R}$ . From (3) and  $(H_1)$ , we have

$$\begin{aligned} \left| \int_{-\infty}^t T_A(t, s) P(s) f(s, \xi(s)) ds \right| &\leq \int_{-\infty}^t K e^{-\alpha(t-s)} e^{\varepsilon|s|} \mu ds \\ &= K \mu e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} e^{\varepsilon|s|} ds. \end{aligned} \tag{45}$$

For  $t \geq 0$ ,

$$\begin{aligned} e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} e^{\varepsilon|s|} ds &= e^{-\alpha t} \left( \int_{-\infty}^0 e^{\alpha s} e^{-\varepsilon s} ds + \int_0^t e^{\alpha s} e^{\varepsilon s} ds \right) \\ &= e^{-\alpha t} \left( \frac{1}{\alpha - \varepsilon} + \frac{1}{\alpha + \varepsilon} (e^{(\alpha+\varepsilon)t} - 1) \right) \\ &\leq \frac{1}{\alpha + \varepsilon} e^{\varepsilon t} + \frac{1}{\alpha - \varepsilon} \\ &\leq \frac{1}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{1}{\alpha - \varepsilon}; \end{aligned} \tag{46}$$

for  $t \leq 0$ ,

$$\begin{aligned} e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} e^{\varepsilon|s|} ds &= e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} e^{-\varepsilon s} ds \\ &= \frac{1}{\alpha - \varepsilon} e^{-\varepsilon t} \\ &\leq \frac{1}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{1}{\alpha - \varepsilon}. \end{aligned} \tag{47}$$

Hence, for any  $t \in \mathbb{R}$ , we have

$$e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} e^{\varepsilon|s|} ds \leq \frac{1}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{1}{\alpha - \varepsilon}. \tag{48}$$

Therefore, for any  $t \in \mathbb{R}$ , we have

$$\left| \int_{-\infty}^t T_A(t, s) P(s) f(s, \xi(s)) ds \right| \leq \frac{K\mu}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{K\mu}{\alpha - \varepsilon}. \tag{49}$$

By the same calculation, for any  $t \in \mathbb{R}$ , we have

$$\left| \int_t^{+\infty} T_A(t, s) Q(s) f(s, \xi(s)) ds \right| \leq \frac{K\mu}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{K\mu}{\alpha - \varepsilon}. \tag{50}$$

In Lemma 9,  $x^+(t)$  and  $x^-(t)$  are uniquely determined by  $a$ ; we denote them by  $x^+(t, a)$  and  $x^-(t, a)$ , respectively. Let  $\rho = (2K\mu)/(\alpha - \varepsilon)$ , denote by  $G$  the closed sphere on  $\mathbb{R}^n$  whose center is at the origin of the coordinate system and whose radius is  $\rho$ . For any  $a \in G$ , we define a mapping  $\mathcal{T}_2 : G \rightarrow \mathbb{R}^n$  as follows:

$$\begin{aligned} \mathcal{T}_2 a &= \int_{-\infty}^0 T_A(0, s) P(s) f(s, x^-(s, a)) ds \\ &\quad - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x^+(s, a)) ds. \end{aligned} \tag{51}$$

It follows from (3) and  $(H_1)$  that

$$\begin{aligned} |\mathcal{T}_2 a| &\leq \int_{-\infty}^0 K e^{\alpha s} e^{\varepsilon|s|} \mu ds + \int_0^{+\infty} K e^{-\alpha s} e^{\varepsilon|s|} \mu ds \\ &= \frac{K\mu}{\alpha - \varepsilon} + \frac{K\mu}{\alpha - \varepsilon} \\ &= \frac{2K\mu}{\alpha - \varepsilon} = \rho, \end{aligned} \tag{52}$$

which implies that  $\mathcal{T}_2$  maps  $G$  onto itself. Now we are going to show that  $\mathcal{T}_2$  is continuous. For any  $a_1, a_2 \in G$ , from (3) and  $(H_2)$ , we have

$$\begin{aligned} |\mathcal{T}_2 a_1 - \mathcal{T}_2 a_2| &\leq \int_{-\infty}^0 K e^{\alpha s} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |x^-(s, a_1) - x^-(s, a_2)| ds \\ &\quad + \int_0^{+\infty} K e^{-\alpha s} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |x^+(s, a_1) - x^+(s, a_2)| ds \\ &\leq \int_{-\infty}^0 K r e^{\alpha s} |x^-(s, a_1) - x^-(s, a_2)| ds \\ &\quad + \int_0^{+\infty} K r e^{-\alpha s} |x^+(s, a_1) - x^+(s, a_2)| ds. \end{aligned} \tag{53}$$

From (3) and the condition (iii) of Lemma 9, for  $t \geq 0$ , we have

$$\begin{aligned} |x^+(t, a_1) - x^+(t, a_2)| &\leq T_A(t, 0) P(0) |a_1 - a_2| \\ &\quad + \int_0^t K e^{-\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |x^+(s, a_1) - x^+(s, a_2)| ds \\ &\quad + \int_t^{+\infty} K e^{\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|s|} |x^+(s, a_1) - x^+(s, a_2)| ds \\ &\leq K e^{-\alpha t} |a_1 - a_2| \\ &\quad + \int_0^t K r e^{-\alpha(t-s)} |x^+(s, a_1) - x^+(s, a_2)| ds \\ &\quad + \int_t^{+\infty} K r e^{\alpha(t-s)} |x^+(s, a_1) - x^+(s, a_2)| ds. \end{aligned} \tag{54}$$

Multiplying by  $e^{-\varepsilon t}$  on both sides of the above inequality, for  $t \geq 0$ , we get

$$\begin{aligned} e^{-\varepsilon t} |x^+(t, a_1) - x^+(t, a_2)| &\leq K e^{-(\alpha+\varepsilon)t} |a_1 - a_2| \\ &\quad + \int_0^t K r e^{-(\alpha+\varepsilon)(t-s)} (e^{-\varepsilon s} |x^+(s, a_1) - x^+(s, a_2)|) ds \\ &\quad + \int_t^{+\infty} K r e^{(\alpha-\varepsilon)(t-s)} (e^{-\varepsilon s} |x^+(s, a_1) - x^+(s, a_2)|) ds. \end{aligned} \tag{55}$$



By Lemma 9, for  $t \geq 0$ ,  $e^{-\varepsilon t} |x^+(t, a_1) - x^+(t, a_2)|$  is a bounded function. And it follows from Lemma 8 that

$$e^{-\varepsilon t} |x^+(t, a_1) - x^+(t, a_2)| \leq K |a_1 - a_2| (1 - \gamma)^{-1} e^{-[(\alpha + \varepsilon) - (1 - \gamma)^{-1} Kr]t}, \tag{56}$$

where  $\gamma = Kr/(\alpha + \varepsilon) + Kr/(\alpha - \varepsilon)$ . From (H<sub>3</sub>) and  $(\alpha + \varepsilon)^{-1} < (\alpha - \varepsilon)^{-1}$ , we get

$$\gamma \leq 2Kr(\alpha - \varepsilon)^{-1} < \frac{1}{2}. \tag{57}$$

Therefore, for  $\alpha - 2Kr > 0$ , we have

$$e^{-\varepsilon t} |x^+(t, a_1) - x^+(t, a_2)| \leq 2K |a_1 - a_2| e^{-[(\alpha + \varepsilon) - 2Kr]t} \quad (t \geq 0). \tag{58}$$

Hence,

$$|x^+(t, a_1) - x^+(t, a_2)| \leq 2K |a_1 - a_2| e^{-(\alpha - 2Kr)t} \quad (t \geq 0). \tag{59}$$

Similarly,

$$|x^-(t, a_1) - x^-(t, a_2)| \leq 2K |a_1 - a_2| e^{(\alpha - 2Kr)t} \quad (t \leq 0). \tag{60}$$

So from (53), it follows that

$$\begin{aligned} |\mathcal{F}_2 a_1 - \mathcal{F}_2 a_2| &\leq \int_{-\infty}^0 Kre^{\alpha s} \cdot 2K |a_1 - a_2| e^{(\alpha - 2Kr)s} ds \\ &\quad + \int_0^{+\infty} Kre^{-\alpha s} \cdot 2K |a_1 - a_2| e^{-(\alpha - 2Kr)s} ds \\ &\leq \int_{-\infty}^0 2K^2 r |a_1 - a_2| e^{2(\alpha - Kr)s} ds \\ &\quad + \int_0^{+\infty} 2K^2 r |a_1 - a_2| e^{-2(\alpha - Kr)s} ds \\ &= \frac{2K^2 r}{\alpha - Kr} |a_1 - a_2|, \end{aligned} \tag{61}$$

which show that  $\mathcal{F}_2$  is a continuous mapping. By fixed point theorem,  $\mathcal{F}_2$  has at least one fixed point on  $G$ . We denote this fixed point by  $a_0$ , then

$$a_0 = \mathcal{F}_2 a_0 = \int_{-\infty}^0 T_A(0, s) P(s) f(s, x^-(s, a_0)) ds - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x^+(s, a_0)) ds. \tag{62}$$

As  $P^2(s) = P(s)$ ,  $P(t)T_A(t, s) = T_A(t, s)P(s)$ ,  $P(s) + Q(s) = \text{Id}$ , we obtain

$$\begin{aligned} P(0) a_0 &= \int_{-\infty}^0 T_A(0, s) P(s) f(s, x^-(s, a_0)) ds, \\ Q(0) a_0 &= - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x^-(s, a_0)) ds. \end{aligned} \tag{63}$$

From Lemma 9, we have

$$\begin{aligned} x^+(0, a_0) &= P(0) a_0 - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x^+(s, a_0)) ds, \\ x^-(0, a_0) &= Q(0) a_0 + \int_{-\infty}^0 T_A(0, s) P(s) f(s, x^-(s, a_0)) ds. \end{aligned} \tag{64}$$

Hence,

$$x^+(0, a_0) = x^-(0, a_0) = a_0. \tag{65}$$

By the existence and uniqueness of the initial value problem, we conclude that  $x^+(t, a_0) = x^-(t, a_0)$ . We can denote it by  $x_0(t)$ . Hence,

$$\begin{aligned} x_0(0) &= a_0 \\ &= \int_{-\infty}^0 T_A(0, s) P(s) f(s, x(s, a_0)) ds \\ &\quad - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x(s, a_0)) ds. \end{aligned} \tag{66}$$

From the above equation, it follows from (44) that

$$\begin{aligned} x_0(t) &= \int_{-\infty}^t T_A(t, s) P(s) f(s, x(s)) ds \\ &\quad - \int_t^{+\infty} T_A(t, s) Q(s) f(s, x(s)) ds. \end{aligned} \tag{67}$$

From (49) and (50), we have

$$|x_0(t)| \leq \frac{2K\mu}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{2K\mu}{\alpha - \varepsilon}, \quad (-\infty < t < +\infty), \tag{68}$$

which implies that  $x_0(t)$  satisfies (5); that is,  $x_0(t) = O(e^{\varepsilon|t|})$ .

Now we are going to prove that the solution of (2) which satisfies (5) is unique. We assume that system (2) has another solution  $x_0^*(t)$  satisfying (5). From (44),  $x_0^*(t)$  can be written as

$$\begin{aligned} x_0^*(t) &= T_A(t, 0) \left[ x_0^*(0) \right. \\ &\quad \left. - \left( \int_{-\infty}^0 T_A(0, s) P(s) f(s, x_0^*(s)) ds \right. \right. \\ &\quad \left. \left. - \int_0^{+\infty} T_A(0, s) Q(s) f(s, x_0^*(s)) ds \right) \right] \\ &\quad + \left( \int_{-\infty}^t T_A(t, s) P(s) f(s, x_0^*(s)) ds \right. \\ &\quad \left. - \int_t^{+\infty} T_A(t, s) Q(s) f(s, x_0^*(s)) ds \right). \end{aligned} \tag{69}$$



From (49) and (50), we get

$$\begin{aligned} \left| \int_{-\infty}^t T_A(t,s) P(s) f(s, x_0^*(s)) ds \right| &\leq \frac{K\mu}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{K\mu}{\alpha - \varepsilon}, \\ \left| \int_t^{+\infty} T_A(t,s) Q(s) f(s, x_0^*(s)) ds \right| &\leq \frac{K\mu}{\alpha - \varepsilon} e^{\varepsilon|t|} + \frac{K\mu}{\alpha - \varepsilon}. \end{aligned} \tag{70}$$

It follows from  $|x_0^*(t)| = O(e^{\varepsilon|t|})$  and Lemma 10 that

$$\begin{aligned} x_0^* - \left( \int_{-\infty}^0 T_A(0,s) P(s) f(s, x_0^*(s)) ds \right. \\ \left. - \int_0^{+\infty} T_A(0,s) Q(s) f(s, x_0^*(s)) ds \right) = 0. \end{aligned} \tag{71}$$

Therefore,

$$\begin{aligned} x_0^*(t) = \int_{-\infty}^t T_A(t,s) P(s) f(s, x_0^*(s)) ds \\ - \int_t^{+\infty} T_A(t,s) Q(s) f(s, x_0^*(s)) ds. \end{aligned} \tag{72}$$

From (67), (72), and (H<sub>2</sub>), we have

$$\begin{aligned} |x_0(t) - x_0^*(t)| &\leq \int_{-\infty}^t K e^{-\alpha(t-s)} e^{\varepsilon|s|} \cdot r e^{-\varepsilon|t|} |x_0(s) - x_0^*(s)| ds \\ &+ \int_t^{+\infty} K e^{\alpha(t-s)} e^{\varepsilon|t|} \cdot r e^{-\varepsilon|t|} |x_0(s) - x_0^*(s)| ds \\ &= \int_{-\infty}^t K r e^{-\alpha(t-s)} |x_0(s) - x_0^*(s)| ds \\ &+ \int_t^{+\infty} K r e^{\alpha(t-s)} |x_0(s) - x_0^*(s)| ds. \end{aligned} \tag{73}$$

Let  $L = \sup_{t \in \mathbb{R}} e^{-\varepsilon|t|} |x_0(t) - x_0^*(t)$ . Since  $x_0(t)$  and  $x_0^*(t)$  satisfy (5),  $e^{-\varepsilon|s|} |x_0(s) - x_0^*(s)|$  is bounded. Thus, for  $t \geq 0$ , we have

$$\begin{aligned} e^{-\varepsilon|t|} |x_0(t) - x_0^*(t)| \\ = e^{-\varepsilon t} |x_0(t) - x_0^*(t)| \end{aligned}$$

$$\begin{aligned} &\leq \int_{-\infty}^t K e^{-(\alpha+\varepsilon)(t-s)} r [e^{-\varepsilon s} |x_0(s) - x_0^*(s)|] ds \\ &+ \int_t^{+\infty} K e^{(\alpha-\varepsilon)(t-s)} r [e^{-\varepsilon s} |x_0(s) - x_0^*(s)|] ds \\ &= \int_{-\infty}^0 K e^{-(\alpha+\varepsilon)(t-s)} r [e^{-\varepsilon s} |x_0(s) - x_0^*(s)|] ds \\ &+ \int_0^t K e^{-(\alpha+\varepsilon)(t-s)} r [e^{-\varepsilon s} |x_0(s) - x_0^*(s)|] ds \\ &+ \int_t^{+\infty} K e^{(\alpha-\varepsilon)(t-s)} r [e^{-\varepsilon s} |x_0(s) - x_0^*(s)|] ds \\ &\leq L \left( \frac{Kr}{\alpha + \varepsilon} e^{-(\alpha-\varepsilon)t} + \frac{Kr}{\alpha + \varepsilon} (1 - e^{-(\alpha+\varepsilon)t}) + \frac{Kr}{\alpha - \varepsilon} \right) \\ &\leq L \left( \frac{Kr}{\alpha + \varepsilon} + \frac{Kr}{\alpha + \varepsilon} + \frac{Kr}{\alpha - \varepsilon} \right) \\ &\leq L \left( \frac{3Kr}{\alpha - \varepsilon} \right) \\ &\leq \frac{3}{4}L \quad (\text{by (H}_3)). \end{aligned} \tag{74}$$

Similarly, we can prove that

$$e^{-\varepsilon|t|} |x_0(t) - x_0^*(t)| \leq \frac{3}{4}L, \quad \text{when } t \leq 0. \tag{75}$$

Therefore,

$$L = \sup_{t \in \mathbb{R}} e^{-\varepsilon|t|} |x_0(t) - x_0^*(t)| \leq \frac{3}{4}L, \quad (-\infty < t < +\infty). \tag{76}$$

That is,  $L \leq (3/4)L$ , which implies  $L = 0$ . Consequently,  $x_0(t) = x_0^*(t)$ . This completes the proof of Theorem 4.  $\square$

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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