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*Research Article*

# Norm and Essential Norm of Composition Followed by Differentiation from Logarithmic Bloch Spaces to $H_\mu^\infty$

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In this note we express the norm of composition followed by differentiation $DC_\varphi$ from the logarithmic Bloch and the little logarithmic Bloch spaces to the weighted space $H_\mu^\infty$ on the unit disk and give an upper and a lower bound for the essential norm of this operator from the logarithmic Bloch space to $H_\mu^\infty$.

## 1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, and $H^\infty$ be the space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

An analytic function $f \in H(\mathbb{D})$ is said to belong to the logarithmic Bloch space $\mathscr{LB}$ if

$$\|f\|_{\mathscr{LB}} = \sup\left\{(1-|z|)\ln\left(\frac{2e}{1-|z|}\right)\left|f'(z)\right| : z \in \mathbb{D}\right\} < \infty \tag{1}$$

and to the little logarithmic Bloch space $\mathscr{LB}_0$ if

$$\lim_{|z|\to 1^-}(1-|z|)\ln\left(\frac{2e}{1-|z|}\right)\left|f'(z)\right| = 0. \tag{2}$$

It can be easily proved that $\mathscr{LB}$ is a Banach space, under the norm

$$\|f\|_{\mathscr{L}} = \left|f(0)\right| + \|f\|_{\mathscr{LB}}, \tag{3}$$

and that $\mathscr{LB}_0$ is a closed subspace of $\mathscr{LB}$. Some sources for results and references about the logarithmic Bloch functions are the papers of Yoneda [1], Stević [2], and the authors of [3–8].

Let $\mu$ be a weight, that is, a positive continuous function on $\mathbb{D}$. The weighted space $H_\mu^\infty$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{D}}\mu(z)\left|f(z)\right| < \infty, \tag{4}$$

where $\mu$ is a weight.

Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. The composition operator $C_\varphi$ is defined by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}). \tag{5}$$

Let $D$ be the differentiation operator. The product $DC_\varphi$ is defined by

$$DC_\varphi(f) = (f\circ\varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D}). \tag{6}$$

The operator $DC_\varphi$ is probably studied for the first time by Hibschweiler and Portnoy in [9], where the boundedness and compactness of $DC_\varphi$ between Bergman and Hardy spaces are investigated. In [10], Stević calculated the norm of the operator $DC_\varphi$ from the classical Bloch space to $H_\mu^\infty$. Recently there has been some interest in calculating operator norms and essential norms of composition and related operators (see, e.g., [11–18] and the references therein). Motivated by the papers [10, 19], we continue here this line of research by calculating $\|DC_\varphi\|_{\mathscr{LB}\to H_\mu^\infty}$.

Suppose that $X_1$ and $X_2$ are Banach spaces and $L : X_1 \to X_2$ is a bounded linear operator. The essential norm $\|L\|_{e,X_1\to X_2}$ of $L$ is its distance to the compact operators. More precisely,

$$\|L\|_{e,X_1\to X_2} = \inf\left\{\|L-K\|_{X_1\to X_2} : K \text{ a compact operator of } X_1 \text{ into } X_2\right\}, \tag{7}$$

where $\|\cdot\|_{X_1\to X_2}$ denotes the operator norm. If $X_1 = X_2$, it is simply denoted by $\|\cdot\|_e$. Since the set of all compact operators is a closed subset of the set of bounded operators, it follows that an operator $L$ is compact if and only if $\|L\|_{e,X_1\to X_2} = 0$.

Essential norm formulas for composition operators are known in various settings. When $C_\varphi$ acts from the Hardy space $H^2(\mathbb{D})$ to itself, Shapiro [20] gives a formula for $\|C_\varphi\|_e$ in terms of the Nevanlinna counting function for $\varphi$. In [21], Donaway gives upper and lower estimates for $\|C_\varphi\|_e$ when $C_\varphi$ maps the Bloch, Dirichlet, or a Besov type space to itself. The essential norm of the $DC_\varphi$ operator from $\alpha$-Bloch spaces to $H_\mu^\infty$ space was estimated recently by Stević in [10]. In this note we give upper and lower estimates for $\|DC_\varphi\|_{e,\mathscr{LB}\to H_\mu^\infty}$.

## 2. The Operator Norm of $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0) \to H_\mu^\infty$

In this section we prove a nice formula. Namely, we calculate the norm of the operator $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0) \to H_\mu^\infty$.

**Theorem 1.** *Assume $\mu$ is a weight on $\mathbb{D}$. Then the following are equivalent:*

(a) $DC_\varphi : \mathscr{LB} \to H_\mu^\infty$ *is a bounded operator;*

(b) $DC_\varphi : \mathscr{LB}_0 \to H_\mu^\infty$ *is a bounded operator;*

(c) $\sup_{z\in\mathbb{D}}(\mu(z)|\varphi'(z)|)/((1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))) < \infty$

*Moreover, one has*

$$\left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty} = \left\|DC_\varphi\right\|_{\mathscr{LB}\to H_\mu^\infty} = \sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|\right)\ln\left(2e/\left(1-\left|\varphi(z)\right|\right)\right)}. \tag{8}$$

*Proof.* $(a) \Rightarrow (b)$. By the fact $\mathscr{LB}_0 \subset \mathscr{LB}$ and the definition of operator norm, we easily obtain that $DC_\varphi : \mathscr{LB}_0 \to H_\mu^\infty$ is a bounded operator and

$$\left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty} \le \left\|DC_\varphi\right\|_{\mathscr{LB}\to H_\mu^\infty}. \tag{9}$$

$(b) \Rightarrow (c)$. Suppose that $DC_\varphi$ is a bounded operator from $\mathscr{LB}_0$ to $H_\mu^\infty$. Taking the test function $f(z) = z \in \mathscr{LB}_0$, we easily have

$$\begin{aligned}\mu(w)\left|\varphi'(w)\right| &\le \left\|\varphi'\right\|_{H_\mu^\infty} = \left\|DC_\varphi(z)\right\|_{H_\mu^\infty}\\ &\le \left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty}\|z\|_{\mathscr{L}}\\ &= \left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty}\ln 2e,\end{aligned} \tag{10}$$

for every $w \in \mathbb{D}$. It implies that (c) holds when $\varphi(z) = 0$.

Fixing $w \in \mathbb{D}\setminus\{0\}$, we consider the function

$$f_w(z) = \frac{1}{\overline{w}}\ln\ln\left(\frac{2e}{1-\overline{w}z}\right) - \frac{1}{\overline{w}}\ln\ln 2e. \tag{11}$$

Since $r(x) = x\ln(2e/x)$ is increasing on $(0,2]$ and $f_w(0) = 0$, we have

$$\begin{aligned}\|f_w\|_{\mathscr{L}} &= \sup_{z\in\mathbb{D}}(1-|z|)\ln\left(\frac{2e}{1-|z|}\right)\\ &\quad\times\frac{1}{|\ln(2e/(1-\overline{w}z))|}\frac{1}{|1-\overline{w}z|}\\ &\le \sup_{z\in\mathbb{D}}\frac{(1-|z|)\ln(2e/(1-|z|))}{(1-|\overline{w}z|)\ln(2e/(1-|\overline{w}z|))}\\ &\quad\times\frac{(1-|\overline{w}z|)\ln(2e/(1-|\overline{w}z|))}{|1-\overline{w}z|\ln(2e/(1-\overline{w}z))} \le 1.\end{aligned} \tag{12}$$

Moreover, since

$$\begin{aligned}&(1-|z|)\ln\frac{2e}{1-|z|}\left|f_w'(z)\right|\\ &\quad\le\frac{(1-|z|)\ln(2e/(1-|z|))}{(1-|\overline{w}z|)\ln(2e/(1-|\overline{w}z|))}\\ &\quad\le\frac{(1-|z|)\ln(2e/(1-|z|))}{(1-|w|)\ln 2e}\longrightarrow 0,\end{aligned} \tag{13}$$

as $|z| \to 1^-$, it follows that $f_w \in \mathscr{LB}_0$ for every $w \in \mathbb{D}\setminus\{0\}$. Thus, for each $t \in (0,1)$ we obtain that

$$\begin{aligned}\left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty} &\ge \left\|DC_\varphi\left(f_{t(\varphi(w)/|\varphi(w)|)}\right)\right\|_{H_\mu^\infty}\\ &= \sup_{z\in\mathbb{D}}\mu(z)\left|\varphi'(z)f'_{t(\varphi(w)/|\varphi(w)|)}(\varphi(z))\right|\\ &\ge\frac{\mu(w)\left|\varphi'(w)\right|}{\left(1-t\left|\varphi(w)\right|\right)\ln\left(2e/\left(1-t\left|\varphi(w)\right|\right)\right)},\end{aligned} \tag{14}$$

for every $\varphi(w) \ne 0$. Letting $t \to 1^-$, we obtain that

$$\left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty} \ge \frac{\mu(w)\left|\varphi'(w)\right|}{\left(1-\left|\varphi(w)\right|\right)\ln\left(2e/\left(1-\left|\varphi(w)\right|\right)\right)}, \tag{15}$$

for every $\varphi(w) \ne 0$. It implies that (c) also holds when $\varphi(z) \ne 0$.

$(c) \Rightarrow (a)$. For every $f \in \mathscr{LB}$, we easily obtain that

$$\begin{aligned} \left\|DC_\varphi f\right\|_{H_\mu^\infty} &\le \sup_{z\in D}\mu(z)\left|\left(DC_\varphi f\right)(z)\right| \\ &= \sup_{z\in D}\mu(z)\left|\varphi'(z) f'(\varphi(z))\right| \\ &\le \sup_{z\in D}\frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}\|f\|_{\mathscr{L}}. \end{aligned} \tag{16}$$

Hence $DC_\varphi : \mathscr{LB} \to H_\mu^\infty$ is a bounded operator. Also, we obtain

$$\left\|DC_\varphi\right\|_{\mathscr{LB}\to H_\mu^\infty} \le \sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}. \tag{17}$$

Moreover, from (9), (10), (15), and (17), we obtain

$$\begin{aligned} \left\|DC_\varphi\right\|_{\mathscr{LB}_0\to H_\mu^\infty} &= \left\|DC_\varphi\right\|_{\mathscr{LB}\to H_\mu^\infty} \\ &= \sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}. \end{aligned} \tag{18}$$

□

## 3. Estimates of Essential Norm of $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$

In this section we will estimate the essential norm of $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$. For this purpose we need some lemmas.

**Lemma 2.** *If $f \in \mathscr{LB}$, then $|f(z)| \le (1/2 + \ln\ln(e/(1-|z|)))\|f\|_{\mathscr{L}}$.*

This can be done in exactly the same way as in the proof of [3, Lemma 2.1].

**Lemma 3.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $\mu$ be a weight on $\mathbb{D}$. Assume that $DC_\varphi$ is a bounded operator from $\mathscr{LB}$ (or $\mathscr{LB}_0$) to $H_\mu^\infty$; then $DC_\varphi$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathscr{LB}$ (or $\mathscr{LB}_0$), which converges to 0 uniformly on compact subsets of $\mathbb{D}$, one has $\|DC_\varphi(f_n)\|_{H_\mu^\infty} \to 0$ as $n \to \infty$.*

*Proof. Necessity*. Suppose that $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$ is compact. Let $\{f_n\}$ be a bounded sequence in $\mathscr{LB}$ (or $\mathscr{LB}_0$) with $f_n \to 0$ uniformly on compact subsets of $D$. Assume that there is a subsequence $\{f_{n_k}\}$ and an $\epsilon_0 > 0$ such that $\|DC_\varphi f_{n_k}\| \ge \epsilon_0$ for all $k = 1, 2, 3, \ldots$. Since $DC_\varphi$ is compact, we can find a further subsequence $\{f_{n_{k_j}}\}$ and a function $f \in H_\mu^\infty$ such that $\lim_{j\to\infty}\|DC_\varphi f_{n_{k_j}} - f\|_{H_\mu^\infty} = 0$. Then we obtain that, for $z \in D$,

$$\left|\left(DC_\varphi f_{n_{k_j}} - f\right)(z)\right| \le \frac{\left\|DC_\varphi f_{n_{k_j}} - f\right\|_{H_\mu^\infty}}{\mu(z)}. \tag{19}$$

Hence $DC_\varphi f_{n_{k_j}} - f \to 0$ uniformly on compact subsets of $D$. Also, since $f_{n_{k_j}} \to 0$ uniformly on compact subsets of $D$, $DC_\varphi f_{n_{k_j}} \to 0$ uniformly on compact subsets of $D$. It follows that $f = 0$ and hence $\lim_{j\to\infty}\|DC_\varphi f_{n_{k_j}}\|_{H_\mu^\infty} = 0$, contradicting the fact that $\|DC_\varphi f_{n_k}\| \ge \epsilon_0$ for all $k = 1, 2, 3, \ldots$. Therefore we must have that $\lim_{n\to\infty}\|DC_\varphi(f_n)\|_{H_\mu^\infty} = 0$.

*Sufficiency*. Let $\{f_n\}$ be a bounded sequence in $\mathscr{LB}$ (or $\mathscr{LB}_0$). Then Lemma 2 and Montel's Theorem tell us that $\{f_n\}$ forms a normal family, and hence there exists a subsequence $\{f_{n_k}\}$ converging uniformly on compact sets to some function $f$. It is easy to see that $f$ must be in $\mathscr{LB}$ ($\mathscr{LB}_0$). Then $\{f_{n_k} - f\}$ is a bounded sequence in $\mathscr{LB}$ (or $\mathscr{LB}_0$) converging to 0 uniformly on compact subsets of $\mathbb{D}$ and by the hypothesis guarantees that $DC_\varphi f_{n_k} \to DC_\varphi f$ in $H_\mu^\infty$. Thus $DC_\varphi$ is compact. □

**Lemma 4.** *Let $\mu$ be a weight on $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$ with $\|\varphi\|_\infty < 1$. Suppose that $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$ is bounded. Then $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$ is compact.*

*Proof.* Suppose that $\{f_n\}$ is a bounded sequence in $\mathscr{LB}$ (or $\mathscr{LB}_0$) which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Cauchy's inequality we easily obtain that $\{f_n'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. Since $DC_\varphi$ is bounded, one can take the test function $f(z) = z$ to see that $\varphi' \in H_\mu^\infty$. Then we obtain that

$$\left\|DC_\varphi f_n\right\|_{H_\mu^\infty} \le \left\|\varphi'\right\|_{H_\mu^\infty}\sup_{w\in\varphi(\mathbb{D})}\left|f_n'(w)\right| \longrightarrow 0, \tag{20}$$

as $n \to \infty$, since $\varphi(\mathbb{D})$ is contained in the disk $|w| \le \|\varphi\|_\infty < 1$, which is a compact subset of $\mathbb{D}$. Hence, by Lemma 3, the operator $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$ is compact. □

**Lemma 5.** *Let $f \in \mathscr{LB}$. Then $\|f_t\|_{\mathscr{L}} \le \|f\|_{\mathscr{L}}, 0 < t < 1$, where $f_t(z) = f(tz)$.*

Since $r(x) = (1-x)\ln(2e/(1-x))$ is decreasing on $[0, 1)$, one may easily prove the result.

**Theorem 6.** *Let $\mu$ be a weight on $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Suppose that $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$ is bounded. Then*

$$\begin{aligned}&\frac{1}{2}\limsup_{|\varphi(z)|\to 1^-}\frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}\\&\quad\le\left\|DC_\varphi\right\|_{e,\mathscr{LB}_0\to H_\mu^\infty}\le\left\|DC_\varphi\right\|_{e,\mathscr{LB}\to H_\mu^\infty}\\&\quad\le 2\limsup_{|\varphi(z)|\to 1^-}\frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}.\end{aligned}\tag{21}$$

*Proof.* If $\|\varphi\|_\infty < 1$, by Lemma 4, it follows that $DC_\varphi : \mathscr{LB}$ (or $\mathscr{LB}_0$) $\to H_\mu^\infty$ is compact which is equivalent to $\|DC_\varphi\|_{e,\mathscr{LB}_0\to H_\mu^\infty} = \|DC_\varphi\|_{e,\mathscr{LB}\to H_\mu^\infty} = 0$. On the other hand, it is clear that in this case the condition $|\varphi(z)| \to 1$ is vacuous, so that it is understood that

$$\limsup_{|\varphi(z)|\to 1^-}\frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}=0.\tag{22}$$

Now suppose that $\|\varphi\|_\infty = 1$. Assume that $\{z_n\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let

$$\begin{aligned}f_n(z)=&\frac{1}{2\overline{\varphi(z_n)}a_n}\left(\ln\ln\frac{2e}{1-\overline{\varphi(z_n)}z}\right)^2\\&-\frac{1}{2\overline{\varphi(z_n)}a_n}(\ln\ln 2e)^2,\end{aligned}\tag{23}$$

where $a_n = \ln\ln(2e/(1-|\varphi(z_n)|^2))$. Then we have $f_n(0) = 0$,

$$f_n'(\varphi(z_n))=\frac{1}{\left(1-|\varphi(z_n)|^2\right)\ln\left(2e/\left(1-|\varphi(z_n)|^2\right)\right)}.\tag{24}$$

Clearly $f_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. It follows that

$$\begin{aligned}\|f_n\|_{\mathscr{L}}=&\sup_{z\in\mathbb{D}}(1-|z|)\ln\frac{2e}{1-|z|}\frac{1}{a_n}\left|\ln\ln\frac{2e}{1-\overline{\varphi(z_n)}z}\right|\\&\times\frac{1}{\left|\ln\left(2e/\left(1-\overline{\varphi(z_n)}z\right)\right)\right|}\frac{1}{\left|1-\overline{\varphi(z_n)}z\right|}\\\le&\sup_{z\in\mathbb{D}}\frac{2\pi+\ln(2\pi+\ln(2e/(1-|\varphi(z_n)|)))}{\ln\ln\left(2e/\left(1-|\varphi(z_n)|^2\right)\right)}\\&\times\frac{(1-|z|)\ln(2e/(1-|z|))}{\left(1-\left|\overline{\varphi(z_n)}z\right|\right)\ln\left(2e/\left(1-\left|\overline{\varphi(z_n)}z\right|\right)\right)}\\&\times\frac{\left(1-\left|\overline{\varphi(z_n)}z\right|\right)\ln\left(2e/\left(1-\left|\overline{\varphi(z_n)}z\right|\right)\right)}{\left|1-\overline{\varphi(z_n)}z\right|\ln\left(2e/\left|1-\overline{\varphi(z_n)}z\right|\right)}\\\le&\frac{2\pi+\ln(2\pi+\ln(2e/(1-|\varphi(z_n)|)))}{\ln\ln\left(2e/\left(1-|\varphi(z_n)|^2\right)\right)}.\end{aligned}\tag{25}$$

Thus, $\limsup_{n\to\infty}\|f_n\|_{\mathscr{L}} \le 1$. Let $g_n = f_n/\|f_n\|_{\mathscr{L}}$. Then $\|g_n\|_{\mathscr{L}} = 1$ and $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since $g_n \in \mathscr{LB}_0$, then it follows that $g_n$ converges to 0 weakly in $\mathscr{LB}_0$. Thus, for any compact operator $K : \mathscr{LB}_0 \to H_\mu^\infty$, $\lim_{n\to\infty}\|Kg_n\|_{H_\mu^\infty} = 0$. Therefore

$$\begin{aligned}\left\|DC_\varphi-K\right\|_{\mathscr{LB}_0\to H_\mu^\infty}&=\sup_{\|f\|_{\mathscr{L}}\le1}\left\|\left(DC_\varphi-K\right)f\right\|_{H_\mu^\infty}\\&\ge\limsup_{n\to\infty}\left\|\left(DC_\varphi-K\right)g_n\right\|_{H_\mu^\infty}\\&\ge\limsup_{n\to\infty}\left\|DC_\varphi g_n\right\|_{H_\mu^\infty}.\end{aligned}\tag{26}$$

Hence

$$\begin{aligned}&\left\|DC_\varphi\right\|_{e,\mathscr{LB}_0\to H_\mu^\infty}\\&\quad\ge\limsup_{n\to\infty}\left\|DC_\varphi g_n\right\|_{H_\mu^\infty}\\&\quad=\limsup_{n\to\infty}\sup_{z\in\mathbb{D}}\left|\mu(z)\,g_n'(\varphi(z))\,\varphi(z)\right|\\&\quad\ge\limsup_{n\to\infty}\frac{1}{\|f_n\|_{\mathscr{L}}}\left|\mu(z_n)\,f_n'(\varphi(z_n))\,\varphi(z_n)\right|\\&\quad\ge\limsup_{n\to\infty}\frac{\mu(z_n)\,|\varphi(z_n)|}{\left(1-|\varphi(z_n)|^2\right)\ln\left(2e/\left(1-|\varphi(z_n)|^2\right)\right)}\\&\quad=\frac{1}{2}\limsup_{n\to\infty}\frac{\mu(z_n)\,|\varphi(z_n)|}{(1-|\varphi(z_n)|)\ln(2e/(1-|\varphi(z_n)|))}.\end{aligned}\tag{27}$$

Thus the first inequality in (21) follows. The second inequality in (21) is obvious. Now we prove the third one.

Let $s \in (0,1)$ be fixed and $\rho_n = 1-1/(n+1)$, $n = 1, 2, \ldots$. By Lemma 4 we obtain that the operator $DC_{\rho_n\varphi} : \mathscr{LB} \to H_\mu^\infty$ is compact for every $n$. It follows that

$$\begin{aligned}\left\|DC_\varphi\right\|_{e,\mathscr{LB}\to H_\mu^\infty}&\le\left\|DC_\varphi-DC_{\rho_n\varphi}\right\|_{\mathscr{LB}\to H_\mu^\infty}\\&=\sup_{\|f\|_{\mathscr{L}}\le1}\left\|(DC_\varphi-DC_{\rho_n\varphi})(f)\right\|_{H_\mu^\infty}\\&=\sup_{\|f\|_{\mathscr{L}}\le1}\sup_{|\varphi(z)|\le s}\mu(z)\left|\varphi'(z)\right|\\&\quad\times\left|f'(\varphi(z))-\rho_nf'(\rho_n\varphi(z))\right|\\&\quad+\sup_{\|f\|_{\mathscr{L}}\le1}\sup_{|\varphi(z)|>s}\mu(z)\left|\varphi'(z)\right|\\&\quad\times\left|f'(\varphi(z))-\rho_nf'(\rho_n\varphi(z))\right|\triangleq I_1+I_2.\end{aligned}\tag{28}$$

By Cauchy's inequality, we obtain that

$$\begin{aligned}
I_1 \le & \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|\varphi(z)|\le s} \left\|\varphi'\right\|_{H_\mu^\infty} \left|f'(\varphi(z)) - f'(\rho_n\varphi(z))\right| \\
& + \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|\varphi(z)|\le s} \left\|\varphi'\right\|_{H_\mu^\infty} (1-\rho_n) \left|f'(\rho_n\varphi(z))\right| \\
\le & \left(1-\rho_n\right) \left\|\varphi'\right\|_{H_\mu^\infty} \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|w|\le s} \left|f''(w)\right| \\
& + \left(1-\rho_n\right) \left\|\varphi'\right\|_{H_\mu^\infty} \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|w|\le s} \left|f'(w)\right| \\
\le & \left(1-\rho_n\right) \left\|\varphi'\right\|_{H_\mu^\infty} \sup_{\|f\|_{\mathscr{L}}\le 1} \frac{2}{1-s} \max_{|z|\le(1+s)/2} \left|f'(z)\right| \\
& + \left(1-\rho_n\right) \left\|\varphi'\right\|_{H_\mu^\infty} \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|w|\le s} \left|f'(w)\right| \\
\le & \left(1-\rho_n\right) \left\|\varphi'\right\|_{H_\mu^\infty} \sup_{\|f\|_{\mathscr{L}}\le 1} \left(1 + \frac{2}{1-s}\right) \\
& \times \max_{|z|\le(1+s)/2} \frac{(1-|z|)\ln(2e/(1-|z|))\left|f'(z)\right|}{(1-|z|)\ln(2e/(1-|z|))} \\
\le & \frac{1}{n+1} \left\|\varphi'\right\|_{H_\mu^\infty} \left(1+\frac{2}{1-s}\right) \frac{2}{(1-s)\ln(4e/(1-s))}.
\end{aligned} \tag{29}$$

On the other hand, by Lemma 5, we obtain that

$$\begin{aligned}
I_2 \le & \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)\left|\varphi'(z)\right| \|f\|_{\mathscr{L}}}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))} \\
& + \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)\left|\varphi'(z)\right| \left\|f_{\rho_n}\right\|_{\mathscr{L}}}{(1-\rho_n|\varphi(z)|)\ln(2e/(1-\rho_n|\varphi(z)|))} \\
\le & 2 \sup_{\|f\|_{\mathscr{L}}\le 1} \sup_{|\varphi(z)|>s} \frac{\mu(z)\left|\varphi'(z)\right| \|f\|_{\mathscr{L}}}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))} \\
\le & 2 \sup_{|\varphi(z)|>s} \frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))},
\end{aligned} \tag{30}$$

where $f_{\rho_n}(z) = f(\rho_n z)$. Hence, for for all $s \in (0,1)$ and all $n$, we have

$$\begin{aligned}
& \left\|DC_\varphi\right\|_{e,\mathscr{LB}\to H_\mu^\infty} \\
& \quad \le \frac{1}{n+1} \left\|\varphi'\right\|_{H_\mu^\infty} \left(1+\frac{2}{1-s}\right) \frac{2}{(1-s)\ln(4e/(1-s))} \\
& \qquad + 2 \sup_{|\varphi(z)|>s} \frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}.
\end{aligned} \tag{31}$$

Letting $n \to \infty$ and then letting $s \to 1^-$, we obtain that

$$\begin{aligned}
& \left\|DC_\varphi\right\|_{e,\mathscr{LB}\to H_\mu^\infty} \\
& \quad \le 2 \limsup_{|\varphi(z)|\to 1^-} \frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))}.
\end{aligned} \tag{32}$$

The proof of the theorem is finished. □

**Corollary 7.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$, $\mu$ be a weight on $\mathbb{D}$, and $DC_\varphi$ be a bounded operator from $\mathscr{LB}$ (or $\mathscr{LB}_0$) to $H_\mu^\infty$. Then $DC_\varphi$ is a compact operator from $\mathscr{LB}$ (or $\mathscr{LB}_0$) to $H_\mu^\infty$ if and only if*

$$\limsup_{|\varphi(z)|\to 1^-} \frac{\mu(z)\left|\varphi'(z)\right|}{(1-|\varphi(z)|)\ln(2e/(1-|\varphi(z)|))} = 0. \tag{33}$$

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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