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Research Article

Nonlinear Variation of Parameters Formula for Impulsive Differential Equations with Initial Time Difference and Application

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This paper establishes variation of parameters formula for impulsive differential equations with initial time difference. As an application, one of the results is used to investigate stability properties of solutions.

1. Introduction

It is now well recognized that impulsive differential equations are suitable mathematical models for many processes and phenomena in biology, physics, technology, and so forth. That is why in recent years the mathematical theory of such systems has gained increasing significance. We notice that most of the studies about initial value problems of impulsive differential equations are investigated only for perturbation or change of dependent variable keeping the initial time unchanged. However, in dealing with real world phenomena, it is impossible not to make errors in the starting time. When we consider such a change of initial time for each solution, we need to deal with the problem of comparing between any two solutions which start at different times.

At present, the investigation of differential systems with initial time difference has attracted a lot of attention. There are two methods of comparing the differences of the two solutions. One is the differential inequalities technique and comparison principle; the other is variation of parameters. For the pioneering works in this area we can refer to the papers [1, 2]. Ever since then, many results for various differential and difference systems have been obtained. The results obtained by the former method can be seen in [3–10]; and those done by the latter can be found in [11–15]. However, up till now, to the best of our knowledge, there are few results for impulsive differential equations with initial time difference. To be specific, there are no results on variation of parameters formula for impulsive differential equations relative to initial time difference. The method of variation of parameters is an important and fruitful technique since it is a practical tool in the investigation of the properties of solutions. It has been applied to the study of the relations of unperturbed and perturbed systems with different initial conditions.

In this paper, we will develop variation of parameters formula for impulsive differential equations with initial time changed and investigate Lipschitz stability by using one of the results obtained. The remainder of this paper is organized in the following manner. Some preliminaries are presented in Section 2, and various types of nonlinear variation of parameters formulae are established in Section 3. Finally, as an application, one of the results is applied to impulsive differential equations and the stability properties are obtained.

2. Preliminaries

Let $R^+ = [0, +\infty)$ and let R^n denote the n-dimensional Euclidean space with appropriate norm $\|\cdot\|$.

Consider the following unperturbed impulsive differential equations

$$\begin{aligned}
&x' = f(t,x), \quad t \neq t_k,\\
&x(t_0^+) = x_0, \qquad (1)\\
&x(t_k^+) = x(t_k) + I_{t_k}(x(t_k)),\\
&x' = f(t,x), \quad t \neq t_k,\\
&x(\tau_0^+) = y_0, \qquad (2)\\
&x(t_k^+) = x(t_k) + I_{t_k}(x(t_k)), \quad \text{whenever } t_k \geq \tau_0,
\end{aligned}$$

together with the perturbed ones of (2)

$$\begin{aligned}
&y' = F(t,y), \quad t \neq t_k,\\
&y(\tau_0^+) = y_0, \qquad (3)\\
&y(t_k^+) = y(t_k) + I_{t_k}(y(t_k)), \quad \text{whenever } t_k \geq \tau_0,
\end{aligned}$$

where

(1) $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty} t_k = \infty$, $k = 1, 2, \ldots$;

(2) $\tau_0 > t_0$, $\eta = \tau_0 - t_0$;

(3) $\bar{t}_k = t_k - \eta \geq 0$;

(4) $S_1 = \{t_k\}$, $S_2 = \{\bar{t}_k\}$, $S = S_1 \cup S_2$;

(5) $t \in R^+$, $x \in \Omega \subset R^n$, Ω – open;

(6) $f, F : R^+ \times \Omega \to R^n$;

(7) $I_{t_k} : \Omega \to R^n$;

(8) $f(t,0) = 0$, $I_{t_k}(0) = 0$, for all t_k.

We are concerned in this paper with the variation of parameters formula for impulsive differential equations relative to initial time difference. Before we can proceed, we will introduce the following lemmas [12], which are necessary for completing our main results.

Lemma 1. *Let the following conditions be fulfilled:*

(A_1) *the function $f : R^+ \times \Omega \to R^n$ is continuous in $(t_{k-1}, t_k] \times \Omega$, $k = 1, 2, \ldots$ and for every k and $x_0 \in R^n$, there exists a finite limit of $f(t,x)$ as $(t,x) \to (t_k, x_0)$, $t > t_k$;*

(A_2) *the function f is locally Lipschitzian in x on $R^+ \times \Omega$;*

(A_3) *for $k = 1, 2, \ldots$ the mapping $\psi_k : \Omega \to \Omega$, $x \to z$, $z = \psi_k(x) \equiv x + I_k(x)$ is a homeomorphism;*

(A_4) *the system (1) had a solution $\phi(t)$ defined in $[\alpha, \beta]$, $(\alpha, \beta \neq t_k,\ k = 1, 2, \ldots)$.*

Then there exist a number $\epsilon > 0$ and a set

$$V = \{(t,x) \in R^+ \times \Omega,\ \alpha \leq t \leq \beta,\ |x - \phi(t^+)| < \epsilon\}, \qquad (4)$$

such that,

(i) *for every $(t_0, x_0) \in V$, there exists a unique solution $x(t, t_0, x_0)$ of the system (1) which is defined on $[\alpha, \beta]$;*

(ii) *the function $x(t, t_0, x_0)$ is continuous for*

$$t \in [\alpha, \beta], \qquad (t_0, x_0) \in V, \quad t, t_0 \notin S_1; \qquad (5)$$

(iii) *for $k = 1, 2, \ldots$, $x_0 \in \Omega$, t, t_0 belonging to the interval of existence of solution $x(t, t_0, x_0)$ of (1), $t \notin S_1$,*

$$\lim_{\substack{\xi \to t_0\\ \rho \to x_0}} x(t, \xi, \rho) = x(t, t_0, x_0). \qquad (6)$$

Lemma 2. *Let the following conditions be fulfilled:*

(A_5) *the function $f : R^+ \times \Omega \to R^n$ is continuous in $(t_{k-1}, t_k] \times \Omega$, $k = 1, 2, \ldots$, and $f_x(t,x)$ is continuous in $(t_{k-1}, t_k) \times \Omega$, $k = 1, 2, \ldots$;*

(A_6) *for every $x_0 \in \Omega$, $k = 1, 2, \ldots$, there exist finite limits of functions f and f_x as $(t,x) \to (t_k, x_0)$, $t > t_k$;*

(A_7) *for $k = 1, 2, \ldots$ the mapping $\psi_k : \Omega \to \Omega$, $x \to z$, $z = \psi_k(x) \equiv x + I_k(x)$ is a diffeomorphism and for $x \in \Omega$*

$$\det\left(I + \frac{\partial I_k}{\partial x}(x)\right) \neq 0, \quad k = 1, 2, \ldots. \qquad (7)$$

Then,

(i) *there exists $\delta > 0$ such that the solution $x(t, t_0, x_0)$ of (1) has continuous derivatives $\partial x/\partial t$, $\partial x/\partial t_0$, $\partial x/\partial x_0$, in the domain*

$$\begin{aligned}
V : \alpha < t < \beta, \qquad &\alpha < t_0 < \beta, \quad t, t_0 \neq t_k,\ k = 1, 2, \ldots\\
&|x_0 - \phi(t_0^+)| < \delta;
\end{aligned} \qquad (8)$$

(ii) *the derivative $\Phi(t, t_0, x_0) = (\partial x/\partial x_0)(t, t_0, x_0)$ is a solution of the initial value problem*

$$\begin{aligned}
&u' = f_x(t, \phi(t))u, \quad t \neq t_k,\\
&\Delta u = \frac{\partial I_k}{\partial x}(\phi(t_k))u, \quad t = t_k, \qquad (9)\\
&u(t_0^+) = I,
\end{aligned}$$

where $\phi(t)$ is the solution of (1) in $[\alpha, \beta]$ $\alpha, \beta \neq t_k$, $k = 1, 2, \ldots$;

(iii) *the derivative $\partial x/\partial t_0$ satisfies the relation*

$$\begin{aligned}
\frac{\partial x}{\partial t_0}(t, t_0, x_0) &= -\frac{\partial x}{\partial x_0}(t, t_0, x_0) f(t_0, x_0)\\
&= -\Phi(t, t_0, x_0) f(t_0, x_0).
\end{aligned} \qquad (10)$$

3. Nonlinear Variation of Parameters Formula

We will present, in this section, the nonlinear variation of parameters formula for impulsive differential equations relative to initial time difference. It is very useful for investigating the stability properties of solutions.

Theorem 3. *Let the system* (1) *satisfy the conditions of Lemma 2 and let $x(t, t_0, x_0)$ be a solution of* (1). *Then for any solution $y(t) = y(t, \tau_0, y_0)$ of the system* (3). *The following formula is valid:*

$$\begin{aligned} &y(t+\eta, \tau_0, y_0) \\ &\quad = x(t, t_0, x_0) + \int_0^1 \Phi(t, t_0, \sigma(s))(y_0 - x_0)\,ds \\ &\qquad + \int_{t_0}^t \widetilde{f}(s, \widetilde{y}(s), \eta)\,ds \\ &\qquad + \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \widetilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k))\right) ds \\ &\qquad \cdot I_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k)). \end{aligned} \tag{11}$$

Proof. Set $p(s) = x(t, s, \widetilde{y}(s))$, where $\widetilde{y}(s) = y(s+\eta, \tau_0, y_0)$, $t_0 < s < t$. Then for $s \notin S$, we have

$$\begin{aligned} p'(s) = {} & \frac{\partial x}{\partial s}(t, s, \widetilde{y}(s)) \\ & + \frac{\partial x}{\partial \widetilde{y}}(t, s, \widetilde{y}(s)) F(s+\eta, \widetilde{y}(s)) \equiv \widetilde{f}(s, \widetilde{y}(s), \eta). \end{aligned} \tag{12}$$

When $s \in S$, we have two cases.

Case 1. Consider

$$\begin{aligned} &\Delta p(s)\big|_{s=\bar{t}_k} \\ &\quad = x\left(t, \bar{t}_k^+, \widetilde{y}(\bar{t}_k^+)\right) - x\left(t, \bar{t}_k^-, \widetilde{y}(\bar{t}_k^-)\right) \\ &\quad = x\left(t, \bar{t}_k, \widetilde{y}(\bar{t}_k) + I_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k))\right) - x(t, \bar{t}_k, \widetilde{y}(\bar{t}_k)) \\ &\quad = \int_0^1 \Phi\left(t, \bar{t}_k, \widetilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k))\right) ds \cdot I_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k)). \end{aligned} \tag{13}$$

Case 2. Consider

$$\Delta p(s)\big|_{s=t_k\in S_1-S_2} = x\left(t, t_k^+, \widetilde{y}(t_k^+)\right) - x\left(t, t_k^-, \widetilde{y}(t_k^-)\right) = 0. \tag{14}$$

Integrating (12) from t_0 to t and using (13) and (14), we have

$$\begin{aligned} \widetilde{y}(t) = {} & x(t, t_0, y_0) + \int_{t_0}^t \widetilde{f}(s, \widetilde{y}(s), \eta)\,ds \\ & + \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \widetilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k))\right) ds \\ & \cdot I_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k)). \end{aligned} \tag{15}$$

Now let $q(s) = x(t, t_0, \sigma(s))$, where $\sigma(s) = y_0 s + (1-s)x_0$, $0 \le s \le 1$. Then we have

$$\frac{dq(s)}{ds} = \frac{\partial x}{\partial \sigma}(t, t_0, \sigma(s))(y_0 - x_0). \tag{16}$$

Integrating (16) from 0 to 1, we arrive at

$$x(t, t_0, y_0) = x(t, t_0, x_0) + \int_0^1 \frac{\partial x}{\partial \sigma}(t, t_0, \sigma(s))(y_0 - x_0)\,ds. \tag{17}$$

Combining (15) and (17) yields

$$\begin{aligned} &y(t+\eta, \tau_0, y_0) \\ &\quad = x(t, t_0, x_0) + \int_0^1 \Phi(t, t_0, \sigma(s))(y_0 - x_0)\,ds \\ &\qquad + \int_{t_0}^t \widetilde{f}(s, \widetilde{y}(s), \eta)\,ds \\ &\qquad + \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \widetilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k))\right) ds \\ &\qquad \cdot I_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k)). \end{aligned} \tag{18}$$

The proof is complete. □

Corollary 4. *Suppose that the assumptions of Theorem 3 hold except that $F(t+\eta, y)$ being replaced with $f(t, y) + R(t+\eta, y)$; then the following formula is valid:*

$$\begin{aligned} &y(t+\eta, \tau_0, y_0) \\ &\quad = x(t, t_0, x_0) + \int_0^1 \Phi(t, t_0, \sigma(s))(y_0 - x_0)\,ds \\ &\qquad + \int_{t_0}^t \Phi(t, s, \widetilde{y}(s)) R(s+\eta, \widetilde{y}(s))\,ds \\ &\qquad + \sum_{t_0<\bar{t}_k<t} \int_0^1 \Phi\left(t, \bar{t}_k, \widetilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k))\right) ds \\ &\qquad \cdot I_{\bar{t}_k+\eta}(\widetilde{y}(\bar{t}_k)). \end{aligned} \tag{19}$$

Theorem 5. *Suppose that the assumptions of Theorem 3 hold; then the following formula is valid:*

$$\begin{aligned} &y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0) \\ &\quad = x(t, t_0, \eta) + \int_{t_0}^t \widetilde{H}(s, w(s), \eta)\,ds \\ &\qquad + \sum_{\substack{t_0<\bar{t}_k<t \\ \bar{t}_k\in S_2-S_1}} \int_0^1 \Phi\left(t, \bar{t}_k, y(\bar{t}_k+\eta) - x(\bar{t}_k)\right. \\ &\qquad\qquad \left. + sI_{\bar{t}_k+\eta}(y(\bar{t}_k+\eta))\right) ds \\ &\qquad \cdot I_{\bar{t}_k+\eta}(y(\bar{t}_k+\eta)) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{t_0<t_k<t\\ t_k\in S_1-S_2}} \int_0^1 \Phi\Big(t,t_k,y(t_k+\eta)-x(t_k)\\
&\qquad\qquad -sI_{t_k}(x(t_k))\Big)\,ds\cdot I_{t_k}(x(t_k))\\
&+ \sum_{\substack{t_0<t_k<t\\ t_k\in S_1\cap S_2}} \int_0^1 \Phi\Big(t,t_k,y(t_k+\eta)-x(t_k)+sI_{t_k+\eta}(y(t_k+\eta))\\
&\qquad\qquad -sI_{t_k}(x(t_k))\Big)\,ds\\
&\cdot\Big(I_{t_k+\eta}(y(t_k+\eta))-I_{t_k}(x(t_k))\Big).
\end{aligned}
\tag{20}
$$

Proof. Set $p(s)=x(t,s,w(s))$, where $w(s)=y(s+\eta,\tau_0,y_0)-x(s,t_0,x_0)$, $t_0<s<t$. Then for $s\notin S$, we have

$$
\begin{aligned}
p'(s) &= \frac{\partial x}{\partial s}(t,s,w(s))\\
&\quad+\frac{\partial x}{\partial w}(t,s,w(s))\,H(s,w(s),\eta)\equiv\widetilde{H}(s,w(s),\eta),
\end{aligned}
\tag{21}
$$

where $H(s,w(s),\eta)=F(s+\eta,w(s)+x(s))-f(s,x(s))$.

If $s\in S$, we have three cases.

Case 1. Consider

$$
\begin{aligned}
&\Delta p(s)\big|_{s=\bar t_k\in S_2-S_1}\\
&\quad= x\big(t,\bar t_k^{+},w(\bar t_k^{+})\big)-x\big(t,\bar t_k^{-},w(\bar t_k^{-})\big)\\
&\quad= x\big(t,\bar t_k,y(\bar t_k^{+}+\eta)-x(\bar t_k^{+})\big)-x\big(t,\bar t_k^{-},y(\bar t_k^{-}+\eta)\\
&\qquad\qquad -x(\bar t_k^{-})\big)\\
&\quad= x\big(t,\bar t_k,y(\bar t_k+\eta)+I_{\bar t_k+\eta}(y(\bar t_k+\eta))-x(\bar t_k)\big)\\
&\qquad -x(t,\bar t_k,y(\bar t_k+\eta)-x(\bar t_k))\\
&\quad= \int_0^1 \Phi\big(t,\bar t_k,y(\bar t_k+\eta)-x(\bar t_k)+sI_{\bar t_k+\eta}(y(\bar t_k+\eta))\big)\,ds\\
&\qquad\cdot I_{\bar t_k+\eta}(y(\bar t_k+\eta)).
\end{aligned}
\tag{22}
$$

Case 2. Consider

$$
\begin{aligned}
&\Delta p(s)\big|_{s=t_k\in S_1-S_2}\\
&\quad= x(t,t_k^{+},w(t_k^{+}))-x(t,t_k^{-},w(t_k^{-}))\\
&\quad= x(t,t_k,y(t_k^{+}+\eta)-x(t_k^{+}))-x(t,t_k,y(t_k^{-}+\eta)-x(t_k^{-}))\\
&\quad= x\big(t,t_k,y(t_k+\eta)-I_{t_k}(x(t_k))-x(t_k)\big)\\
&\qquad -x(t,t_k,y(t_k+\eta)-x(t_k))\\
&\quad= -\int_0^1 \Phi\big(t,t_k,y(t_k+\eta)-x(t_k)-sI_{t_k}(x(t_k))\big)\,ds\\
&\qquad\cdot I_{t_k}(x(t_k)).
\end{aligned}
\tag{23}
$$

Case 3. Consider

$$
\begin{aligned}
&\Delta p(s)\big|_{s=t_k\in S_1\cap S_2}\\
&\quad= x(t,t_k^{+},w(t_k^{+}))-x(t,t_k^{-},w(t_k^{-}))\\
&\quad= x(t,t_k,y(t_k^{+}+\eta)-x(t_k^{+}))-x(t,t_k,y(t_k^{-}+\eta)\\
&\qquad\qquad -x(t_k^{-}))\\
&\quad= x\big(t,t_k,y(t_k+\eta)+I_{t_k+\eta}(y(t_k+\eta))-I_{t_k}(x(t_k))\\
&\qquad -x(t_k)\big)-x(t,t_k,y(t_k+\eta)-x(t_k))\\
&\quad= \int_0^1 \Phi\big(t,t_k,y(t_k+\eta)-x(t_k)\\
&\qquad\qquad +sI_{t_k+\eta}(y(t_k+\eta))-sI_{t_k}(x(t_k))\big)\,ds\\
&\qquad\cdot\big(I_{t_k+\eta}(y(t_k+\eta))-I_{t_k}(x(t_k))\big).
\end{aligned}
\tag{24}
$$

Integrating (21) from t_0 to t and using (22) and (24), we have

$$
\begin{aligned}
&y(t+\eta,\tau_0,y_0)-x(t,t_0,x_0)\\
&\quad= x(t,t_0,\eta)+\int_{t_0}^{t}\widetilde{H}(s,w(s),\eta)\,ds\\
&\qquad+\sum_{\substack{t_0<\bar t_k<t\\ \bar t_k\in S_2-S_1}}\int_0^1\Phi\big(t,\bar t_k,y(\bar t_k+\eta)-x(\bar t_k)\\
&\qquad\qquad +sI_{\bar t_k+\eta}(y(\bar t_k+\eta))\big)\,ds\cdot I_{\bar t_k+\eta}(y(\bar t_k+\eta))\\
&\qquad-\sum_{\substack{t_0<t_k<t\\ t_k\in S_1-S_2}}\int_0^1\Phi\big(t,t_k,y(t_k+\eta)-x(t_k)-sI_{t_k}(x(t_k))\big)\,ds\\
&\qquad\cdot I_{t_k}(x(t_k))\\
&\qquad+\sum_{\substack{t_0<t_k<t\\ t_k\in S_1\cap S_2}}\int_0^1\Phi\big(t,t_k,y(t_k+\eta)-x(t_k)\\
&\qquad\qquad +sI_{t_k+\eta}(y(t_k+\eta))-sI_{t_k}(x(t_k))\big)\,ds\\
&\qquad\cdot\big(I_{t_k+\eta}(y(t_k+\eta))-I_{t_k}(x(t_k))\big).
\end{aligned}
\tag{25}
$$

The proof is complete. □

4. Application

In this section, we turn to the Lipschitz stability of system (3):

$$
\begin{aligned}
y' &= F(t,y), \quad t\neq t_k,\\
y(\tau_0^{+}) &= y_0,\\
y(t_k^{+}) &= y(t_k)+I_{t_k}(y(t_k)), \quad \text{whenever } t_k\geq\tau_0,
\end{aligned}
\tag{26}
$$

where $F(t+\eta,y)=f(t,y)+R(t+\eta,y)$.

Definition 6. The solution $y(t+\eta,\tau_0,y_0)$ of the system (3) is said to be initial time difference Lipschitz stable (ITDLS) with respect to the solution $x(t,t_0,x_0)$ for $t \geq t_0$, where $x(t,t_0,x_0)$ is any solution of the system (1), if and only if there exists an $M = M(\tau_0)$ such that

$$\left\| y\left(t+\eta,\tau_0,y_0\right) - x\left(t,t_0,x_0\right)\right\| \leq M\left(\left\|y_0 - x_0\right\| + \tau_0 - t_0\right). \tag{27}$$

Theorem 7. *Let the following conditions be fulfilled:*

(B_1) *the assumptions of* Corollary 4 *hold;*

(B_2) *the zero solution of* (1) *is Lipschitz stable;*

(B_3) $\|\Phi(t,s,\tilde{y}(s))R(s+\eta,\tilde{y}(s))\| \leq \gamma(s)\|\tilde{y}(s)\|$ *for* $t_0 < s \leq t$;

(B_4) $\|\Phi(t,t_0,\sigma(s))\| \leq M_1(\|y_0 - x_0\| + \eta)/\|y_0 - x_0\|$ *and* M_1 *is a constant;*

(B_5) $\|I_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k))\| \leq \beta_k\|\tilde{y}(\bar{t}_k)\|$ *and* $\beta_k \geq 0$ *are constants;*

(B_6) $\|\Phi(t,\bar{t}_k,\tilde{y}(\bar{t}_k) + sI_{\bar{t}_k+\eta}(\tilde{y}(\bar{t}_k)))\| \leq \alpha_k$ *and* $\alpha_k \geq 0$ *are constants;*

(B_7) $\int_{t_0}^{\infty}\gamma(s)ds < \infty$, $\gamma(s) \in C[R^+,R^+]$ *and* $\prod_{t_0<\bar{t}_k<t}(1+\alpha_k\beta_k) < \infty$.

Then the solution $y(t+\eta,\tau_0,y_0)$ *of the system* (3) *is ITDLS with respect to the solution* $x(t,t_0,x_0)$.

Proof. From Corollary 4, it follows that

$$\begin{aligned}
&\tilde{y}(t) - x\left(t,t_0,x_0\right) \\
&\quad = \int_0^1 \Phi\left(t,t_0,\sigma(s)\right)\left(y_0 - x_0\right)ds \\
&\qquad + \int_{t_0}^{t}\Phi\left(t,s,\tilde{y}(s)\right)R\left(s+\eta,\tilde{y}(s)\right)ds \\
&\qquad + \sum_{t_0<\bar{t}_k<t}\int_0^1 \Phi\left(t,\bar{t}_k,\tilde{y}\left(\bar{t}_k\right) + sI_{\bar{t}_k+\eta}\left(\tilde{y}\left(\bar{t}_k\right)\right)\right)ds \\
&\qquad \cdot I_{\bar{t}_k+\eta}\left(\tilde{y}\left(\bar{t}_k\right)\right).
\end{aligned} \tag{28}$$

Taking the norm and using the triangle inequality on both sides, we have

$$\begin{aligned}
&\left\|\tilde{y}(t) - x\left(t,t_0,x_0\right)\right\| \\
&\quad \leq \int_0^1 \left\|\Phi\left(t,t_0,\sigma(s)\right)\right\|\left\|\left(y_0 - x_0\right)\right\|ds \\
&\qquad + \int_{t_0}^{t}\left\|\Phi\left(t,s,\tilde{y}(s)\right)R\left(s+\eta,\tilde{y}(s)\right)\right\|ds \\
&\qquad + \sum_{t_0<\bar{t}_k<t}\int_0^1 \left\|\Phi\left(t,\bar{t}_k,\tilde{y}\left(\bar{t}_k\right) + sI_{\bar{t}_k+\eta}\left(\tilde{y}\left(\bar{t}_k\right)\right)\right)\right\|ds \\
&\qquad \cdot \left\|I_{\bar{t}_k+\eta}\left(\tilde{y}\left(\bar{t}_k\right)\right)\right\|.
\end{aligned} \tag{29}$$

From conditions (B_2)–(B_5), we obtain

$$\begin{aligned}
&\left\|\tilde{y}(t) - x\left(t,t_0,x_0\right)\right\| \\
&\quad \leq M_1\left(\left\|y_0 - x_0\right\| + \eta\right) + \int_{t_0}^{t}\gamma(s)\left\|\tilde{y}(s)\right\|ds \\
&\qquad + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k\left\|\tilde{y}\left(\bar{t}_k\right)\right\|.
\end{aligned} \tag{30}$$

Setting $M^*(t) = \|\tilde{y}(t) - x(t,t_0,x_0)\|$, we have

$$\begin{aligned}
M^*(t) &\leq M_1\left(\left\|y_0 - x_0\right\| + \eta\right) + \int_{t_0}^{t}\gamma(s)M^*(s)ds \\
&\quad + \int_{t_0}^{t}\gamma(s)\left\|x\left(s,t_0,x_0\right)\right\|ds \\
&\quad + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k M^*\left(\bar{t}_k\right) + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k\left\|x\left(\bar{t}_k\right)\right\|.
\end{aligned} \tag{31}$$

Since $\|x(t,t_0,x_0)\| \leq M_2\|x_0\|$, as long as $\|x_0\| < \varepsilon$, then we have

$$\begin{aligned}
&\left\|x\left(t,t_0,x_0\right)\right\| \leq M_2\varepsilon, \\
&M^*(t) \leq M_1\left(\left\|y_0 - x_0\right\| + \eta\right) + \int_{t_0}^{t}\gamma(s)M^*(s)ds \\
&\qquad + M_2\varepsilon\int_{t_0}^{t}\gamma(s)ds + \sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k M^*\left(\bar{t}_k\right) \\
&\qquad + M_2\varepsilon\sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k.
\end{aligned} \tag{32}$$

Applying Gronwall's inequality to (32), we get

$$\begin{aligned}
&M^*(t) \\
&\quad \leq \left\{M_1\left(\left\|y_0 - x_0\right\| + \eta\right)\right. \\
&\qquad \left. + M_2\varepsilon\int_{t_0}^{t}\gamma(s)ds + M_2\varepsilon\sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k\right\} \\
&\qquad \cdot \prod_{t_0<\bar{t}_k<t}\left(1+\alpha_k\beta_k\right)\exp\left\{\int_{t_0}^{t}\gamma(s)ds\right\}.
\end{aligned} \tag{33}$$

Setting $M_3 = \{M_1 + (M_2\varepsilon\int_{t_0}^{t}\gamma(s)ds/\|y_0 - x_0\| + \eta) + (M_2\varepsilon\sum_{t_0<\bar{t}_k<t}\alpha_k\beta_k/\|y_0 - x_0\| + \eta)\}\prod_{t_0<\bar{t}_k<t}(1+\alpha_k\beta_k)\exp\{\int_{t_0}^{t}\gamma(s)ds\}$, we have

$$M^*(t) \leq M_3\left(\left\|y_0 - x_0\right\| + \eta\right). \tag{34}$$

From condition (B_7), it follows that the solution $y(t+\eta,\tau_0,y_0)$ of the system (3) is ITDLS with respect to the solution $x(t,t_0,x_0)$.

The proof is complete. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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