Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 738609, 6 pages
http://dx.doi.org/10.1155/2014/738609

Research Article

Generalized Bilinear Differential Operators, Binary Bell Polynomials, and Exact Periodic Wave Solution of Boiti-Leon-Manna-Pempinelli Equation

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Received 5 May 2014; Accepted 20 June 2014; Published 8 July 2014

Academic Editor: Tiecheng Xia

We introduce how to obtain the bilinear form and the exact periodic wave solutions of a class of (2 + 1)-dimensional nonlinear integrable differential equations directly and quickly with the help of the generalized D_p-operators, binary Bell polynomials, and a general Riemann theta function in terms of the Hirota method. As applications, we solve the periodic wave solution of BLMP equation and it can be reduced to soliton solution via asymptotic analysis when the value of p is 5.

1. Introduction

It is significantly important to research nonlinear evolution equations in exploring physical phenomena in depth [1, 2]. Since the soliton theory has been proposed, the research on seeking the exact solutions of the soliton equations has attracted great attention and made great progress. A series of methods have been proposed, such as Painléve test [3], Bäcklund transformation method [4, 5], Darboux transformation [6], inverse scattering transformation method [7], Lie group method [8, 9], and Hamiltonian method [10, 11]. Particularly, Hirota direct method [12, 13] provides a direct approach to solve a kind of specific bilinear differential equations among the exciting methods. As we all know, once the bilinear forms of nonlinear differential equations are obtained, we can construct the multisoliton solutions, the bilinear Bäcklund transformation, and Lax pairs easily. It is clear that the key of Hirota direct method is to find the bilinear forms of the given differential equations by the Hirota differential D-operators. Recently, Ma put forward generalized bilinear differential operators named D_p-operators in [14] which are used to create bilinear differential equations. Furthermore, different symbols are also used to furnish relations with Bell polynomials in [15], and even for trilinear equations in [16].

In this paper, we would like to explore the relations between multivariate binary Bell polynomials [17–19] and the D_p-operators and to find the bilinear form of Boiti-Leon-Manna-Pempinelli (BLMP) equation [20, 21]. Then, we can obtain the exact periodic wave solution [22–25] of the BLMP equation with the help of a general Riemann theta function in terms of Hirota method.

The paper is structured as follows. In Section 2, we will give a brief introduction about the difference between the Hirota differential D-operators and the generalized D_P-operators. In Section 3, we will explore the relations between multivariate binary Bell polynomials and the D_p-operators. In Section 4, we will use the relation in Section 2 to seek the differential form of the BLMP equation and then take advantage of the Riemann theta function [26, 27] and Hirota method to obtain its exact periodic wave solution which can be reduced to the soliton solution via asymptotic analysis.

2. Hirota Bilinear D-Operators and the Generalized D_p-Operators

It is known to us that Hirota bilinear D-operators play a significant role in Hirota direct method. The D-operators are defined in [14] as the following:

$$D_x^m D_t^n a(x,t)\, b(x,t) = \frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial y^n} \times a(x+y,t+s)b(x-y,t-s)\big|_{s=0,y=0}, \tag{1}$$

where $m,n = 0,1,2,\ldots$. Generally, we have

$$\begin{aligned} &D_x^n D_y^m D_t^s a(x,y,t)\, b(x',y',t') \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \\ &\quad \times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^s a(x,y,t)\, b(x',y',t')\Big|_{x'=x,y'=y,t'=t}, \end{aligned} \tag{2}$$

where $m,n,s = 0,1,2,\ldots$.

For instance, for the Boussinesq equation

$$u_{tt} - u_{xx} - 3\left(u^2\right)_{xx} - u_{xxxx} = 0, \tag{3}$$

under $u = 2(\ln F)_{xx}$, we have

$$\begin{aligned} &-F_t^2 + F_{tt}F + F_x^2 + \frac{3}{F^2}F_x^4 - F_{xx}F \\ &\quad - \frac{6}{F}F_x^2 F_{xx} + 4F_x F_{xxx} - F_{xxxx}F = 0; \end{aligned} \tag{4}$$

we can get its bilinear form with D-operators

$$\left(D_t^2 - D_x^2 - D_x^4\right) F\cdot F = 0. \tag{5}$$

However, based on the Hirota D-operators, Professor Ma put forward a kind of generalized bilinear D_p-operators in [14]:

$$\begin{aligned} &D_{p,x}^n D_{p,y}^m D_{p,t}^s f(x,y,t)\, g(x',y',t') \\ &= \left(\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} + \alpha\frac{\partial}{\partial y'}\right)^m \\ &\quad \times \left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial t'}\right)^s f(x,y,t)\, g(x',y',t')\Big|_{x'=x,y'=y,t'=t}, \end{aligned} \tag{6}$$

where, for an integer k, the kth power of α is defined by

$$\alpha^k = (-1)^r(k), \quad \text{if } k \equiv r(k) \bmod p \tag{7}$$

with $0 \leqslant r(k) < p$.

For example, if $p = 2k\,(k \in N)$, the D-operators are Hirota operators.

If $p = 5$, we have

$$\begin{aligned} &\alpha = -1, \quad \alpha^2 = 1, \quad \alpha^3 = -1, \quad \alpha^4 = \alpha^5 = 1, \\ &\alpha^6 = -1, \quad \alpha^7 = 1, \quad \alpha^8 = -1, \quad \alpha^9 = \alpha^{10} = 1, \\ &\vdots \end{aligned} \tag{8}$$

By (6) and (8), it is clear to see that

$$\begin{aligned} D_{5,x}D_{5,t} f\cdot g &= f_{xt}g - f_x g_t - f_t g_x + f g_{xt}, \\ D_{5,x}^2 f\cdot g &= f_{xx}g - 2f_x g_x + f g_{xx}, \\ D_{5,x}^4 f\cdot g &= f_{4x} - 4f_{3x}g_x + 6f_{xx}g_{xx} - 4f_x g_{3x} + f g_{4x}. \end{aligned} \tag{9}$$

Now, under $u = 2(\ln F)_{xx}$, the generalized bilinear Boussinesq equation can be expressed as

$$\left(D_{5,t}^2 - D_{5,x}^2 - D_{5,x}^4\right) F\cdot F = 0. \tag{10}$$

Then, we would like to discuss how to use the D-operator to seek the bilinear differential form of other nonlinear integrable differential equations with the help of binary Bell polynomial.

3. Binary Bell Polynomial

As we all know, Bell proposed three kinds of exponent-form polynomials. Later, Lambert, Gilson, and their partners generalized the third type of Bell polynomials in [28, 29] which is used mainly in this paper.

The multidimensional binary Bell polynomials which we will use are defined as follows:

$$\begin{aligned} Y_{n_1x_1,\ldots,n_lx_l}(y) &= Y_{n_1,\ldots,n_l}\left(y_{r_1},\ldots y_{r_l}\right) \\ &= e^{-y}\partial_{x_1}^{n_1},\ldots,\partial_{x_l}^{n_l}e^{y} \quad (n_1,\ldots,n_l \geq 0), \\ \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,\omega) &= Y_{n_1,\ldots,n_l}(y)\Big|_{(y_{r_1x_1,\ldots,r_lx_l}=(1/2)(1+(-1)^{r_1+\cdots+r_l})v_{r_1x_1,\ldots,r_1x_1}+(1/2)(1-(-1)^{r_1+\cdots+r_l})\omega_{r_1x_1,\ldots,r_1x_1})}, \end{aligned} \tag{11}$$

in which $y_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1},\ldots,\partial_{x_l}^{r_l}y(x_1,\ldots,x_l)$.

In that way, we have

$$\begin{aligned} y_x(v,\omega) &= v_x, \\ y_{2x}(v,\omega) &= v_x^2 + \omega_{xx}, \\ y_{x,y}(v,\omega) &= \omega_{xy} + v_x v_y. \end{aligned} \tag{12}$$

For convenience, we assume that

$$\begin{aligned} F &= e^{\xi(x_1,\ldots,x_l)}, \qquad G = e^{\eta(x_1,\ldots,x_l)}, \\ \xi &= \frac{\omega+v}{2}, \qquad \eta = \frac{\omega-v}{2} \end{aligned} \tag{13}$$

and read that

$$\begin{aligned} &(FG)^{-1} D_{p,x_1}^{n_1},\ldots,D_{p,x_l}^{n_l} F\cdot G \\ &\quad = G^{-1}(\eta) F^{-1}(\xi) D_{p,x_1}^{n_1},\ldots,D_{p,x_l}^{n_l} F(\xi)\cdot G(\eta) \\ &\quad = \sum_{k_1=0}^{n_1}\cdots\sum_{k_l=0}^{n_l}\prod_{i=1}^{l}\alpha^{k_i}\binom{n_i}{k_i}\left(e^{-\xi}\partial_{x_1}^{n_1-k_1},\ldots,\partial_{x_l}^{n_l-k_l}e^{\xi}\right) \\ &\qquad\qquad \times\left(e^{-\eta}\partial_{x_1}^{n_1-k_1},\ldots,\partial_{x_l}^{n_l-k_l}e^{\eta}\right) \\ &\quad = \sum_{k_1=0}^{n_1}\cdots\sum_{k_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{k_i} Y_{n_1-k_1,\ldots,n_l-k_l}(\xi), Y_{k_1,\ldots,k_l}\left(\alpha^{\sum_{r_1}^{r_l} r_i}\eta\right) \\ &\quad = Y_{n_1,\ldots,n_l}\left(y_{r_1,\ldots,r_l} = \xi_{r_1,\ldots,r_l} + \alpha^{\sum_{r_1}^{r_l} r_i}\eta_{r_1,\ldots,r_l}\right) \\ &\quad = Y_{n_1,\ldots,n_l}\left(y_{r_1,\ldots,r_l} = \frac{1}{2}\left(1+\alpha^{\sum_{r_1}^{r_l} r_i}\right)v_{r_1,\ldots,r_l}\right. \\ &\qquad\qquad \left. + \frac{1}{2}\left(1+\alpha^{\sum_{r_1}^{r_l} r_i}\right)\omega_{r_1,\ldots,r_l}\right) \\ &\quad = \mathscr{Y}_{p;n_1x_1,\ldots,n_lx_l}(v,\omega). \end{aligned} \tag{14}$$

We find that the link between $\mathscr{Y}$-polynomials and the D_p-operator can be given as the following through the above deduction:

$$\begin{aligned} &\mathscr{Y}_{p;n_1x_1,\ldots,n_lx_l}\left(v = \frac{\ln F}{G}, \omega = \ln FG\right) \\ &\quad = Y_{n_1,\ldots,n_l}\left(y_{r_1,\ldots,r_l} = \frac{1}{2}\left(1+\alpha^{\sum_{r_1}^{r_l} r_i}\right)vr_1,\ldots,r_l\right. \\ &\qquad\qquad \left. + \frac{1}{2}\left(1+\alpha^{\sum_{r_1}^{r_l} r_i}\right)\omega r_1,\ldots,r_l\right). \end{aligned} \tag{15}$$

Particularly, when $F = G$, we define $\mathscr{P}$-polynomials by

$$\mathscr{P}_{p;n_1x_1,\ldots,n_lx_l}(q) = \mathscr{Y}_{p;n_1x_1,\ldots,n_lx_l}(v=0, \omega = 2\ln F = q). \tag{16}$$

When $p = 5$, we can obtain that

$$\begin{aligned} P_{5;y,t} &= q_{xt}, \\ P_{5;2x} &= q_{xx}, \\ P_{5;4x} &= 3q_{2x}^2 + q_{4x}, \\ P_{5;3x,y} &= q_{3xy} + 3q_{xx}q_{xy}, \\ P_{5;2x,t} &= 0. \end{aligned} \tag{17}$$

Let us now utilize the $\mathscr{P}$-polynomials given above to seek the bilinear form of BLMP equation with the D_p-operators.

4. Boiti-Leon-Manna-Pempinelli Equation

In this section, firstly, we will give the bilinear form of BLMP equation with the help of $\mathscr{P}$-polynomials and the D_p-operators. And then, we construct the exact periodic wave solution of BLMP equation with the aid of the Riemann theta function, Hirota direct method, and the special property of the D_p-operators when acting on exponential functions.

4.1. Bilinear Form. BLMP equation can be written as

$$u_{y,t} + u_{3x,y} - 3u_{2x}u_y - 3u_x u_{x,y} = 0. \tag{18}$$

Setting $u = -q_x$, inserting it into (18), and integrating with respect to x yields

$$q_{y,t} + q_{3x,y} + 3q_{xx}q_{x,y} - \lambda = 0, \tag{19}$$

where λ is an integral constant.

Based on (17), (19) can be expressed as

$$P_{5;y,t}(q) + P_{5;3x,y}(q) - \lambda = 0. \tag{20}$$

From the above, we can get the bilinear form of (18):

$$\left(D_{5,y}D_{5,t} + D_{5,x}^3 D_{5,y}\right)F\cdot F - \lambda\cdot F^2 = 0 \tag{21}$$

with $q = 2\ln F$.

4.2. Periodic Wave Solutions. When acting on exponential functions, we find that D_p-operators have a good property

$$\begin{aligned} &H\left(D_{p,x_1},\ldots,D_{p,x_l}\right)e^{\xi_1}\cdot e^{\xi_2} \\ &\quad = H\left(k_1+\alpha k_2, l_1+\alpha l_2, w_1+\alpha w_2\right)e^{\xi_1+\xi_2}, \end{aligned} \tag{22}$$

if we assume that

$$\xi_i = k_i x + l_i y + w_i t + \xi_i^{(0)} \quad i = 1,2,\ldots. \tag{23}$$

As a result of the property above, we consider Riemann's theta function solution of (18):

$$F = \sum_{n=-\infty}^{\infty} e^{2\pi i n\eta + \pi i n^2\tau}, \tag{24}$$

where $n \in Z, \tau \in C, \operatorname{Im}\tau > 0, \eta = kx + ly + wt$, with k, l, and w being constants to be determined.

Then, we have

$$\begin{aligned}
&H\left(D_{p,x}, D_{p,y}, D_{p,t}\right) F \cdot F \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} H\left(D_{p,x}, D_{p,y}, D_{p,t}\right) e^{2\pi i n\eta + \pi i n^2 \tau} e^{2\pi i m\eta + \pi i m^2 \tau} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} H\left(2\pi i (n+\alpha m) k, 2\pi i (n+\alpha m) l,\right. \\
&\qquad \left. 2\pi i (n+\alpha m) w\right) e^{2\pi i (n+m)\eta + \pi i (n^2+m^2)\tau} \\
&= \sum_{q=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} H\left(\left(2\pi i (1-\alpha) n - \alpha^2 q\right) k,\right.\right. \\
&\qquad \left(2\pi i (1-\alpha) n - \alpha^2 q\right) l, \\
&\qquad \left.\left(2\pi i (1-\alpha) n - \alpha^2 q\right) w\right) \\
&\qquad \left. \times e^{\pi i (n^2 + (n+\alpha q)^2)\tau} \right\} e^{2\pi i (-\alpha q)\eta} \\
&= \sum_{q=-\infty}^{\infty} \overline{H}(q) e^{2\pi i (-\alpha q)\eta},
\end{aligned} \tag{25}$$

where $q = -(1/\alpha)(m+n)$.

To the bilinear form of BLMP equation, $\overline{H}(q)$ satisfies the period characters when $p = 5$. The powers of α obey rule (7), noting that

$$\begin{aligned}
\overline{H}(q) &= \sum_{n=-\infty}^{\infty} H\left(\left(2\pi i (1-\alpha) n - \alpha^2 q\right) k,\right. \\
&\qquad \left(2\pi i (1-\alpha) n - \alpha^2 q\right) l, \\
&\qquad \left.\left(2\pi i (1-\alpha) n - \alpha^2 q\right) w\right) e^{\pi i (n^2 + (n+\alpha q)^2)\tau} \\
&= \sum_{n=-\infty}^{\infty} H\left(2\pi i (2n-q) k, 2\pi i (2n-q) l,\right. \\
&\qquad \left. 2\pi i (2n-q) w\right) e^{\pi i (n^2 + (q-n)^2)\tau} \\
&= \sum_{h=-\infty}^{\infty} H\left(2\pi i (2h - (q-2)) k,\right. \\
&\qquad \left. 2\pi i (2h - (q-2)) l, 2\pi i (2h - (q-2)) w\right) \\
&\qquad \cdot e^{\pi i (h^2 + (q-h-2)^2)\tau} \cdot e^{2\pi i (q-1)\tau} \\
&= \overline{H}(q-2) e^{2\pi i (q-1)\tau},
\end{aligned} \tag{26}$$

where $h = n + \alpha$.

From (26) we can infer that

$$\overline{H}(q) = \begin{cases} \overline{H}(0) e^{\pi i n q \tau}, & q = 2n; \\ \overline{H}(1) e^{\pi i (2n+2n^2)(q+1)\tau}, & q = 2n+1. \end{cases} \tag{27}$$

For (21), we may let

$$\begin{aligned}
\overline{H}(0) &= \sum_{n=-\infty}^{\infty} \left\{ [2\pi i (1-\alpha) n]^2 l \cdot w + [2\pi i (1-\alpha) n k]^3 \right. \\
&\qquad \left. \cdot 2\pi i (1-\alpha) n l - \lambda \right\} e^{2\pi i n^2 \tau} \\
&= \sum_{n=-\infty}^{\infty} \left(-16 n^2 \pi^2 l w + 25 n^4 \pi^4 k^3 l - \lambda\right) e^{2\pi i n^3 \tau} \\
&= 0, \\
\overline{H}(1) &= \sum_{n=-\infty}^{\infty} \left\{ 2\pi \left(i (1-\alpha) n - \alpha^2\right) l \right. \\
&\qquad \cdot 2\pi \left(i (1-\alpha) n - \alpha^2\right) w \\
&\qquad + \left[2\pi \left(i (1-\alpha) n - \alpha^2\right) k\right]^3 \\
&\qquad \left. \cdot 2\pi \left(i (1-\alpha) n - \alpha^2\right) l - \lambda \right\} \\
&\qquad \times e^{2\pi i (n^2 - 2n + 1)\tau} \\
&= \sum_{n=-\infty}^{\infty} \left[-4(2n-1)^2 \pi^2 l w + 16(2n-1)^4 \pi^4 k^3 l - \lambda\right] \\
&\qquad \times e^{\pi i (2n^2 - 2n + 1)\tau} = 0.
\end{aligned} \tag{28}$$

Also, the powers of α obey rule (7). For the sake of computational convenience, we denote that

$$\begin{aligned}
g_1(n) &= e^{2\pi i n^2 \tau}, \\
a_{11} &= \sum_{n=-\infty}^{\infty} -16 n^2 \pi^2 l g_1(n), \\
a_{12} &= \sum_{n=-\infty}^{\infty} \left(256 n^4 \pi^4 k^3 l\right) g_1(n), \\
a_{13} &= \sum_{n=-\infty}^{\infty} g_1(n); \\
g_2(n) &= e^{\pi i (2n^2 - 2n + 1)\tau}, \\
a_{21} &= \sum_{n=-\infty}^{\infty} -4(2n-1)^2 \pi^2 l g_2(n), \\
a_{22} &= \sum_{n=-\infty}^{\infty} \left(16(2n-1)^4 \pi^4 k^3 l\right) g_2(n), \\
a_{23} &= \sum_{n=-\infty}^{\infty} g_2(n).
\end{aligned} \tag{29}$$

By (28), (29), and (30), we can get that

$$a_{11}\omega + a_{12} - \lambda a_{13} = 0,$$
$$a_{21}\omega + a_{22} - \lambda a_{23} = 0. \tag{30}$$

In view of (30), it is easy to see that

$$\omega = \frac{a_{13}a_{22} - a_{23}a_{12}}{a_{11}a_{23} - a_{13}a_{21}},$$
$$\lambda = \frac{a_{12}a_{21} - a_{11}a_{22}}{a_{11}a_{23} - a_{13}a_{21}}. \tag{31}$$

Thus, we obtain the periodic wave solution of BLMP equation:

$$u = -2(\ln F)_x, \tag{32}$$

where F is given by (24) and ω, λ are satisfied with (31).

Then, assuming $e^{\pi i\tau} = \gamma$, based on (29), we may obtain that

$$\begin{aligned}
a_{11} &= \sum_{n=-\infty}^{\infty} -16n^2\pi^2 l \cdot e^{2\pi i n^2\tau} \\
&= -32\pi^2 l\left(\gamma^2 + 4\gamma^8 + 9\gamma^{18} + \cdots\right), \\
a_{12} &= \sum_{n=-\infty}^{\infty} 256n^4\pi^4k^3 l \cdot e^{2\pi i n^2\tau} \\
&= 2\times 256\pi^4k^3 l\left(\gamma^2 + 4\gamma^8 + 9\gamma^{18} + \cdots\right), \\
a_{13} &= \sum_{n=-\infty}^{\infty} e^{2\pi i n^2\tau} = 1 + 2\gamma^2 + 2\gamma^8 + 2\gamma^{18} + \cdots, \\
a_{21} &= \sum_{n=-\infty}^{\infty} -4(2n-1)^2\pi^2 l \cdot e^{\pi i(2n^2-2n+1)\tau} \\
&= -8\pi^2 l\left(\gamma + 9\gamma^5 + 25\gamma^{13} + \cdots\right), \\
a_{22} &= \sum_{n=-\infty}^{\infty} 16(2n-1)^4\pi^4k^3 l \cdot e^{\pi i(2n^2-2n+1)\tau} \\
&= 32\pi^4k^3 l\left(\gamma + 3^4\gamma^5 + 5^4\gamma^{13} + \cdots\right), \\
a_{23} &= \sum_{n=-\infty}^{\infty} e^{\pi i(2n^2-2n+1)\tau} = 1 + 2\gamma + 2\gamma^5 + 2\gamma^{13} + \cdots
\end{aligned} \tag{33}$$

which lead to

$$\begin{aligned}
a_{11}a_{23} - a_{21}a_{13} &= 8\pi^2 l\gamma + o(\gamma), \\
a_{13}a_{22} - a_{12}a_{23} &= 32\pi^4k^3 l\gamma + o(\gamma), \\
a_{11}a_{22} - a_{12}a_{21} &= o(\gamma).
\end{aligned} \tag{34}$$

So, we have $\omega \to 4\pi^2k^3$ and $\lambda \to 0$, as $\gamma \to 0$.

It is interesting that if we set

$$k_1 = 2\pi i k, \quad l_1 = 2\pi i l, \quad w_1 = 2\pi i w,$$
$$\eta_1 = k_1 x + l_1 y + w_1 t + \pi i\tau, \tag{35}$$

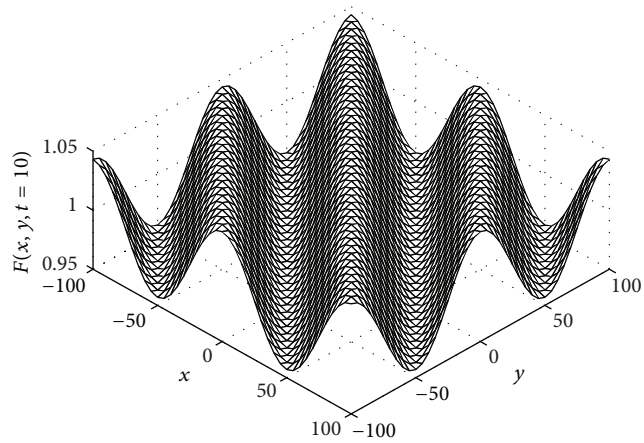

Figure 1: Solution of (18).

which can infer that

$$\begin{aligned}
F &= \sum_{n=-\infty}^{\infty} e^{2\pi i\eta + \pi i n^2\tau}, \\
&= 1 + e^{\pi i\eta}\left(e^{2\pi i\eta} + e^{-2\pi i\eta}\right) + e^{4\pi i\eta}\left(e^{4\pi i\eta} + e^{-4\pi i\eta}\right) + \cdots, \\
&= 1 + e^{\eta_1} + \gamma^2\left(e^{\eta_1} + e^{2\eta_1}\right) + \gamma^6\left(e^{-2\eta_1} + e^{3\eta_1}\right) + \cdots, \\
&\longrightarrow 1 + e^{\eta_1} \quad (\eta_1 \longrightarrow 0).
\end{aligned} \tag{36}$$

From all the above, it can be proved that the periodic wave solution (32) just goes to the soliton solution

$$w_1 = 2\pi i w \longrightarrow -k_1^3. \tag{37}$$

Thus, if we assume that $k = 0.01$, $l = 0.01$, and $\tau = i$ to the solution

$$F = 1 + e^{2\pi i(kx + ly + 4\pi^2k^3t + \tau)}, \tag{38}$$

solution of (18) can be shown in Figure 1.

5. Conclusions and Remarks

In this paper, we obtain the bilinear form of bilinear differential equations by applying the D_p-operators and binary Bell polynomials, which has proved to be a quick and direct method. Furthermore, together with Riemann theta function and Hirota method, we successfully get the exact periodic wave solution and figure of BLMP equation when $p = 5$.

There are many other interesting questions on bilinear differential equations, for example, how to apply the generalized operators into the discrete equations; it is known that researches on the discrete and differential equations are also significant. Besides, we will try to explore other operators to construct more nonlinear evolution equations simply and directly in the near future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by National Natural Science Foundation of China (no. 11271007), Nature Science Foundation of Shandong Province of China (no. ZR2013AQ017), the Strategic Pioneering Program of Chinese Academy of Sciences (no. XDA 10020104), SDUST Research Fund (no. 2012 KYTD105), Open Fund of the Key Laboratory of Ocean Circulation and Waves, Chinese Academy of Science (no. KLOCAW1401), and Graduate Innovation Foundation from Shandong University of Science and Technology (no. YC130321).

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