

Research Article

New Iteration Methods for Time-Fractional Modified Nonlinear Kawahara Equation

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We put side by side the methodology of two comparatively new analytical techniques to get to the bottom of the system of nonlinear fractional modified Kawahara equation. The technique is described and exemplified with a numerical example. The dependability of both methods and the lessening in computations give these methods a wider applicability. In addition, the computations implicated are very simple and undemanding.

1. Introduction

Within the scope of fractional calculus in the recent decade several scholars have modeled physical and engineering problems. Respective scholar while dealing with real world problems found out that it is worth describing these phenomena with the idea of derivatives with fractional order. While searching the literature, we found out that, this concept of noninteger order derivative not only has been intensively used but also has played an essential role in assorted branches of sciences including but not limited to hydrology, chemistry, image processing, electronics and mechanics; the applicability of this philosophy can be found in [1–10]. In the foregone respective decennial, the research of travelling-wave solutions for nonlinear equations has played a crucial character in the examination of nonlinear physical phenomena.

Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction, and convection are very important in nonlinear wave equations. Concepts like solitons, peakons, kinks, breathers, cusps, and compactons have now been thoroughly investigated in the scientific literature [11–13].

Various powerful mathematical methods such as the inverse scattering method, bilinear transformation [14], the tanh-sech method [15, 16], extended tanh method [16], Exp-function method [17–19], sine-cosine method [20] Adomian decomposition method [21], Exp-function method [22], homotopy perturbation method [23] have been proposed for obtaining exact and approximate analytical solutions.

The purpose of this paper is to examine the approximated solution of the nonlinear fractional modified Kawahara equation, using the relatively new analytical method, the Homotopy decomposition method (HDM), and the Sumudu transform method. The fractional partial differential equations under investigation here are given below as

$$\begin{aligned} \partial_t^\alpha u(x, t) + u^2(x, t) u_x(x, t) + p u_{xx}(x, t) \\ + q u_{xxx}(x, t) = 0, \quad 0 < \alpha \leq 1, \end{aligned} \quad (1)$$

subject to the initial condition

$$u(x, 0) = \frac{3p}{\sqrt{-10q}} \operatorname{sech}[Kx]^2, \quad K = \frac{1}{2} \sqrt{\frac{-p}{5q}}. \quad (2)$$

The outstanding of this paper is prearranged as follows. In Section 2 we present a succinct history of the fractional derivative order and their properties. We present the basic ideal of the HDM and the STM for solving high order nonlinear fractional partial differential equations. We present their application to fractional nonlinear differential equations (1) and (2) and numerical results in Section 4. The conclusions are then given in Section 5.

2. Fractional Derivative Order

2.1. Brief History. There exists a vast literature on different definitions of fractional derivatives [24–27]. The most popular ones are the Riemann-Liouville and the Caputo derivatives. For Caputo we have

$${}_0^C D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt. \quad (3)$$

For the case of Riemann-Liouville we have the following definition:

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt. \quad (4)$$

Each fractional derivative presents some advantages and disadvantages [24–27], Jumarie (see [28, 29]) proposed a simple alternative definition to the Riemann-Liouville derivative.

Consider

$$\begin{aligned} D_x^\alpha (f(x)) \\ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t) - f(0)\} dt. \end{aligned} \quad (5)$$

2.2. Properties and Definitions

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0, \infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \quad (6)$$

Properties of the operator can be found in [26, 27], we mention only the following.

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$:

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+\beta} f(x), \quad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \\ J^\alpha x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{aligned} \quad (7)$$

Lemma 3. If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, $\mu \geq -1$, then

$$\begin{aligned} D^\alpha J^\alpha f(x) &= f(x), \\ J^\alpha D_0^\alpha f(x) &= f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \end{aligned} \quad (8)$$

$x > 0$.

Definition 4 (partial derivatives of fractional order). Assume now that $f(\mathbf{x})$ is a function of n variables, x_i , $i = 1, \dots, n$, also of class C on $D \in \mathbb{R}_n$. As an extension of Definition 3 we define partial derivative of order α for $f(x)$ with respect to x_i using the function.

$$a \partial_{\mathbf{x}}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_a^{x_i} (x_i-t)^{m-\alpha-1} \partial_{x_i}^m f(x_j) |_{x_j=t} dt. \quad (9)$$

If it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer order m .

3. Basic Information Regarding the Methodology of the HDM [32–35]

To illustrate the basic idea of this method we consider a general nonlinear nonhomogeneous fractional p differential equation with initial conditions of the following form [30]

$$\frac{\partial^\alpha V(x, t)}{\partial t^\alpha} = L(V(x, t)) + N(V(x, t)) + g(t), \quad \alpha > 0 \quad (10)$$

subject to the initial condition

$$\begin{aligned} D_0^{\alpha-k} U(x, 0) &= f_k, \quad (k = 0, \dots, n-1), \\ D_0^{\alpha-n} V(x, 0) &= 0, \quad n = [\alpha], \\ D_0^k U(x, 0) &= h_k, \quad (k = 0, \dots, n-1), \\ D_0^n V(x, 0) &= 0, \quad n = [\alpha], \end{aligned} \quad (11)$$

where, $\partial^\alpha / \partial t^\alpha$ denotes the Caputo or Riemann-Liouville fraction derivative operator, g is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator [30]. The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator $\partial^\alpha / \partial t^\alpha$ on both sides of (10) to obtain the following: In the case of Riemann-Liouville fractional derivative,

$$\begin{aligned} V(x, t) &= \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha-j+1)} t^{\alpha-j} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[L(V(x, \tau)) + N(V(x, \tau)) \right. \\ &\quad \left. + g(x, \tau) \right] d\tau. \end{aligned} \quad (12)$$

In the case of Caputo fractional derivative,

$$U(x, t) = \sum_{j=1}^{n-1} \frac{h_j(x)}{\Gamma(\alpha - j + 1)} t^j + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[L(V(x, \tau)) + N(V(x, \tau)) + g(x, \tau) \right] d\tau. \quad (13)$$

Or in general by putting

$$\sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha - j + 1)} t^{\alpha-j} = f(x, t) \quad (14)$$

or

$$f(x, t) = \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j. \quad (15)$$

We obtain

$$V(x, t) = T(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[L(V(x, \tau)) + N(V(x, \tau)) + g(x, \tau) \right] d\tau. \quad (16)$$

In the homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p [30]

$$V(x, t, p) = \sum_{n=0}^{\infty} p^n U_n(x, t), \quad (17)$$

$$V(x, t) = \lim_{p \rightarrow 1} V(x, t, p), \quad (18)$$

and the nonlinear term can be decomposed as

$$NV(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(V), \quad (19)$$

where $p \in (0, 1]$ is an embedding parameter. $\mathcal{H}_n(U)$ is the He's polynomials that can be generated by

$$\mathcal{H}_n(V_0, \dots, V_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{j=0}^{\infty} p^j V_j(x, t) \right) \right], \quad n = 0, 1, 2, \dots \quad (20)$$

The homotopy decomposition method is obtained by the combination of homotopy technique with Abel integral and is given by [30]

$$\begin{aligned} & \sum_{n=0}^{\infty} p^n V_n(x, t) - F(x, t) \\ &= \frac{p}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[g(x, \tau) + L \left(\sum_{n=0}^{\infty} p^n V_n(x, \tau) \right) + N \left(\sum_{n=0}^{\infty} p^n V_n(x, \tau) \right) \right] d\tau. \end{aligned} \quad (21)$$

Comparing the terms of same powers of p gives solutions of various orders with the first term:

$$V_0(x, t) = F(x, t). \quad (22)$$

Theorem 5 (see [31]). Assuming that $X \times T \subset \mathbb{R} \times \mathbb{R}^+$ is a Banach space with a well-defined norm $\|\cdot\|$, over which the series sequence of the approximate solution of (10) is defined, and the operator $G(U_n(x, t)) = U_{n+1}(x, t)$ defining the series solution of (14) satisfies the Lipschitzian conditions, that is, $\|G(U_k^*) - G(U_k)\| \leq \varepsilon \|U_k^*(x, t) - U_k(x, t)\|$ for all $(x, t, k) \in X \times T \times \mathbb{N}$, then series solution obtained (17) is unique.

Proof (see [31]). Assume that $U(x, t)$ and $U^*(x, t)$ are the series solution satisfying (10); then

$$U^*(x, t, p) = \sum_{n=0}^{\infty} p^n U_n^*(x, t), \quad (23)$$

with initial guess $T(x, t)$

$$U(x, t, p) = \sum_{n=0}^{\infty} p^n U_n(x, t) \quad (24)$$

also with initial guess $T(x, t)$ therefore

$$\|U_n^*(x, t) - U_n(x, t)\| = 0, \quad n = 0, 1, 2, \dots \quad (25)$$

By the recurrence for $n = 0$, $U_n^*(x, t) = U_n(x, t) = T(x, t)$, assume that for $n > k \geq 0$, $\|U_k^*(x, t) - U_k(x, t)\| = 0$. Then

$$\begin{aligned} \|U_{k+1}^*(x, t) - U_{k+1}(x, t)\| &= \|G(U_k^*) - G(U_k)\| \\ &\leq \varepsilon \|U_k^*(x, t) - U_k(x, t)\| = 0, \end{aligned} \quad (26)$$

which completes the proof. \square

4. Background of Sumudu Transform

Definition 6 (see [34]). The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by

$$S(f(t)) = F_s(u) = \int_0^{\infty} \frac{1}{u} \exp\left[-\frac{t}{u}\right] f(t) dt. \quad (27)$$

Theorem 7 (see [35]). Let $G(u)$ be the Sumudu transform of $f(t)$ such that

- (i) $G(1/s)/s$ is a meromorphic function, with singularities having $\text{Re}[s] \leq \gamma$,
- (ii) there exists a circular region Γ with radius R and positive constants M and K with $|G(1/s)/s| < MR^{-K}$; then the function $f(t)$ is given by

$$\begin{aligned} S^{-1}(G(s)) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \exp[st] G\left(\frac{1}{s}\right) \frac{ds}{s} \\ &= \sum \text{residual} \left[\exp[st] \frac{G(1/s)}{s} \right]. \end{aligned} \quad (28)$$

For the proof see [36].

4.1. Basics of the Sumudu Transform Method. We illustrate the basic idea of this method [34–41], by considering a general fractional nonlinear nonhomogeneous partial differential equation with the initial condition of the form

$$D_t^\alpha U(x, t) = L(U(x, t)) + N(U(x, t)) + f(x, t), \quad \alpha > 0, \quad (29)$$

subject to the initial condition

$$\begin{aligned} D_0^k U(x, 0) &= g_k, \quad (k = 0, \dots, n-1), \\ D_0^n U(x, 0) &= 0, \quad n = [\alpha], \end{aligned} \quad (30)$$

where D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Applying the Sumudu transform on both sides of (29), we obtain

$$\begin{aligned} S[D_t^\alpha U(x, t)] &= S[L(U(x, t))] + S[N(U(x, t))] \\ &\quad + S[f(x, t)]. \end{aligned} \quad (31)$$

Using the property of the Sumudu transform, we have

$$\begin{aligned} S[U(x, t)] &= u^\alpha S[L(U(x, t))] + u^\alpha S[N(U(x, t))] \\ &\quad + u^\alpha S[f(x, t)] + g(x, t). \end{aligned} \quad (32)$$

Now applying the Sumudu inverse on both sides of (19) we obtain

$$\begin{aligned} U(x, t) &= S^{-1}[u^\alpha S[L(U(x, t))] + u^\alpha S[N(U(x, t))]] \\ &\quad + G(x, t) \end{aligned} \quad (33)$$

$G(x, t)$ represents the term arising from the known function $f(x, t)$ and the initial conditions.

Now we apply the HPM:

$$U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t). \quad (34)$$

The nonlinear term can be decomposed as follows:

$$NU(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U) \quad (35)$$

using the He's polynomial $\mathcal{H}_n(U)$ given as

$$\begin{aligned} \mathcal{H}_n(U_0, \dots, U_n) \\ = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{j=0}^{\infty} p^j U_j(x, t) \right) \right], \quad n = 0, 1, 2, \dots \end{aligned} \quad (36)$$

Substituting (35) and (36),

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, t) &= G(x, t) \\ &\quad + p \left[S^{-1} \left[u^\alpha S \left[L \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) \right] \right. \right. \\ &\quad \left. \left. + u^\alpha S \left[N \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) \right] \right] \right], \end{aligned} \quad (37)$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p , the following approximations are obtained:

$$\begin{aligned} p^0 : U_0(x, t) &= G(x, t), \\ p^1 : U_1(x, t) &= S^{-1} [u^\alpha S [L(U_0(x, t)) + H_0(U)]], \\ p^2 : U_2(x, t) &= S^{-1} [u^\alpha S [L(U_1(x, t)) + H_1(U)]], \\ p^3 : U_3(x, t) &= S^{-1} [u^\alpha S [L(U_2(x, t)) + H_2(U)]], \\ p^n : U_n(x, t) &= S^{-1} [u^\alpha S [L(U_{n-1}(x, t)) + H_{n-1}(U)]]. \end{aligned} \quad (38)$$

Finally, we approximate the analytical solution $U(x, t)$ by truncated series

$$U(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N U_n(x, t). \quad (40)$$

The above series solutions generally converge very rapidly.

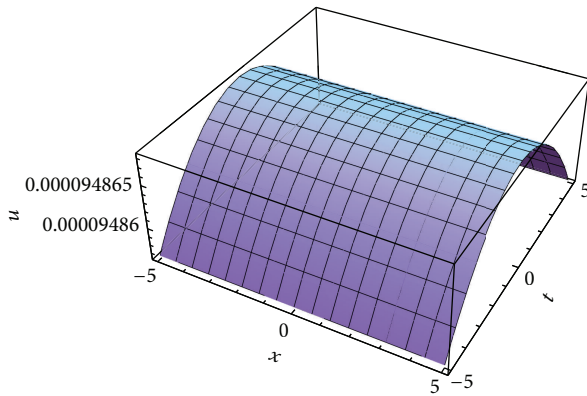


FIGURE 1: Approximated solutions for $\alpha = 0.45$.

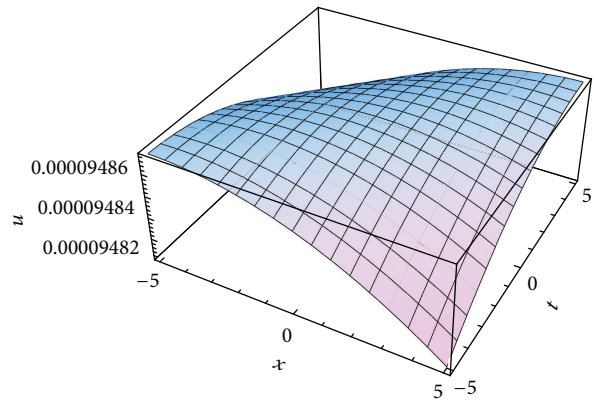


FIGURE 3: Exact solution for $\alpha = 1$.

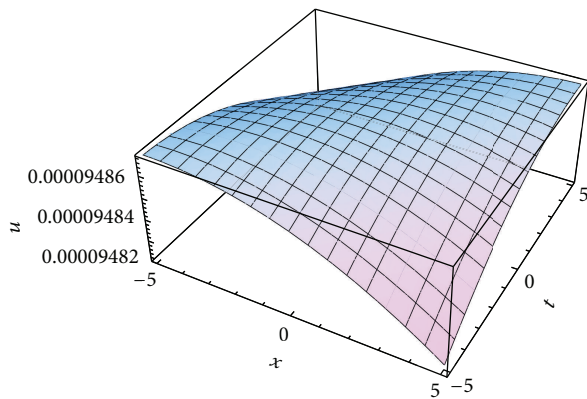


FIGURE 2: Approximated solution for $\alpha = 0.98$.

5. Application

5.1. Application with HDM. In this section we apply this method for solving nonlinear of fractional differential equation (1). Following the steps involve in the HDM, we arrive at the following equation:

$$\begin{aligned} & \sum_{n=0}^{\infty} p^n u_n(x, t) \\ &= u(x, 0) \\ & - \frac{p}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \\ & \times \left(\left(\sum_{n=0}^{\infty} p^n u_n(x, t) \right)^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, t) \right)_x \right. \\ & \quad + P \left(\sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xx} \\ & \quad \left. + q \left(\sum_{n=0}^{\infty} p^n u_n(x, t) \right)_{xxxx} \right) d\tau. \end{aligned} \quad (41)$$

Now comparing the terms of the same power of p we arrive at the following integral equations:

$$\begin{aligned} p^0: u_0(x, t) &= u(x, 0), \quad u_0(x, 0) = u(x, 0), \\ p^1: u_1(x, t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \\ & \times \left(u_0^2(u_0)_x + P(u_0)_{xx} \right. \\ & \quad \left. + q(u_0)_{xxxx} \right) d\tau, \quad u_1(x, 0) = 0, \\ p^n: u_n(x, t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \\ & \times \left(\sum_{j=0}^{n-1} \sum_{k=0}^j u_j u_{j-k} (u_{n-j-1})_x \right. \\ & \quad \left. + P(u_{n-1})_{xx} + q(u_{n-1})_{xxxx} \right) d\tau, \\ & u_n(x, 0) = 0, \quad n \geq 2. \end{aligned} \quad (42)$$

The following solutions are straightforward obtained:

$$u_0(x, t) = \frac{3P}{\sqrt{-10q}} (\operatorname{sech}(Kx))^2. \quad (43)$$

For simplicity we put $a = 3P/\sqrt{-10q}$

$u_1(x, t)$

$$= \frac{3PKt^\alpha (\operatorname{sech}(Kx))^2}{4\Gamma(1+\alpha)\sqrt{-10q}} \\ \times \left(8K(P - 20K^2q) \cosh(Kx) \right. \\ \left. + K(P + 100K^2q) \cosh(3Kx) \right. \\ \left. - K(P + 4K^2q) \cosh(5Kx) + 8a^2 \sinh(Kx) \right),$$

$u_2(x, t)$

$$= \frac{1}{\Gamma^2(\alpha+1)\Gamma(0.5+\alpha)\Gamma(1+3\alpha)} \\ \times \left(2^{-5-2\alpha} a K t^\alpha (\operatorname{sech}(Kx))^{12} \right. \\ \times \left(\Gamma(1+\alpha)\Gamma(1+3\alpha) \right. \\ \times \left(4^{3+\alpha} a^2 (\cosh(Kx))^5 \Gamma(0.5+\alpha) \sinh(Kx) \right. \\ \left. + K \sqrt{\pi} t^\alpha (-64a^4 + 276(KP)^2 \right. \\ \left. - 20832K^4Pq + 1087296K^6q^2 \right. \\ \left. + 2(32a^4 + 165(KP)^2 - 7224K^4Pq \right. \\ \left. - 45360K^6q^2) \cosh(2Kx) \right) \\ \left. - 768K^4q(-17P + 1240K^2q) \cosh(4Kx) \right. \\ \left. - 75K^2P^2 \cosh(6Kx) \right. \\ \left. + 5736K^4Pq \cosh(6Kx) \right. \\ \left. + 217680K^6q^2 \cosh(6Kx) \right. \\ \left. - 20K^2P^2 \cosh(8Kx) \right. \\ \left. - 928K^4Pq \cosh(8Kx) \right. \\ \left. - 8000K^6q^2 \cosh(8Kx) \right. \\ \left. + K^2P^2 \cosh(10Kx) \right. \\ \left. + 8K^4Pq \cosh(10Kx) \right. \\ \left. + 16K^6q^2 \cosh(6Kx) \right. \\ \left. + 592a^2KP \sinh(2Kx) \right. \\ \left. - 86720a^2K^3q \sinh(2Kx) \right. \\ \left. + 176a^2KP \sinh(4Kx) \right. \\ \left. + 35264a^2K^3q \sinh(4Kx) \right. \\ \left. - 80a^2KP \sinh(6Kx) \right. \\ \left. - 2624a^2K^3q \sinh(6Kx) \right. \\ \left. - 2a^2K^2t^{2\alpha}\Gamma(0.5+\alpha) \right. \\ \left. \times \Gamma(1+2\alpha) (\operatorname{sech}(Kx))^4 \right. \\ \left. \times \left(4^{3+\alpha} a^4 - 33 \times 2^{1+2\alpha} (KP)^2 + 147 \right. \right. \\ \left. \left. \times 4^{2+\alpha} K^4Pq - 1113 \times 2^{5+2\alpha} K^6q^2 \right) \right. \\ \left. - 2^{1+2\alpha} \left((32a^4 + 39(KP)^2 - 744K^4Pq \right. \right. \\ \left. \left. - 36000K^6q^2) \right. \right. \\ \left. \times \cosh(2Kx) + 3 \right. \\ \left. \times 2^{9+2\alpha} K^4q \right. \\ \left. \times (-P + 20K^2q) \cosh(4Kx) \right. \\ \left. + 15 \times 4^\alpha K^2P^2 \cosh(6Kx) - 57 \right. \\ \left. \times 2^{3+2\alpha} K^4Pq \cosh(6Kx) - 705 \right. \\ \left. \times 4^{2+\alpha} K^6q^2 \cosh(6Kx) + 2^{1+3\alpha} \right. \\ \left. \times K^2P^2 \cosh(8Kx) + 13 \times 4^{2+\alpha} \right. \\ \left. \times K^4Pq \cosh(8Kx) + 25 \times 2^{5+2\alpha} \right. \\ \left. \times K^6q^2 \cosh(8Kx) \right. \\ \left. - 4^\alpha K^2P^2 \cosh(10Kx) \right. \\ \left. - 2^{3+2\alpha} K^4Pq \cosh(10Kx) \right. \\ \left. - 4^{2+\alpha} K^6q^2 \cosh(10Kx) + 7 \right. \\ \left. \times 4^{2+\alpha} a^2KP \sinh(2Kx) + 65 \right. \\ \left. \times 4^{3+\alpha} a^2K^3q \sinh(2Kx) - 2^{5+2\alpha} a^2 \right. \\ \left. \times KP \sinh(4Kx) - 13 \times 2^{7+2\alpha} a^2K^3 \right. \\ \left. \times q \sinh(4Kx) + 4^{2+\alpha} a^2Kq \sinh(6Kx) \right. \\ \left. + 4^{3+\alpha} a^2K^3q \sinh(6Kx) \right) \tanh(Kx) \Big), \quad (44)$$

and so on; using the package Mathematica, in the same manner one can obtain the rest of the components. But, here, few terms were computed and the asymptotic solution is given by

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) \\ + u_3(x, t) + \dots \quad (45)$$

The figures show the graphical representation of the approximated solution of the system of nonlinear modified fractional Kawahara equation for $P = 0.0001$, $q = -1$. The approximate

solutions of main problem have been depicted in Figures 1, 2, and 3 which were plotted in Mathematica according to different α values.

5.2. Applications with STM. In this subdivision, we take advantage of the line of attack of the Sumudu transform technique to obtain approximated solution of the adapted fractional Karawana equation. According to the steps involved in the Sumudu transform method, we arrive at the next series solution.

Consider

$$u_0(x, t) = \frac{3P}{\sqrt{-10q}} (\operatorname{sech}(Kx))^2$$

$$u_1(x, t) = \frac{3PKt^\alpha (\operatorname{sech}(Kx))^2}{4\Gamma(1+\alpha)\sqrt{-10q}} \quad (46)$$

$$\begin{aligned} & \times \left(8K(P - 20K^2q) \cosh(Kx) \right. \\ & \quad + K(P + 100K^2q) \cosh(3Kx) \\ & \quad \left. - K(P + 4K^2q) \cosh(5Kx) + 8a^2 \sinh(Kx) \right), \end{aligned}$$

$$\begin{aligned} u_2(x, t) &= \frac{1}{\Gamma^2(\alpha+1)\Gamma(0.5+\alpha)\Gamma(1+3\alpha)} \\ & \times \left(2^{-5-2\alpha} aKt^\alpha (\operatorname{sech}(Kx))^{12} \right. \\ & \quad \times \left(\Gamma(1+\alpha)\Gamma(1+3\alpha) \right. \\ & \quad \times \left(4^{3+\alpha} a^2 (\cosh(Kx))^5 \Gamma(0.5+\alpha) \sinh(Kx) \right. \\ & \quad + K\sqrt{\pi}t^\alpha \left(-64a^4 + 276(KP)^2 \right. \\ & \quad \quad - 20832K^4Pq + 1087296K^6q^2 \\ & \quad \quad + 2 \left(32a^4 + 165(KP)^2 - 7224K^4Pq \right. \\ & \quad \quad \quad \left. \left. - 45360K^6q^2 \right) \cosh(2Kx) \right) \\ & \quad - 768K^4q \left(-17P + 1240K^2q \right) \cosh(4Kx) \\ & \quad - 75K^2P^2 \cosh(6Kx) \\ & \quad + 5736K^4Pq \cosh(6Kx) \\ & \quad + 217680K^6q^2 \cosh(6Kx) \\ & \quad - 20K^2P^2 \cosh(8Kx) \\ & \quad - 928K^4Pq \cosh(8Kx) \\ & \quad \left. \left. - 8000K^6q^2 \cosh(8Kx) \right) \right) \end{aligned}$$

$$\begin{aligned} & + K^2P^2 \cosh(10Kx) \\ & + 8K^4Pq \cosh(10Kx) \\ & + 16K^6q^2 \cosh(6Kx) \\ & + 592a^2KP \sinh(2Kx) \\ & - 86720a^2K^3q \sinh(2Kx) \\ & + 176a^2KP \sinh(4Kx) \\ & + 35264a^2K^3q \sinh(4Kx) \\ & - 80a^2KP \sinh(6Kx) \\ & - 2624a^2K^3q \sinh(6Kx) \\ & - 2a^2K^2t^{2\alpha}\Gamma(0.5+\alpha) \\ & \times \Gamma(1+2\alpha) (\operatorname{sech}(Kx))^4 \\ & \times \left(4^{3+\alpha} a^4 - 33 \times 2^{1+2\alpha} (KP)^2 + 147 \right. \\ & \quad \times 4^{2+\alpha} K^4Pq - 1113 \times 2^{5+2\alpha} K^6q^2 \left. \right) \\ & - 2^{1+2\alpha} \left((32a^4 + 39(KP)^2 - 744K^4Pq \right. \\ & \quad \left. - 36000K^6q^2) \right. \\ & \quad \times \cosh(2Kx) + 3 \\ & \quad \times 2^{9+2\alpha} K^4q \\ & \quad \times \left(-P + 20K^2q \right) \cosh(4Kx) \\ & \quad + 15 \times 4^\alpha K^2P^2 \cosh(6Kx) - 57 \\ & \quad \times 2^{3+2\alpha} K^4Pq \cosh(6Kx) - 705 \\ & \quad \times 4^{2+\alpha} K^6q^2 \cosh(6Kx) + 2^{1+3\alpha} \\ & \quad \times K^2P^2 \cosh(8Kx) + 13 \times 4^{2+\alpha} \\ & \quad \times K^4Pq \cosh(8Kx) + 25 \times 2^{5+2\alpha} \\ & \quad \times K^6q^2 \cosh(8Kx) \\ & \quad - 4^\alpha K^2P^2 \cosh(10Kx) \\ & \quad - 2^{3+2\alpha} K^4Pq \cosh(10Kx) \\ & \quad - 4^{2+\alpha} K^6q^2 \cosh(10Kx) + 7 \\ & \quad \times 4^{2+\alpha} a^2KP \sinh(2Kx) + 65 \\ & \quad \times 4^{3+\alpha} a^2K^3q \sinh(2Kx) - 2^{5+2\alpha} a^2 \\ & \quad \times KP \sinh(4Kx) - 13 \times 2^{7+2\alpha} a^2K^3 \\ & \quad \times q \sinh(4Kx) \end{aligned}$$

$$\begin{aligned}
& + 4^{2+\alpha} a^2 K q \sinh(6Kx) \\
& + 4^{3+\alpha} a^2 K^3 q \sinh(6Kx) \\
& \times \tanh(Kx) \Big).
\end{aligned}
\tag{47}$$

Remark 8. It worth noting that, in this investigation, both techniques have provided the same results. However, from their methodologies one can observe that the HDM is very easy to implement and the complexity of the HDM is of order n .

6. Conclusion

We derived approximated solutions of nonlinear fractional Kawahara equations using comparatively innovative analytical modulus operandi, the HDM and STM. We offered the epigrammatic history and some properties of fractional derivative concept. It is established that HDM and STM are authoritative and well-organized instruments of FPDEs. Additionally, the calculations concerned are very simple and uncomplicated.

Conflict of Interests

All authors declare there is no conflict of interests for this paper.

Authors' Contribution

The first draft was written by Abdon Atangana and Necdet Bildik, and the revised form was corrected in detail by S. C. Noutchie. All authors read and submitted the last version.

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