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Research Article

Asymptotic Degree Distribution of a Kind of Asymmetric Evolving Network

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We propose a kind of evolving network which shows tree structure. The model is a combination of preferential attachment model and uniform model. We show that the proportional degree sequence $\{p_k\}_{k>1}$ obeys power law, exponential distribution, and other forms according to the relation of k and parameter m.

1. Introduction

In recent ten years, there has been much interest in understanding the properties of real large-scale complex networks which describe a wide range of systems in nature and society. Examples of such networks appear in communications, biology, social science, economics, and so forth [1]. In pursuit of such understanding, mathematicians and physicists usually use random graphs to model all these real-life networks. In the investigation of various complex networks, the degree distribution is always the main concern because it characterizes the fundamental topological properties of complex networks which show importance in network control, estimation, and sensor [2–8]. Several models were introduced to explain the properties. Bollobás [9] proposed a model with n vertices and m edges. In this model, the degree distribution is approximately Poisson distribution. Later, Barabási and Albert [10] proposed the following model: at each time step, add a new vertex v and a fixed number r of edges originating at v and directed towards vertices chosen at random with probability proportional to their degrees. Based on simulation and heuristic approximation, they predicted that the degree distribution behaves like d^{-3} for all $r \geq 1$. The result was confirmed by Barabási et al. [11, 12]. In order to generate power laws with arbitrary exponents, Dorogovtsev et al. [13] and Drine et al. [14] introduced the following natural generalization of the above model: the destination of the r new edges added at each time step is chosen with probability proportional to the degree plus an initial attractiveness αr; they gave a nonrigorous argument that the degree distribution p_d behaves like $d^{-2-\alpha}$ for large d.

In some real networks, experiments show that the distribution obeys neither power law nor exponential. To explain the phenomenon, we propose a model as follows: starting with a single vertex, at each time step, a new vertex is added and linked to one of the existing vertices, which is chosen according the following rule: at time $m, 2m, 3m, \ldots$, where m is integer, we choose one of the existing vertices with probability proportional to the degree; that is, we have probability k/s_n, where k is the degree of the vertex chosen and s_n is the total degree of vertices; at another time step, we choose one of the existing vertices with equal probability. Related models were also proposed by Krapivsky and Redner [15] and Li [16] to describe the organization of growing networks. In this paper, we will focus on the distribution of evolving network and the distribution of the number of vertices with given degree will be considered in Section 2. In Section 3, we will consider the asymptotic degree distribution.

2. The Number of Vertices with Given Degrees

Let $D_n(k)$ denote the number of vertices with degree k at time n. We will consider the case $k = 1, 2$ in this section and the case $k > 2$ will be considered in the next section. As $k = 1$, we obtain the following result.

Lemma 1. *In the evolving network, the expectation of the number of degree 1 satisfies*

$$ED_n(1) = \frac{2m}{4m-1}n. \tag{1}$$

Proof. Herein after, = denotes asymptotic equivalence as $n \to \infty$. From the way the network is formed, we can see that, for $n > 1$, the number of vertices of degree 1 does not change if we attach a new vertex v_n to a vertex with degree 1 and increases by 1 if we attach v_n to vertices of degree larger than 1 after joining the vertex v_n. Assuming n is multiple of m, that is, $n = km$, where k is integer number, and taking expectation of $D_n(1)$, we obtain

$$\begin{aligned}
\mathrm{ED}_{km}(1) &= \left(1 - \frac{1}{2(km-1)}\right)\mathrm{ED}_{km-1}(1) + 1 \\
&= \left(1 - \frac{1}{2(km-1)}\right) \\
&\quad \times \left[\mathrm{ED}_{km-2}(1)\left(1 - \frac{1}{km-2}\right) + 1\right] + 1 \\
&= \left(1 - \frac{1}{2(km-1)}\right)\left(1 - \frac{1}{km-2}\right)\mathrm{ED}_{km-2}(1) \\
&\quad + \left(1 - \frac{1}{2(km-1)}\right) + 1.
\end{aligned} \tag{2}$$

The first equation shows that when we add a new vertex and link it to one of existing vertices with preferential attachment, the number of vertex increases by 1, while the second equation comes from the uniform attachment. Continuing the iteration and noticing the boundary condition $\mathrm{ED}_1(1) = 0$, we have

$$\begin{aligned}
\mathrm{ED}_{km}(1) &= \sum_{j=1}^{km}\prod_{i=0}^{[j/m]}\left(1 - \frac{1}{2[(k-i)m-1]}\right) \\
&\quad \times \prod_{v=2}^{j}\left(1 - \frac{1}{km-v}\right) \\
&\quad \times \left(\prod_{s=0}^{[j/m]}\left(1 - \frac{1}{(k-i)m-1}\right)\right)^{-1}.
\end{aligned} \tag{3}$$

Considering the term

$$\begin{aligned}
&\prod_{i=0}^{[j/m]}\left(1 - \frac{1}{2[(k-i)m-1]}\right)\prod_{v=2}^{j}\left(1 - \frac{1}{km-v}\right) \\
&\quad \times \left(\prod_{s=0}^{[j/m]}\left(1 - \frac{1}{(k-i)m-1}\right)\right)^{-1},
\end{aligned} \tag{4}$$

we have

$$\begin{aligned}
&e^{\ln\prod_{i=0}^{[j/m]}(1-(1/2[(k-i)m-1]))\prod_{v=2}^{j}(1-(1/(km-v)))-\ln\prod_{s=0}^{[j/m]}(1-(1/((k-i)m-1)))} \\
&\quad = e^{-\sum_{i=0}^{[j/m]}(1/2[(k-i)m-1])-\sum_{v=2}^{j}(1/(km-v))+\sum_{i=0}^{[j/m]}(1/((k-i)m-1))} \\
&\quad = e^{-(1/(2k-1))\int_{t=0}^{[j/m]}(1/(1-(mt/(2k-1))))dt-(1/km)\int_{v=2}^{j}(1/(1-(v/km)))dv+(1/(km-1))\int_{t=0}^{[j/m]}(1/(1-(mt/(km-1))))dt} \\
&\quad = \left(1 - \frac{j}{km}\right)^{(1-(1/2m))}.
\end{aligned} \tag{5}$$

We obtain that

$$\begin{aligned}
\mathrm{ED}_{km}(1) &= \sum_{j=1}^{km}\left(1 - \frac{j}{km}\right)^{(1-(1/2m))} \\
&= \frac{2m}{4m-1}\cdot km
\end{aligned} \tag{6}$$

when n is not a multiple of m, assuming $n = k'm + s$, $1 < s < m$, where k' is an integer number; we also obtain

$$\begin{aligned}
&\mathrm{ED}_{k'm+s}(1) \\
&\quad = \mathrm{ED}_{k'm+s-1}(1)\left(1 - \frac{1}{k'm+s-1}\right) + 1 \\
&\quad = \cdots
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{1}{k'm + s - 1}\right) \\
&\quad \times \left(1 - \frac{1}{k'm + s - 2}\right) \cdots \left(1 - \frac{1}{k'm}\right) \mathrm{ED}_{k'm}^{m}(1) \\
&\quad + \left(1 - \frac{1}{k'm + s - 1}\right) \\
&\quad \times \left(1 - \frac{1}{k'm + s - 2}\right) \cdots \left(1 - \frac{1}{k'm + 1}\right) \\
&\quad + \cdots + \left(1 - \frac{1}{k'm + s - 1}\right)\left(1 - \frac{1}{k'm + s - 2}\right) \\
&\quad + \left(1 - \frac{1}{k'm + s - 1}\right).
\end{aligned} \tag{7}
$$

When n is large enough, we can see that

$$\mathrm{ED}_{k'm}(1) = \mathrm{ED}_{k'm+s}(1)(1 + o(1)). \tag{8}$$

As a result, we have

$$\mathrm{ED}_n(1) = \frac{2m}{4m - 1} n. \tag{9}$$

□

Now we discuss the number of degree 2 in the network; we have the following.

Lemma 2. *For $n > 2$,*

$$ED_n(2) = \frac{2m - 1}{8m - 2} n. \tag{10}$$

Proof. We prove the case that n is a multiple of m and assume $n = km$, where k is an integer number; considering the expectation of $D_n(2)$, we have

$$
\begin{aligned}
\mathrm{ED}_{km}(2) &= \left(1 - \frac{2}{2(km - 1)}\right) \mathrm{ED}_{km-1}(2) + \frac{\mathrm{ED}_{km-1}(1)}{2(km - 1)} \\
&= \left(1 - \frac{1}{km - 1}\right) \left[\mathrm{ED}_{km-2}(2)\left(1 - \frac{1}{km - 2}\right) \right. \\
&\qquad \left. + \frac{\mathrm{ED}_{km-2}(1)}{km - 2}\right] + \frac{\mathrm{ED}_{km-1}(1)}{2(km - 1)}.
\end{aligned} \tag{11}
$$

Noticing the boundary condition $\mathrm{ED}_2(2) = 0$ and Lemma 1, we have

$$
\begin{aligned}
\mathrm{ED}_{km}(2) = &\left(\sum_{j=1}^{km} \prod_{i=1}^{j} \left(1 - \frac{1}{km - i}\right) \right. \\
&\left. - \frac{1}{2} \sum_{j=1}^{k} \prod_{v=1}^{jm} \left(1 - \frac{1}{km - v}\right)\right) \frac{2m}{4m - 1}.
\end{aligned} \tag{12}
$$

By the estimation $\ln(1 + x) = x$ and the fact that

$$
\begin{aligned}
\prod_{i=1}^{j} \left(1 - \frac{1}{km - i}\right) &= 1 - \frac{j}{km}, \\
\prod_{v=0}^{jm} \left(1 - \frac{1}{km - v}\right) &= 1 - \frac{j}{k},
\end{aligned} \tag{13}
$$

we obtain that

$$
\begin{aligned}
\mathrm{ED}_{km}(2) &= \frac{2m}{4m - 1} \left[\sum_{j=1}^{km} \left(1 - \frac{j}{km}\right) - \frac{1}{2} \sum_{j=0}^{k} \left(1 - \frac{j}{k}\right)\right] \\
&= km \frac{2m}{4m - 1} \left[\int_0^1 (1 - x)\, dx - \frac{1}{2} \int_0^1 (1 - x)\, dx\right] \\
&= \frac{2m - 1}{8m - 2} \cdot km.
\end{aligned} \tag{14}
$$

The case n, which is not a multiple of m, is the same as Lemma 1, just a little tedious. □

3. Asymptotic Degree Distribution of Network

Let

$$p_k(n) = \frac{D_n(k)}{n} \tag{15}$$

denote the proportion of vertices with degree k at time n. Considering the expectation of $D_n(k)$, we have the following theorem.

Theorem 3. *For arbitrary $k > 1$ and n, the expectation of the number of degree k satisfies*

$$ED_n(k) = \frac{2m}{4m - 1} \prod_{i=2}^{k} \frac{2m + i - 3}{4m + j - 2} n. \tag{16}$$

Proof. The case $k = 1, 2$ is just the result of Lemmas 1 and 2. Assume the result is true for k; that is,

$$\mathrm{ED}_n(k) = \frac{2m}{4m - 1} \prod_{i=2}^{k} \frac{2m + i - 3}{4m + j - 2} n. \tag{17}$$

We will prove the result is true for $k+1$. We just prove the case n is a multiple of m; that is, $n = lm$, where l is integer number. From the network constructed, we have

$$\begin{aligned}
&\mathrm{ED}_{ml}(k+1)\\
&\quad=\left(1-\frac{k+1}{2(ml-1)}\right)\mathrm{ED}_{ml-1}(k+1)\\
&\qquad+\frac{k}{2(ml-1)}\mathrm{ED}_{ml-1}(k)\\
&\quad=\left(1-\frac{k+1}{2(ml-1)}\right)\\
&\qquad\times\left[\mathrm{ED}_{ml-2}(k+1)\left(1-\frac{1}{ml-2}\right)+\frac{\mathrm{ED}_{ml-2}(k)}{ml-2}\right]\\
&\qquad+\frac{k}{2(ml-1)}\mathrm{ED}_{ml-1}(k).
\end{aligned}\tag{18}$$

Continuing the iteration and noticing the boundary condition $\mathrm{ED}_s(k+1)=0,\ s<k$, we obtain that

$$\begin{aligned}
&\mathrm{ED}_{ml}(k+1)\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\\
&\qquad\times\sum_{j=1}^{lm}\prod_{i=0}^{[j/m]}\left(1-\frac{k+1}{2(l-i)m-1}\right)\prod_{v=1}^{j}\left(1-\frac{1}{lm-v}\right)\\
&\qquad\times\left(\prod_{i=0}^{[j/m]}\left(1-\frac{1}{(l-i)m-1}\right)\right)^{-1}+\frac{k-2}{2}\frac{2m}{4m-1}\\
&\qquad\times\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\sum_{j=1}^{l}\prod_{i=0}^{j}\left(1-\frac{k+1}{2(l-i)m-1}\right)\\
&\qquad\times\prod_{v=1}^{j}\left(1-\frac{1}{lm-v}\right)\times\left(\prod_{i=0}^{j}\left(1-\frac{1}{(l-i)m-1}\right)\right)^{-1}.
\end{aligned}\tag{19}$$

Noticing the fact that

$$\begin{aligned}
&\prod_{i=0}^{[j/m]}\left(1-\frac{k+1}{2(l-i)m-1}\right)\prod_{v=1}^{j}\left(1-\frac{1}{lm-v}\right)\\
&\qquad\times\left(\prod_{i=0}^{[j/m]}\left(1-\frac{1}{(l-i)m-1}\right)\right)^{-1}\\
&\quad=\left(1-\frac{j}{lm}\right)^{(((k-1)/2m)+1)},
\end{aligned}\tag{20}$$

$$\begin{aligned}
&\prod_{i=0}^{j}\left(1-\frac{k+1}{2(l-i)m-1}\right)\prod_{v=1}^{jm}\left(1-\frac{1}{lm-v}\right)\\
&\qquad\times\left(\prod_{i=0}^{j}\left(1-\frac{1}{(l-i)m-1}\right)\left(1-\frac{j}{lm}\right)^{(((k-1)/2m)+1)}\right)^{-1}\\
&\quad=\left(1-\frac{j}{l}\right)^{(((k-1)/2m)+1)}.
\end{aligned}\tag{21}$$

We obtain

$$\begin{aligned}
&\mathrm{ED}_{lm}(k+1)\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\\
&\qquad\times\sum_{j=1}^{lm}\left(1-\frac{j}{lm}\right)^{(((k-1)/2m)+1)}+\frac{k-2}{2}\frac{2m}{4m-1}\\
&\qquad\times\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\sum_{j=1}^{l}\left(1-\frac{j}{l}\right)^{(((k-1)/2m)+1)}\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\int_0^1(1-x)^{(((k-1)/2m)+1)}dx\cdot lm\\
&\qquad+\frac{k-2}{2}\frac{2m}{4m-1}\\
&\qquad\times\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\int_0^1(1-x)^{(((k-1)/2m)+1)}dx\cdot l\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\left(1+\frac{k-2}{2m}\right)\\
&\qquad\cdot\frac{2m}{4m+k-1}\cdot lm\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k+1}\frac{2m+i-3}{4m+j-2}\cdot lm.
\end{aligned}\tag{22}$$

The result is true for $k+1$. □

From Theorem 3, we can see that $\lim_{n\to\infty}(\mathrm{ED}_n(k)/n)$ exists; we denote it by p_k. Now we consider the relation of p_k and $p_k(n)$; we introduce the following lemma.

Lemma 4. *There exists a bound constant $C(k)$ such that for arbitrary $a>0$,*

$$P\left(\left|D_n(k)-ED_n(k)\right|\geq a\right)\leq 2e^{-a^2/2C(k)^2n}.\tag{23}$$

Proof. Let $\mathfrak{F}_n=\sigma(D_1(1),\ldots,D_k(1),D_k(2),\ldots D_k(k),\ldots,D_n(1),\ldots D_n(k)\ldots,D_n(n))$ denote the *σ-algbra*. For $m=0,1,\ldots,n$, we define

$$M_m=E\left(D_k(n)\mid\mathfrak{F}_m\right).\tag{24}$$

By the tower property of conditional expectation and the fact that the σ-*algbra* $\mathfrak{F}_n$ can be deduced from $\mathfrak{F}_{n+1}$, we obtain that, for $m < n$,

$$\begin{aligned} E\left(M_{m+1} \mid \mathfrak{F}_m\right) &= E\left[E\left(D_n(k) \mid \mathfrak{F}_{m+1}\right) \mid \mathfrak{F}_m\right] \\ &= E\left(D_n(k) \mid \mathfrak{F}_m\right) \\ &= M_m. \end{aligned} \tag{25}$$

Noticing the fact that

$$E\left[\left|M_m\right|\right] = EM_m = \mathrm{E}D_n(k) < n < \infty, \tag{26}$$

we have $\{M_m\}_{m=0}^n$ as a martingale sequence. According to the definition of the σ-*algbra*, we know the $\mathfrak{F}_0$ has no information of the network and $\mathfrak{F}_n$ has the whole information, so we have

$$\begin{aligned} M_0 &= E\left[D_n(k) \mid \mathfrak{F}_0\right] = \mathrm{E}D_n(k), \\ M_n &= E\left[D_n(k) \mid \mathfrak{F}_n\right] = D_n(k). \end{aligned} \tag{27}$$

Therefore, we have

$$D_n(k) - \mathrm{E}D_n(k) = M_n - M_0 = \sum_{j=0}^{n}\left(M_{j+1} - M_j\right). \tag{28}$$

Now we prove that there exists a bound constant $C(k)$, such that $|M_{j+1} - M_j| \le C(k)$. We will prove the result by induction. For the case $k = 1$, we have

$$\begin{aligned} &\left|M_{j+1} - M_j\right| \\ &\quad = \left|E\left(D_n(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_n(1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \Big|E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) \\ &\qquad - E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_j\right) \\ &\qquad + E\left(D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(1) \mid \mathfrak{F}_j\right)\Big| \\ &\quad = \Big|E\left(E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_{n-1}\right) \mid \mathfrak{F}_{j+1}\right) \\ &\qquad - E\left(E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_{n-1}\right) \mid \mathfrak{F}_j\right) \\ &\qquad + E\left(D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(1) \mid \mathfrak{F}_j\right)\Big| \\ &\quad = \left(1 - \frac{1}{n-1}\right) \\ &\qquad \times \left|E\left(D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(1) \mid \mathfrak{F}_j\right)\right|. \end{aligned} \tag{29}$$

Continuing the iteration and noticing the fact that $E(D_m(1) \mid \mathfrak{F}_{j+1}) - E(D_m(1) \mid \mathfrak{F}_j) = 0$, for $m < j$, we obtain that

$$\begin{aligned} &\left|M_{j+1} - M_j\right| \\ &\quad = \prod_{i=j}^{n-1}\left(1 - \frac{1}{i}\right)\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{2(sm-1)}\right) \\ &\qquad \times \left(\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{sm-1}\right)\right)^{-1} \\ &\qquad \cdot \left|E\left(D_{j+1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{j+1}(1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \prod_{i=j}^{n-1}\left(1 - \frac{1}{i}\right)\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{2(sm-1)}\right) \\ &\qquad \times \left(\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{sm-1}\right)\right)^{-1} \\ &\qquad \cdot \left|\left(D_{j+1}(1) - D_j(1)\right) - E\left(D_{j+1}(1) - D_j(1) \mid \mathfrak{F}_j\right)\right|. \end{aligned} \tag{30}$$

Obviously,

$$\begin{gathered} \left|D_{j+1}(1) - D_j(1)\right| \le 1, \\ \prod_{i=j}^{n-1}\left(1 - \frac{1}{i}\right)\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{2(sm-1)}\right) \\ \times \left(\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{sm-1}\right)\right)^{-1} \le 1, \end{gathered} \tag{31}$$

so we have

$$\left|M_{j+1} - M_j\right| \le 2. \tag{32}$$

Assume the result is true for k; that is, there exists a bound constant $C(k)$, such that

$$\left|M_{j+1} - M_j\right| \le C(k). \tag{33}$$

For $k + 1$, by the definition of M_{j+1}, we have

$$\begin{aligned} &\left|M_{j+1} - M_j\right| \\ &\quad = \left|E\left(D_n(k+1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_n(k+1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \left(1 - \frac{1}{n-1}\right) \\ &\qquad \times \left|E\left(D_n(k+1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(k+1) \mid \mathfrak{F}_j\right)\right| \\ &\qquad + \frac{1}{n-1}\left[E\left(D_{n-1}(k) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(k) \mid \mathfrak{F}_j\right)\right]. \end{aligned} \tag{34}$$

Continuing the iteration and using the assumption for k, we obtain that

$$\begin{aligned}
&\left|M_{j+1}-M_j\right| \\
&\quad\le \prod_{v=j+1}^{n-1}\left(1-\frac{1}{v}\right)\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{k+1}{2(im-1)}\right) \\
&\quad\times\left(\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{1}{(im-1)}\right)\right)^{-1} \\
&\quad\cdot\left|E\left(D_{j+1}(k+1)\mid\mathfrak{F}_{j+1}\right)-E\left(D_{j+1}(k+1)\mid\mathfrak{F}_j\right)\right| \\
&\quad+\sum_{v=1}^{n-j-1}\prod_{s=1}^{v}\left(1-\frac{1}{n-s}\right)\frac{1}{n-v-1}C(k) \\
&\quad+\sum_{v=0}^{[(n-j-1)/m]}\prod_{s=1}^{n-([n/m]-v)m}\left(1-\frac{1}{n-s}\right) \\
&\quad\cdot\prod_{s=0}^{v}\left(1-\frac{k+1}{2(([n/m]-s)m-1)}\right) \\
&\quad\times\left(\prod_{s=0}^{j}\left(1-\frac{1}{([n/m]-s)m-1}C(k)\right)\right)^{-1}.
\end{aligned}\tag{35}$$

Noticing the fact that $1-((k+1)/2j)<1-(1/j)$ and

$$\begin{aligned}
&\prod_{v=j+1}^{n-1}\left(1-\frac{1}{v}\right)\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{k+1}{2(im-1)}\right) \\
&\quad\times\left(\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{1}{(im-1)}\right)\right)^{-1}\le 1,
\end{aligned}\tag{36}$$

we obtain that

$$\begin{aligned}
\left|M_{j+1}-M_j\right| &\le 2+\frac{n-j-1}{n-1}C(k)+\frac{[(n-j-1)/m]}{n-1}C(k) \\
&\le 2+C(k)\left(1+\frac{1}{m}\right).
\end{aligned}\tag{37}$$

We just let $C(k+1)=2+C(k)(1+(1/m))$ and the result for $k+1$ is proved. By Asume-Hoeffding's inequality, we have the following for arbitrary $a>0$:

$$P\left(\left|D_n(k)-\mathrm{ED}_n(k)\right|\ge a\right)\le 2e^{-a^2/2C(k)^2n}.\tag{38}$$

□

Theorem 5. *For a fixed k, one has*

$$\lim_{n\to\infty}p_k(n)=p_k\quad a.e.\tag{39}$$

Proof. By the Borel-Cantelli Lemma, we need to prove the following for arbitrary ε:

$$\sum_{n=1}^{\infty}P\left(\left|p_k(n)-p_k\right|>\varepsilon\right)<\infty.\tag{40}$$

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\left|p_k(n)-p_k\right|>\varepsilon\right) \\
&\quad=\sum_{n=1}^{\infty}P\left(\left|\frac{D_n(k)}{n}-\frac{\mathrm{ED}_n(k)}{n}+\frac{\mathrm{ED}_n(k)}{n}-p_k\right|>\varepsilon\right) \\
&\quad\le\sum_{n=1}^{\infty}P\left(\left|\frac{D_n(k)}{n}-\frac{\mathrm{ED}_n(k)}{n}\right|\ge\frac{\varepsilon}{2}\right) \\
&\qquad+\sum_{n=1}^{\infty}P\left(\left|\frac{\mathrm{ED}_n(k)}{n}-p_k\right|\ge\frac{\varepsilon}{2}\right).
\end{aligned}\tag{41}$$

Noticing that $\lim_{n\to\infty}\mathrm{ED}_k(n)/n=p_k$ and using Lemma 4, we obtain that there exists N, such that

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\left|p_k(n)-p_k\right|>\varepsilon\right) \\
&\quad\le\sum_{n=1}^{\infty}P\left(\left|D_n(k)-\mathrm{ED}_n(k)\right|\ge\frac{\varepsilon}{2}n\right)+N \\
&\quad\le\sum_{n=1}^{\infty}2e^{-(\varepsilon^2/4)n}+N \\
&\quad<\infty.
\end{aligned}\tag{42}$$

□

Remark 6. As a result, we can see that the distribution p_k obeys the following rule.

When $m\ll k$, $p_k\propto k^{-(2m+1)}$, the degree distribution obeys power law; when $m\gg k$, $p_k\propto 2^{-k}$, the degree distribution obeys exponential distribution; otherwise, $p_k=(2m/(4m-1))\prod_{j=2}^{k}((2m+j-3)/(4m+j-2))$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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