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Research Article

Notes on Lipschitz Properties of Nonlinear Scalarization Functions with Applications

Fang Lu and Chun-Rong Chen

College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China

Correspondence should be addressed to Chun-Rong Chen; chencr1981@163.com

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Various kinds of nonlinear scalarization functions play important roles in vector optimization. Among them, the one commonly known as the Gerstewitz function is good at scalarizing. In linear normed spaces, the globally Lipschitz property of such function is deduced via primal and dual spaces approaches, respectively. The equivalence of both expressions for globally Lipschitz constants obtained by primal and dual spaces approaches is established. In particular, when the ordering cone is polyhedral, the expression for calculating Lipschitz constant is given. As direct applications of the Lipschitz property, several sufficient conditions for Hölder continuity of both single-valued and set-valued solution mappings to parametric vector equilibrium problems are obtained using the nonlinear scalarization approach.

1. Introduction

In the development of vector optimization, the theory and the methods of scalarization have always played important roles [1–5]. The linear scalarization is historically the first method proposed and the most widely known and used. Besides this, the nonlinear scalarization is also fully developed. Several well-known nonlinear scalarization functions were introduced, such as the Hiriart-Urruty function [6] and the Gerstewitz (Tammer) function [7, 8]. Among them, the function ξ_q (see Definition 1) commonly known as the Gerstewitz function in vector optimization [7, 9, 10] is a powerful tool, which was introduced in [11] and has further been mentioned in [12, 13]. It has many good properties, such as continuity, sublinearity, convexity, monotonicity, and Lipschitz property. These properties have been fully exploited in the literature [5, 7–10, 14–17] to deal with various problems with vector objectives, such as existence and continuity of solutions, optimality conditions, gap functions, duality, vector variational principles, well posedness, vector minimax inequalities, and vector network equilibrium problems.

However, as far as we know, the locally and globally Lipschitz properties of ξ_q have not been noticed until recently. Tammer and Zălinescu [8] studied Lipschitz continuity properties of such kind of functions and gave some applications for deriving necessary optimality conditions for vector optimization problems. For other close works about this aspect, one can refer to Durea and Tammer [14] and Nam and Zălinescu [18]. Chen and Li [15, 16] deduced the globally Lipschitz property of ξ_q by the dual space approach and applied it to discussing Hölder continuity of solutions to parametric vector (quasi)equilibrium problems. Motivated by the work reported in [8, 15, 16], in this paper we further discuss the globally Lipschitz property of ξ_q in linear normed spaces via the primal space approach (see Proposition 7). The equivalence of both expressions for globally Lipschitz constants L and L' obtained by primal and dual spaces approaches is established (see Proposition 8). The expressions for L' and L are related to a minimization problem (P) and a maximization problem (D), respectively, and hence, the equivalence that $L' = L$ means that the property of strong duality (i.e., $\inf(P) = L' = L = \sup(D)$) holds between primal and dual problems. Furthermore, the above discussions are extended to general Gerstewitz function φ_{-v}, and exact characterizations to the globally Lipschitz property for φ_{-v} are discussed, which would further complete the theory of [8, 18]. In addition, when the ordering cone is

polyhedral, the expression for calculating Lipschitz constant is also given.

Vector equilibrium problems (VEPs, for short), also known as generalized Ky Fan inequalities recently, contain many important models as special cases, such as vector variational inequalities, vector complementarity problems, and vector optimization problems (see, e.g., [9, 19, 20] and the references therein). As a significant topic of stability analysis, Hölder or Lipschitz continuity of solutions to parametric VEPs is of considerable interest. Recently, in this field, Hölder continuity of both single-valued and set-valued solution mappings to parametric VEPs have been intensively studied in [16, 17, 21–32], respectively. In addition, recent and related papers about stability of multifunctions published in [33, 34] are worth noticing.

Scalarization approaches have been used as efficient methods to study semicontinuity and Hölder continuity of parametric VEPs. Among them, scalarizing approaches were applied by using linear functionals [27, 35] or nonlinear scalarization functions [15, 16, 24]. We notice that nonlinear scalarization methods by virtue of several nonlinear scalarization functions to deal with solution stability have received some attention; for example, Sach and Tuan [36, 37] have used Gerstewitz-like scalarization functions to study both upper and lower semicontinuities of solution mappings for parametric VEPs. The globally Lipschitz property of the nonlinear scalarization function ξ_q seems to be good at dealing with stability and sensitivity analysis of VEPs [15–17]. It is necessary to further exploit applications of the globally Lipschitz property of ξ_q together with other useful properties for studying Hölder continuity of parametric VEPs. Motivated by the work reported in [25, 31, 38], this paper also aims to give some applications of the properties of ξ_q to the Hölder continuity of solutions for parametric VEPs. To our aim, the nonlinear scalarization function ξ_q as a fundamental tool will play key roles such that, its globally Lipschitz property, monotonicity, and sublinearity will be fully exploited. The results obtained are new and generalizations of known ones [25, 31] for the corresponding scalar cases, and our approach is totally based on the techniques of nonlinear scalarization.

The rest of the paper is organized as follows. In Section 2, we first summarize basic properties of the nonlinear scalarization function ξ_q, then discuss the globally Lipschitz property of ξ_q in linear normed spaces via the primal space approach, establish the equivalence that $L' = L$, and finally, extend the discussions to the general case $\varphi_{-\gamma}$. In Section 3, as applications of the Lipschitz property of ξ_q, we study Hölder continuity of both single-valued and set-valued solution mappings to parametric VEPs based on the nonlinear scalarization approach. The last section gives some conclusions.

2. Lipschitz Properties of Nonlinear Scalarization Functions

In this section, we first recall the nonlinear scalarization function ξ_q in vector optimization. Its main properties, especially, the globally Lipschitz property, are summarized.

Let X and Y be linear normed spaces, and let $K \subset Y$ be a pointed, closed, and convex cone with nonempty interior $\operatorname{int} K$. Let Y^* be the topological dual space of Y, $\langle \cdot, \cdot \rangle$ the natural pairing between Y and Y^*, and K^* the dual cone of K; that is, $K^* := \{\lambda \in Y^* \mid \langle \lambda, k \rangle \geq 0, \forall\, k \in K\}$.

Definition 1 (see [7, 9, 10]). Given a fixed point $q \in \operatorname{int} K$, the nonlinear scalarization function $\xi_q : Y \to \mathbb{R}$ is defined by

$$\xi_q(y) := \min\{t \in \mathbb{R} \mid y \in tq - K\}. \tag{1}$$

In the special case of $Y = \mathbb{R}^l$, $K = \mathbb{R}^l_+$, and $q = (1, \ldots, 1) \in \operatorname{int} \mathbb{R}^l_+$, the function ξ_q can be expressed in the equivalent form $\xi_q(y) = \max_{1 \leq i \leq l} \{y_i\}, \forall y = (y_1, \ldots, y_l) \in \mathbb{R}^l$.

It is well known that ξ_q is continuous, positively homogeneous, subadditive, and convex on Y, and it is monotone (i.e., $y^2 - y^1 \in K \Rightarrow \xi_q(y^1) \leq \xi_q(y^2)$) and strictly monotone (i.e., $y^2 - y^1 \in \operatorname{int} K \Rightarrow \xi_q(y^1) < \xi_q(y^2)$) (see [9]). Note, however, that the function ξ_q is not strongly monotone (i.e., $y^2 - y^1 \in K \setminus \{0_Y\} \not\Rightarrow \xi_q(y^1) < \xi_q(y^2)$).

Proposition 2 (see [7, 9]). *For any fixed $q \in \operatorname{int} K$, $y \in Y$, and $r \in \mathbb{R}$,*

(i) $\xi_q(y) < r \Leftrightarrow y \in rq - \operatorname{int} K$ *(i.e., $\xi_q(y) \geq r \Leftrightarrow y \notin rq - \operatorname{int} K$);*

(ii) $\xi_q(y) \leq r \Leftrightarrow y \in rq - K$ *(i.e., $\xi_q(y) > r \Leftrightarrow y \notin rq - K$);*

(iii) $\xi_q(y) = r \Leftrightarrow y \in rq - \operatorname{bd} K$, *where* $\operatorname{bd} K$ *denotes the topological boundary of K;*

(iv) $\xi_q(rq) = r$, *in particular,* $\xi_q(0_Y) = 0$;

(v) $\xi_q(y + rq) = \xi_q(y) + r$ *(translation property).*

Let $q \in \operatorname{int} K$ be a fixed point. The set

$$K^q := \{\lambda \in K^* \mid \langle \lambda, q \rangle = 1\} \tag{2}$$

is a weak*-compact set of Y^*. Clearly, K^q is a weak*-compact base of K^*; that is, K^q is convex and weak*-compact such that $0_{Y^*} \notin K^q$ and $K^* = \bigcup_{t \geq 0} tK^q$. Notice that [14, Lemma 2.4] we have $K^q = \partial \xi_q(0_Y)$, where $\partial \xi_q(y)$ denotes the classical (Fenchel) subdifferential of ξ_q at $y \in Y$. Generally, for every $y \in Y$, $\partial \xi_q(y) = \{\lambda \in K^q \mid \langle \lambda, y \rangle = \xi_q(y)\} \subset K^q$.

Proposition 3 (see [15, 16]). *Let $q \in \operatorname{int} K$. Then for any $y \in Y$, $\xi_q(y) = \max_{\lambda \in K^q} \langle \lambda, y \rangle$.*

Remark that the form of (K, q)-max scalarizing function $\phi(y) := \max_{\lambda \in K^q} \langle \lambda, y \rangle$ is also widely developed and has many applications, such as stability analysis of vector equilibrium problems [24] and optimality conditions for vector optimization problems [39].

Another famous scalarizing function is the oriented distance function $\Delta(y) := d_{-K}(y) - d_{Y \setminus (-K)}(y)$ introduced in [6], where $d_A(y) := \inf\{\|y - a\| \mid a \in A\}$ denotes the distance from $y \in Y$ to the set $A \subset Y$. It is well known that this function has very good general properties (see [3, Proposition 3.2]), especially, which is always 1-Lipschitz.

Remark 4. The oriented distance function Δ is not applicable for studying Hölder continuity of solution mappings to parametric vector equilibrium problems (PVEP) and (PVAEP) in this paper, unlike the Gerstewitz function ξ done in the next section. This is because the properties like Proposition 2 (ii) and (v) with $r \neq 0$ are not satisfied by the function Δ, while they play important roles in our study.

Let $\mathbb{B}(x_0, \delta)$ be the closed ball centered at x_0 and radius $\delta > 0$. It is said that $g : X \to Y$ is Lipschitz around $x_0 \in X$ iff there exist $L > 0$ and $\delta > 0$ such that $\|g(x) - g(x')\| \leq L\|x - x'\|$, $\forall x, x' \in \mathbb{B}(x_0, \delta)$, and g is locally Lipschitz on $A \subset X$ if and only if g is Lipschitz around each $x_0 \in A$. g is called (globally) Lipschitz on A if and only if $\|g(x) - g(x')\| \leq L\|x - x'\|$, $\forall x, x' \in A$.

Proposition 5 (see [15, 16]). *ξ_q is globally Lipschitz on Y, and its Lipschitz constant is $L := \sup_{\lambda \in K^q} \|\lambda\| \in [1/\|q\|, +\infty[$. In particular, under the scalar case of $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, the Lipschitz constant of ξ_q is $L = (1/q)$ $(q > 0)$.*

Remark 6. Let $q \in \operatorname{int} K$ and $a \in Y$. The nonlinear scalarization function $\xi_{q,a} : Y \to \mathbb{R}$ is defined as $\xi_{q,a}(y) := \min\{t \in \mathbb{R} \mid y \in a + tq - K\}$. It is easy to see that $\xi_{q,a}$ is still globally Lipschitz on Y with Lipschitz constant $L = \sup_{\lambda \in K^q} \|\lambda\|$, because $\xi_{q,a}(y) = \max_{\lambda \in K^q} \langle \lambda, y - a \rangle$.

Assume that Y is a separated locally convex space, and $K \subset Y$ is a proper, closed, and convex cone with $\operatorname{int} K \neq \emptyset$. Let $q \in \operatorname{int} K$. Then it follows from the proof of [8, Theorem 3.1(ii)] (see inequality (6) therein) that ξ_q is Lipschitz on Y; namely,

$$\left|\xi_q(y) - \xi_q(y')\right| \leq p_V(y - y'), \quad \forall y, y' \in Y, \tag{3}$$

where $p_V : Y \to \mathbb{R}$ is the Minkowski functional associated with V and $V \subset Y$ is a symmetric closed and convex neighborhood of 0_Y such that $q + V \subset K$.

When Y is a linear normed space and $V := \tau_{K,q}\mathbb{B}$ for some $\tau_{K,q} > 0$ ($\mathbb{B}$ denotes the closed unit ball), because

$$\begin{aligned} p_V(x) &:= \inf\left\{\alpha > 0 \mid \frac{x}{\alpha} \in V\right\} \\ &= \inf\left\{\alpha > 0 \mid \frac{x}{\alpha} \in \tau_{K,q}\mathbb{B}\right\} = \frac{\|x\|}{\tau_{K,q}}, \end{aligned} \tag{4}$$

we get from (3) that, for all $y, y' \in Y$,

$$\left|\xi_q(y) - \xi_q(y')\right| \leq \frac{\|y - y'\|}{\tau_{K,q}}. \tag{5}$$

Whence,

$$\left|\xi_q(y) - \xi_q(y')\right| \leq \frac{\|y - y'\|}{\tau_{K,q}^{\max}}, \quad \forall y, y' \in Y, \tag{6}$$

where $\tau_{K,q}^{\max} := \sup\{\tau > 0 \mid q + \tau\mathbb{B} \subset K\} = \sup\{\tau > 0 \mid \mathbb{B}(q, \tau) \subset K\}$. Therefore, the following conclusion holds for Y a linear normed space.

Proposition 7. *ξ_q is globally Lipschitz on Y with Lipschitz constant $L' := 1/\tau_{K,q}^{\max}$.*

Note that $L' = 1/\tau_{K,q}^{\max} = \inf\{1/\tau > 0 \mid q + \tau\mathbb{B} \subset K\}$.

Tammer and Zălinescu [8] recently have studied the Lipschitz property of the Gerstewitz function under more general settings than ours, using the primal space approach, which is different from the dual space approach adopted by us [15, 16]. In this paper, we limit our discussions in linear normed spaces to get more exact characterizations and more clear geometrical interpretations.

Whether the two Lipschitz constants $L = L'$ hold, we show it as follows.

Proposition 8. *Consider $\sup_{\lambda \in K^q} \|\lambda\| = 1/\tau_{K,q}^{\max}$ (i.e., $L = L'$).*

Proof. Let $L := \sup_{\lambda \in K^q} \|\lambda\|$ and $L' := 1/\tau_{K,q}^{\max}$. Firstly, we prove that $L \geq L'$.

By the definition of $\tau_{K,q}^{\max}$, for any given $\epsilon > 0$, we have $q + (\tau_{K,q}^{\max} + \epsilon)\mathbb{B} \not\subset K$. Thus there exists $\bar{b} \in \mathbb{B}$ such that $q + (\tau_{K,q}^{\max} + \epsilon)\bar{b} \notin K$. It follows from [2, Lemma 3.21(a)] that $K = \{k \in Y \mid \langle \lambda, k \rangle \geq 0, \forall \lambda \in K^*\}$, as K is a closed and convex cone. Whence, there is an $\tilde{\lambda} \in K^*$ satisfying $\langle \tilde{\lambda}, q + (\tau_{K,q}^{\max} + \epsilon)\bar{b} \rangle < 0$. Obviously, $\tilde{\lambda} \neq 0_{Y^*}$. As $q \in \operatorname{int} K$, $\langle \tilde{\lambda}, q \rangle > 0$. Without loss of generality, we may assume that $\langle \tilde{\lambda}, q \rangle = 1$; that is, $\tilde{\lambda} \in K^q$. Hence, we can deduce that

$$\begin{aligned} 0 &> \left\langle \tilde{\lambda}, q + \left(\tau_{K,q}^{\max} + \epsilon\right)\bar{b} \right\rangle \\ &= \left\langle \tilde{\lambda}, q \right\rangle + \left\langle \tilde{\lambda}, \left(\tau_{K,q}^{\max} + \epsilon\right)\bar{b} \right\rangle \\ &= 1 + \left(\tau_{K,q}^{\max} + \epsilon\right)\left\langle \tilde{\lambda}, \bar{b} \right\rangle \\ &\geq 1 - \left(\tau_{K,q}^{\max} + \epsilon\right)\left\|\tilde{\lambda}\right\| \cdot \left\|\bar{b}\right\| \\ &\geq 1 - \left(\tau_{K,q}^{\max} + \epsilon\right)\left\|\tilde{\lambda}\right\|. \end{aligned} \tag{7}$$

Thus,

$$\frac{1}{\tau_{K,q}^{\max} + \epsilon} < \left\|\tilde{\lambda}\right\| \leq L. \tag{8}$$

By the arbitrariness of $\epsilon > 0$, we obtain that $L' \leq L$.

Secondly, we prove that $L \leq L'$.

Take any $\lambda \in K^q$ and $\tau \in \Gamma := \{t > 0 \mid q + t\mathbb{B} \subset K\}$. Then, $\langle \lambda, q + \tau\mathbb{B} \rangle \geq 0$; namely, for any $b \in \mathbb{B}$, $\langle \lambda, q + \tau b \rangle \geq 0$. As $\langle \lambda, q \rangle = 1$ and $\tau > 0$, we deduce that $1/\tau + \langle \lambda, b \rangle \geq 0$. Hence, $1/\tau \geq -\langle \lambda, b \rangle = \langle \lambda, -b \rangle$. By the symmetry of $\mathbb{B}$ and the arbitrariness of $b \in \mathbb{B}$, we have $1/\tau \geq \langle \lambda, b \rangle$, $\forall b \in \mathbb{B}$. Whence, we get

$$\inf_{\tau \in \Gamma} \frac{1}{\tau} \geq \sup_{\lambda \in K^q, b \in \mathbb{B}} \langle \lambda, b \rangle = \sup_{\lambda \in K^q} \|\lambda\|, \tag{9}$$

where the last equality holds by the Cauchy-Schwarz inequality and that $b \in \mathbb{B}$. Note that $L' = 1/\tau_{K,q}^{\max} = \inf\{1/\tau > 0 \mid q + \tau\mathbb{B} \subset K\}$. Thus, we obtain that $L' \geq L$.

Based on the above analysis, the equivalence that $L = L'$ holds.

It is obvious that $\tau_{K,q}^{\max} = \operatorname{dist}(q, \operatorname{bd}K)$, where the latter one denotes the distance from $q \in \operatorname{int} K$ to the boundary of K; that is, $\operatorname{dist}(q, \operatorname{bd}K) := \inf_{\kappa \in \operatorname{bd}K} \|q - \kappa\|$. Thus, we also have that $L' = 1/\operatorname{dist}(q, \operatorname{bd}K)$ (see also [14, Lemma 2.4]). However, the expression of the form $L' = 1/\tau_{K,q}^{\max} = \inf\{1/\tau > 0 \mid q + \tau\mathbb{B} \subset K\}$ has its advantage: it is convenient to describe the dual characterization concerning $L = \sup_{\lambda \in K^q} \|\lambda\|$ (see also Remark 13).

Due to the observation above, we could give another proof of the second part of Proposition 8 (i.e., the weak duality between the problems (P) and (D): $\inf(P) = L' \geq L = \sup(D)$; see Remark 13 below). As $q + \operatorname{dist}(q, \operatorname{bd}K)\mathbb{B} \subset K$, then, for any $b \in \mathbb{B}$, $q + \operatorname{dist}(q, \operatorname{bd}K)b \in K$. It follows from the symmetry of $\mathbb{B}$ and the arbitrariness of $b \in \mathbb{B}$ that $q/\operatorname{dist}(q, \operatorname{bd}K) - b \in K$. Take any $\lambda \in K^q$. Then $0 \leq \langle \lambda, q/\operatorname{dist}(q, \operatorname{bd}K) - b\rangle = \langle \lambda, q\rangle/\operatorname{dist}(q, \operatorname{bd}K) - \langle \lambda, b\rangle = 1/\operatorname{dist}(q, \operatorname{bd}K) - \langle \lambda, b\rangle$. Hence $L' = 1/\operatorname{dist}(q, \operatorname{bd}K) \geq \sup_{\lambda \in K^q, b \in \mathbb{B}} \langle \lambda, b\rangle = \sup_{\lambda \in K^q} \|\lambda\| = L$. □

We give several examples to illustrate Proposition 8.

Example 9 (see [16, Example 2.1]). Let $Y = \mathbb{R}^2$ and $K = \{(y_1, y_2) \in \mathbb{R}^2 \mid 1/4y_1 \leq y_2 \leq 2y_1\}$. Obviously, $K^* = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_2 \geq -(1/2)\lambda_1, \lambda_2 \geq -4\lambda_1\}$, and $K \subset K^*$.

Take $q = (2, 3) \in \operatorname{int} K$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 2\lambda_1 + 3\lambda_2 = 1, \lambda_1 \in [-1/10, 2]\}$. We have calculated that the Lipschitz constant $L = \sup_{\lambda \in K^q} \|\lambda\| = \|(2, -1)\| = \sqrt{5}$. Now we calculate another Lipschitz constant $L' = 1/\tau_{K,q}^{\max}$. Notice that the distance from a point $(x_0, y_0) \in \mathbb{R}^2$ to the line $y = kx + b$ (resp., $Ax + By + C = 0$) is

$$d = \frac{|kx_0 - y_0 + b|}{\sqrt{k^2 + 1}} \quad \left(\text{resp., } d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}\right). \tag{10}$$

It is easy to verify that

$$\tau_{K,q}^{\max} = \left\|(2, 3) - \left(\frac{8}{5}, \frac{16}{5}\right)\right\| = \frac{|2 \times 2 - 3|}{\sqrt{2^2 + 1}} = \frac{\sqrt{5}}{5}. \tag{11}$$

Thus, we also get that the Lipschitz constant $L' = 1/\tau_{K,q}^{\max} = \sqrt{5}$.

Example 10. Let $Y = \mathbb{R}^2$ and $K = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq -y_1, y_2 \geq 0\}$. It is clear that $K^* = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \geq 0, \lambda_2 \geq \lambda_1\}$ and $K^* \subset K$.

Case i. Take $q = (-1, 2) \in \operatorname{int} K \setminus K^*$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid -\lambda_1 + 2\lambda_2 = 1, \lambda_1 \in [0, 1]\}$. Thus, $L = \sup_{\lambda \in K^q} \|\lambda\| = \|(1, 1)\| = \sqrt{2}$. Moreover, $\tau_{K,q}^{\max} = \|(-1, 2) - (-3/2, 3/2)\| = |(-1) \times (-1) - 2|/\sqrt{(-1)^2 + 1} = \sqrt{2}/2$, and hence $L' = 1/\tau_{K,q}^{\max} = \sqrt{2}$.

Case ii. Take $q = (1, 1/2) \in \operatorname{int} K \setminus K^*$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid 2\lambda_1 + \lambda_2 = 2, \lambda_1 \in [0, 2/3]\}$. Thus, $L = \sup_{\lambda \in K^q} \|\lambda\| = \|(0, 2)\| = 2$. Moreover, $\tau_{K,q}^{\max} = \|(1, 1/2) - (1, 0)\| = 1/2$, and hence $L' = 1/\tau_{K,q}^{\max} = 2$.

Case iii. Take $q = (1/2, 3/2) \in \operatorname{int} K \cap K^*$. Then, $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 + 3\lambda_2 = 2, \lambda_1 \in [0, 1/2]\}$. Thus, $L = \sup_{\lambda \in K^q} \|\lambda\| = \|(1/2, 1/2)\| = \sqrt{2}/2$. Moreover, $\tau_{K,q}^{\max} = \|(1/2, 3/2) - (-1/2, 1/2)\| = |(-1) \times (1/2) - 3/2|/\sqrt{(-1)^2 + 1} = \sqrt{2}$, and hence $L' = 1/\tau_{K,q}^{\max} = \sqrt{2}/2$.

Example 11. Let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Then $K^* = K$. Take $q = (q_1, q_2) \in \operatorname{int} K$. We have $K^q = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 q_1 + \lambda_2 q_2 = 1, \lambda_1 \in [0, 1/q_1]\}$. Thus, $L = \sup_{\lambda \in K^q} \|\lambda\| = \max\{1/q_1, 1/q_2\}$. Meanwhile, $\tau_{K,q}^{\max} = \min\{q_1, q_2\}$, so $L' = 1/\tau_{K,q}^{\max} = 1/\min\{q_1, q_2\} = \max\{1/q_1, 1/q_2\}$.

Now we calculate $\tau_{K,q}^{\max}$ when K has the following explicit structure (it is said to be a polyhedral convex cone in [40, Definition 2.1.7]), which contains the above examples as special cases. Herein we use the notation $x^T y$, instead of $\langle x, y\rangle$, to denote the standard inner product on $\mathbb{R}^n$.

Proposition 12. *If the polyhedral ordering cone $K \subset \mathbb{R}^n$ with nonempty interior is described by linear inequalities, $K := \{x \in \mathbb{R}^n \mid a_i^T x \leq 0, i = 1, \ldots, m\}$, where $a_i \in \mathbb{R}^n$ and $a_i \neq 0_{\mathbb{R}^n}$, then for given $q \in \operatorname{int} K$, $\tau_{K,q}^{\max} = \min_{1 \leq i \leq m}\{-a_i^T q/\|a_i\|\}$. Thus, $L = L' = \max_{1 \leq i \leq m}\{-\|a_i\|/a_i^T q\}$.*

Proof. Obviously, K is a closed convex cone. By the definition of $\tau_{K,q}^{\max}$, we wish to maximize $\tau > 0$ subject to the constraint $\mathbb{B}(q, \tau) = \{q + r \mid \|r\| \leq \tau\} \subset K$; that is, $a_i^T x \leq 0, i = 1, \ldots, m$ for all $x \in \mathbb{B}(q, \tau)$. Therefore, $\mathbb{B}(q, \tau) \subset K$ if and only if

$$\sup\left\{a_i^T (q + r) \mid \|r\| \leq \tau\right\} \leq 0. \tag{12}$$

Since $\sup\{a_i^T r \mid \|r\| \leq \tau\} = \tau\|a_i\|$, we can rewrite (12) as

$$a_i^T q + \tau \|a_i\| \leq 0, \quad i = 1, \ldots, m. \tag{13}$$

Thus, we have

$$\tau \leq -\frac{a_i^T q}{\|a_i\|}, \quad i = 1, \ldots, m. \tag{14}$$

Because $q \in \operatorname{int} K$, $a_i^T q < 0, i = 1, \ldots, m$. Whence, we obtain that

$$\begin{aligned} \tau_{K,q}^{\max} &= \max\left\{\tau > 0 \mid \tau \leq -\frac{a_i^T q}{\|a_i\|},\ i = 1, \ldots, m\right\} \\ &= \min_{1 \leq i \leq m}\left\{-\frac{a_i^T q}{\|a_i\|}\right\} > 0. \end{aligned} \tag{15}$$

This completes the proof. □

Remark 13. Proposition 8 implies that Propositions 5 and 7 (or see [14, Lemma 2.4]) are equivalent. Proposition 8 also shows that the property of strong duality (i.e., $\inf(P) = L' =$

$L = \sup(D)$) holds between the following primal problem (P) and dual problem (D) for fixed $q \in \operatorname{int} K$:

$$(P)\begin{cases}\inf & \dfrac{1}{\tau}\\ \text{s.t.} & \tau > 0,\\ & q + \tau\mathbb{B} \subset K,\end{cases}$$
$$(D)\begin{cases}\sup & \|\lambda\|\\ \text{s.t.} & \lambda \in K^*,\\ & \langle \lambda, q\rangle = 1.\end{cases} \tag{16}$$

Clearly, the Lipschitz constants L' and L are deduced via two approaches: the primal space and the dual space approaches, respectively.

Remark 14. Relative to L, the expression for Lipschitz constant L' is natural and exhibits a clear geometrical interpretation. In the setting of linear normed space Y, if $r > 0$ is the largest radius such that the closed ball $\mathbb{B}(q, r)$ centered at given $q \in \operatorname{int} K$ with radius r lies in the ordering cone K of Y, that is, $\mathbb{B}(q, r) \subset K$, then $1/r$ is the Lipschitz constant L'. Clearly, the value of r coincides with the distance from q to the boundary of K. Based on the geometrical interpretation of L', we know that the choice of $q \in \operatorname{int} K$, namely, the location of q, will directly confirm the modulus of Lipschitz continuity of ξ_q. It is obvious that the Lipschitz constant L' becomes larger whenever q is closer to the boundary of K.

As a direct application to the proof of Proposition 8, we give a note on Lipschitz properties of the directional minimal time function [18].

Given a vector $v \in Y$, $v \neq 0_Y$, and a nonempty closed set $\Omega \subset Y$ and $\Omega \neq Y$, the directional minimal time function with direction v and target set Ω is defined by

$$T_v(y; \Omega) := \inf\{t \geq 0 \mid y + tv \in \Omega\}. \tag{17}$$

This class of functions is similar to the class of nonlinear scalarization functions that has been extensively used to study vector optimization problems (see [7, 8]):

$$\varphi_v(y; \Omega) := \inf\{t \in \mathbb{R} \mid y + tv \in \Omega\}. \tag{18}$$

Obviously, ξ_q is a special but popular case of φ_v, by taking $\Omega := -K$ and $v := -q \in -\operatorname{int} K$.

Recall that the recession cone of Ω is given by

$$\Omega_\infty := \{u \in Y \mid \omega + tu \in \Omega, \forall \omega \in \Omega, \forall t \in \mathbb{R}_+\}. \tag{19}$$

It is known that Ω_∞ is a convex cone, and Ω_∞ is also closed as Ω is closed (see [8]). From [18, Proposition 2.1] we see that if $v \in \Omega_\infty$, then $T_v(y; \Omega) = \max\{\varphi_v(y; \Omega), 0\}, \forall y \in Y$.

We recall a result of the globally Lipschitz property for T_v [18, Proposition 4.1].

Proposition 15 (see [18]). *Suppose that $v \in \operatorname{int} \Omega_\infty$. Then T_v is globally Lipschitz on Y with Lipschitz constant $\ell' := \inf\{1/r > 0 \mid \mathbb{B}(v, r) \subset \Omega_\infty\} = 1/\operatorname{dist}(v, \operatorname{bd}\Omega_\infty)$.*

Notice that $\ell' = 1/\tau^{\max}_{\Omega_\infty, v}$, where $\tau^{\max}_{\Omega_\infty, v} := \sup\{\tau > 0 \mid v + \tau\mathbb{B} \subset \Omega_\infty\} = \sup\{\tau > 0 \mid \mathbb{B}(v, \tau) \subset \Omega_\infty\}$. Based on a similar proof to that of $L = L'$, letting $K := \Omega_\infty$, which is a closed and convex cone, and $q := v$, we can get

$$\ell' = \ell := \sup\{\|\lambda\| \mid \lambda \in \Omega^*_\infty, \langle \lambda, v\rangle = 1\}, \tag{20}$$

where Ω^*_∞ is the dual cone of Ω_∞. Thus, we have the following equivalent proposition.

Proposition 16. *If $v \in \operatorname{int} \Omega_\infty$, then T_v is globally Lipschitz on Y with Lipschitz constant ℓ.*

Remark 17. The relation (20) implies that the strong duality (i.e., $\inf(P') = \ell' = \ell = \sup(D')$) holds between the following primal problem (P') and dual problem (D') for fixed $v \in \operatorname{int} \Omega_\infty$:

$$(P')\begin{cases}\inf & \dfrac{1}{r}\\ \text{s.t.} & r > 0,\\ & \mathbb{B}(v, r) \subset \Omega_\infty,\end{cases}$$
$$(D')\begin{cases}\sup & \|\lambda\|\\ \text{s.t.} & \lambda \in \Omega^*_\infty,\\ & \langle \lambda, v\rangle = 1.\end{cases} \tag{21}$$

In what follows, we deduce more exact characterizations to the globally Lipschitz property for $\varphi_{-v}(y; \Omega) := \inf\{t \in \mathbb{R} \mid y \in tv + \Omega\}$, which has been studied in [8, Theorem 3.1]. These results would further complete the theory of [8, 18] and could be applied in many aspects of vector optimization.

Proposition 18. *If $v \in \operatorname{int} K$ and Ω satisfies the free-disposal assumption $\Omega - K = \Omega$, then φ_{-v} is globally Lipschitz on Y with Lipschitz constant $L_\varphi := \sup_{\lambda \in K^v} \|\lambda\|$ (or equivalently, $L'_\varphi := 1/\tau^{\max}_{K,v} = \inf\{1/\tau > 0 \mid \mathbb{B}(v, \tau) \subset K\} = 1/\operatorname{dist}(v, \operatorname{bd}K)$).*

Proof. According to [8, Theorem 3.1(ii)], φ_{-v} is finite on Y. Moreover, by [8, Theorem 3.1(i)], we get, for any $y, y' \in Y$, $\varphi_{-v}(y; \Omega) \leq \varphi_{-v}(y'; \Omega) + \varphi_{-v}(y - y'; -K)$. Thus, it follows from Proposition 3 that

$$\begin{aligned}\varphi_{-v}(y; \Omega) - \varphi_{-v}(y'; \Omega) &\leq \varphi_{-v}(y - y'; -K)\\ &= \xi_v(y - y')\\ &= \max_{\lambda \in K^v} \langle \lambda, y - y'\rangle\\ &\leq \sup_{\lambda \in K^v} \|\lambda\| \cdot \|y - y'\|\\ &= L_\varphi \|y - y'\|.\end{aligned} \tag{22}$$

This implies $|\varphi_{-v}(y; \Omega) - \varphi_{-v}(y'; \Omega)| \leq L_\varphi \|y - y'\| = L'_\varphi \|y - y'\|$ (see Proposition 8), because of the symmetry between y and y'. □

Remark 19. Compared to [8, Theorem 3.1(ii)], the exact expressions for globally Lipschitz constants L_φ and L'_φ ($L_\varphi =$

L'_φ) are given in Proposition 18. In addition, they are proved based on different ideas, respectively. Our simplified proof mainly relies on known results of [8, Theorem 3.1(i)] and Proposition 3.

Corollary 20. *If $v \in \operatorname{int} K$ and Ω satisfies the free-disposal assumption $\Omega + K = \Omega$, then φ_v is globally Lipschitz on Y with Lipschitz constant L_φ (or equivalently, L'_φ).*

Note that T_v also holds in Corollary 20, which is new to [18]. This is because the free-disposal condition $\Omega = \Omega + K$ yields that $K \subset \Omega_\infty$, so $v \in \Omega_\infty$; then $T_v = \max\{\varphi_v, 0\}$.

Corollary 21. *If $v \in \operatorname{int} \Omega_\infty$, then φ_v is globally Lipschitz on Y with Lipschitz constant ℓ (or equivalently, ℓ').*

Proof. The free-disposal condition $\Omega = \Omega - K$ shows that $K \subset -\Omega_\infty$. As Ω_∞ is a closed and convex cone because Ω is closed, hence $-\Omega_\infty$ is the largest closed convex cone K verifying the free-disposal assumption $\Omega = \Omega - K$. So, the conclusion follows by applying Proposition 18 with $K := -\Omega_\infty$.

Similarly, the conclusion also follows by applying Corollary 20 with $K := \Omega_\infty$. □

Corollary 22. *If $-v \in \operatorname{int} \Omega_\infty$, then φ_{-v} is globally Lipschitz on Y with Lipschitz constant $\ell_\varphi := \sup\{\|\lambda\| \mid \lambda \in \Omega^*_\infty, \langle \lambda, v\rangle = -1\}$ (or equivalently, $\ell'_\varphi := \inf\{1/r > 0 \mid \mathbb{B}(v, r) \subset -\Omega_\infty\} = 1/\operatorname{dist}(v, \operatorname{bd}(-\Omega_\infty)) = 1/\tau^{\max}_{-\Omega_\infty, v}$).*

Remark that from [8, Corollary 3.4] and [18, Proposition 4.2] we see that the function φ_{-v} (resp., T_v, φ_v) is finite-valued and Lipschitz if and only if $-v \in \operatorname{int} \Omega_\infty$ (resp., $v \in \operatorname{int} \Omega_\infty$, $v \in \operatorname{int} \Omega_\infty$).

When $\Omega = -K$ and $v \in \operatorname{int} K$, $\varphi_{-v}(y; \Omega) = \xi_v(y)$, Proposition 18 and Corollary 22 reduce to Proposition 5 or Proposition 7. When $\Omega = K$ and $v \in \operatorname{int} K$, Corollaries 20 and 21 reduce to the case that $\xi'_v(y) := \varphi_v(y; K) = \inf\{t \in \mathbb{R} \mid y \in -tv + K\}$.

3. Applications to the Hölder Continuity

The globally Lipschitz property of the nonlinear scalarization function ξ_q seems to be good at dealing with stability and sensitivity analysis of vector optimization problems, such as [15–17]. In this section, we will give some direct applications of this property to the Hölder continuity of solutions for parametric vector equilibrium problems. The proofs of the results obtained are applications of the corresponding ones in [25, 31] for the scalar problems, by using the mentioned scalarization function ξ_q. The results are new and generalizations of known ones for the corresponding scalar cases.

3.1. A Single-Valued Case. In this subsection, let (X, d_X) be a linear metric space, let Y be a linear normed space, let Λ and Ω be nonempty subsets of metric spaces, and let $C \subset Y$ be a pointed, closed, and convex cone with $\operatorname{int} C \neq \emptyset$. Let $K : \Lambda \rightrightarrows X$ be a set-valued mapping with nonempty, closed, and convex values and let $F : X \times X \times \Omega \to Y$ be a vector-valued mapping.

For the parameters $\lambda \in \Lambda$ and $\mu \in \Omega$, we consider the following parametric vector equilibrium problem (PVEP) of finding $\overline{x} \in K(\lambda)$ such that

$$F(\overline{x}, y, \mu) \notin -\operatorname{int} C, \quad \forall y \in K(\lambda). \tag{23}$$

Remark that when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the model (PVEP) reduces to the scalar one (PKFI) studied in [25].

Let $S(\lambda, \mu)$ be the subset of $K(\lambda)$ of the solutions of (PVEP). For the reference point $(\overline{\lambda}, \overline{\mu}) \in \Lambda \times \Omega$, we always assume that $S(\lambda, \mu) \neq \emptyset$ for every $\lambda \in U(\overline{\lambda})$ and $\mu \in U(\overline{\mu})$, where $U(v)$ denotes some neighborhood of the reference point v.

Definition 23 (classical notion). A set-valued mapping $G : \Omega \rightrightarrows X$ is said to be $\ell \cdot \alpha$-Hölder continuous at μ_0, if and only if there is a neighborhood $U(\mu_0)$ of μ_0 such that, $\forall \mu_1, \mu_2 \in U(\mu_0)$,

$$G(\mu_1) \subseteq G(\mu_2) + \ell d^\alpha(\mu_1, \mu_2)\, \mathbb{B}_X, \tag{24}$$

where $\ell \geq 0$ and $\alpha > 0$, and $\mathbb{B}_X$ denotes the unit ball of X.

In particular, when X is a linear normed space, the vector-valued mapping $g : \Omega \to X$ is said to be $\ell \cdot \alpha$-Hölder continuous at μ_0, if and only if $\|g(\mu_1) - g(\mu_2)\| \leq \ell d^\alpha(\mu_1, \mu_2)$.

We say that g (or G) is ℓ-Lipschitz continuous at μ_0 if anf only if g (or G) is $\ell \cdot 1$-Hölder continuous at μ_0.

Next, we introduce the concept of strong C-convexity for a vector-valued mapping, which extends [25, Definition 2.1] from real-valued to vector-valued case.

Definition 24. Let (X, d) be a linear metric space. A vector-valued mapping $g : X \to Y$ is said to be $h \cdot \alpha$-strongly C-convex with respect to $e \in \operatorname{int} C$ on X, if and only if there exists $e \in \operatorname{int} C$ such that

$$\begin{aligned} &tg(x) + (1-t)\, g(y) \\ &\quad - g(tx + (1-t)\, y) - ht(1-t)\, d^\alpha(x, y)\, e \in C, \end{aligned} \tag{25}$$

for all $x, y \in X$ and all $t \in]0, 1[$, where $h, \alpha > 0$.

Note that $e \in \operatorname{int} C$ plays the role of the "modulus of strong C-convexity" of the mapping g. Clearly, as in the scalar case, strong C-convexity implies (strict) C-convexity. As shown in [25], the strong C-convexity of g plays important roles.

Lemma 25. *If $g : X \to Y$ is $h \cdot \alpha$-strongly C-convex with respect to $e \in \operatorname{int} C$ on X, then the real-valued function $x \mapsto \xi_e(g(x))$ is $h \cdot \alpha$-strongly convex on X.*

Proof. For any $x, y \in X$ and all $t \in]0, 1[$, we have

$$\begin{aligned} &\xi_e(g(tx + (1-t)\, y)) \\ &\quad \leq \xi_e(tg(x) + (1-t)\, g(y) - ht(1-t)\, d^\alpha(x, y)\, e) \\ &\quad = \xi_e(tg(x) + (1-t)\, g(y)) - ht(1-t)\, d^\alpha(x, y) \\ &\quad \leq t\xi_e(g(x)) + (1-t)\, \xi_e(g(y)) - ht(1-t)\, d^\alpha(x, y), \end{aligned} \tag{26}$$

where the first and the third inequalities follow from the monotonicity and sublinearity of ξ_e, respectively, and the second equality follows from Proposition 2(v). Whence, the composite function $\xi_e \circ g$ is $h \cdot \alpha$-strongly convex on X. □

Now we state and prove the following results.

Theorem 26. *For problem (PVEP), assume that the solutions exist in a neighborhood of considered point $(\bar{\lambda}, \bar{\mu}) \in \Lambda \times \Omega$. Suppose that the following conditions hold.*

(i) *$K(\cdot)$ is $h \cdot \alpha$-Hölder continuous at $\bar{\lambda}$.*

(ii) *There exist neighborhoods $U(\bar{\lambda})$ of $\bar{\lambda}$ and $U(\bar{\mu})$ of $\bar{\mu}$ such that, for each $\mu \in U(\bar{\mu})$, $F(\cdot, \cdot, \mu)$ is pseudomonotone on $K(U(\bar{\lambda}))$, where $K(U(\bar{\lambda})) := \bigcup_{u \in U(\bar{\lambda})} K(u)$; that is, $\forall x, y \in K(U(\bar{\lambda}))$: $x \neq y$, $F(x, y, \mu) \notin -\operatorname{int} C \Rightarrow F(y, x, \mu) \in -C$.*

(iii) *For any $x \in K(U(\bar{\lambda}))$, $\mu \in U(\bar{\mu})$, $F(x, x, \mu) \in -\operatorname{bd} C$.*

(iv) *For any $y \in K(U(\bar{\lambda}))$, $\mu \in U(\bar{\mu})$, $F(\cdot, y, \mu)$ is ℓ-Lipschitz continuous on $K(U(\bar{\lambda}))$, and for any $x \in K(U(\bar{\lambda}))$, $\mu \in U(\bar{\mu})$, $F(x, \cdot, \mu)$ is $k \cdot \beta$-strongly C-convex with respect to $e \in \operatorname{int} C$ on X as well as ℓ'-Lipschitz continuous on $K(U(\bar{\lambda}))$.*

(v) *$F(x, y, \cdot)$ is $\widetilde{m} \cdot \gamma$-Hölder continuous at $\bar{\mu}$ with θ-relative to $K(U(\bar{\lambda}))$; that is, $\forall \mu_1, \mu_2 \in U(\bar{\mu})$, $\forall x, y \in K(U(\bar{\lambda}))$: $x \neq y$, $\|F(x, y, \mu_1) - F(x, y, \mu_2)\| \leq \widetilde{m} d_X^{\theta}(x, y) d^{\gamma}(\mu_1, \mu_2)$.*

(vi) *$\beta > \theta > 0$.*

Then for every $(\lambda, \mu) \in U(\bar{\lambda}) \times U(\bar{\mu})$, the solution of (PVEP) is unique, $x(\lambda, \mu)$, and this function satisfies the Hölder condition

$$\begin{aligned} & d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \\ & \quad \leq \left(\frac{2\gamma_e(\ell + \ell')h}{k} \right)^{1/\beta} d^{\alpha/\beta}(\lambda_1, \lambda_2) \\ & \quad + \left(\frac{2\gamma_e \widetilde{m}}{k} \right)^{1/(\beta-\theta)} d^{\gamma/(\beta-\theta)}(\mu_1, \mu_2), \end{aligned} \tag{27}$$

where $\gamma_e := \sup_{s \in C^e} \|s\| \in [1/\|e\|, +\infty[$ (or $\gamma_e = 1/\tau_{C,e}^{\max}$) is the Lipschitz constant of ξ_e on Y.

Proof. It follows from Proposition 2(i) that, for given $e \in \operatorname{int} C$,

$$S(\lambda, \mu) = \{x \in K(\lambda) \mid \xi_e(F(x, y, \mu)) \geq 0, \forall y \in K(\lambda)\}. \tag{28}$$

Thus, we could apply Theorem 3.1 of [25] by replacing f therein with $\xi_e \circ F$. Now we need to check all conditions of $\xi_e \circ F$.

First, by virtue of the globally Lipschitz property of ξ_e (see Proposition 5 or 7), on one hand, for $\mu \in U(\bar{\mu})$ and $y \in K(U(\bar{\lambda}))$, $\forall x_1, x_2 \in K(U(\bar{\lambda}))$,

$$\begin{aligned} & |\xi_e(F(x_1, y, \mu)) - \xi_e(F(x_2, y, \mu))| \\ & \quad \leq \gamma_e \|F(x_1, y, \mu) - F(x_2, y, \mu)\| \leq \gamma_e \ell d_X(x_1, x_2), \end{aligned} \tag{29}$$

and for $\mu \in U(\bar{\mu})$ and $x \in K(U(\bar{\lambda}))$, $\forall y_1, y_2 \in K(U(\bar{\lambda}))$,

$$\begin{aligned} & |\xi_e(F(x, y_1, \mu)) - \xi_e(F(x, y_2, \mu))| \\ & \quad \leq \gamma_e \|F(x, y_1, \mu) - F(x, y_2, \mu)\| \leq \gamma_e \ell' d_X(y_1, y_2). \end{aligned} \tag{30}$$

On the other hand, $\forall \mu_1, \mu_2 \in U(\bar{\mu})$, $\forall x, y \in K(U(\bar{\lambda}))$: $x \neq y$,

$$\begin{aligned} & |\xi_e(F(x, y, \mu_1)) - \xi_e(F(x, y, \mu_2))| \\ & \quad \leq \gamma_e \|F(x, y, \mu_1) - F(x, y, \mu_2)\| \\ & \quad \leq \gamma_e \widetilde{m} d_X^{\theta}(x, y) d^{\gamma}(\mu_1, \mu_2). \end{aligned} \tag{31}$$

Letting $f := \xi_e \circ F$, whence the Lipschitz or Hölder constants of [25, Theorem 3.1] are fulfilled with $l := \gamma_e \ell$, $l' := \gamma_e \ell'$ and $m := \gamma_e \widetilde{m}$.

Second, using Proposition 2(i)-(ii), condition (ii) implies that

$$\xi_e(F(x, y, \mu)) \geq 0 \Longrightarrow \xi_e(F(y, x, \mu)) \leq 0, \tag{32}$$

which shows that $\xi_e \circ F(\cdot, \cdot, \mu)$ is pseudomonotone on $K(U(\bar{\lambda}))$. In addition, it follows from Proposition 2(iii) and condition (iii) that, for any $x \in K(U(\bar{\lambda}))$, $\mu \in U(\bar{\mu})$, $\xi_e(F(x, x, \mu)) = 0$.

Third, by virtue of Lemma 25, it is clear that $\xi_e \circ F(x, \cdot, \mu)$ is $k \cdot \beta$-strongly convex on X. Thus, all conditions of [25, Theorem 3.1] are satisfied, and hence the conclusion follows. □

Apply Theorem 3.2 of [25] by replacing f therein with $\xi_e \circ F$, and combining with similar analysis of Theorem 26, we obtain the following result readily.

Theorem 27. *Suppose that the conditions of Theorem 26 are satisfied except (ii) and (iv), which are replaced by the following ones, respectively.*

(ii′) *There exist neighborhoods $U(\bar{\lambda})$ of $\bar{\lambda}$ and $U(\bar{\mu})$ of $\bar{\mu}$ such that, for each $\mu \in U(\bar{\mu})$, $F(\cdot, \cdot, \mu)$ is monotone with respect to ξ_e on $K(U(\bar{\lambda}))$; that is, $\forall x, y \in K(U(\bar{\lambda}))$: $x \neq y$, $\xi_e(F(x, y, \mu)) + \xi_e(F(y, x, \mu)) \leq 0$.*

(iv′) *For any $x \in K(U(\bar{\lambda}))$, $\mu \in U(\bar{\mu})$, $F(x, \cdot, \mu)$ is $k \cdot \beta$-strongly C-convex with respect to $e \in \operatorname{int} C$ on X as well as ℓ'-Lipschitz continuous on $K(U(\bar{\lambda}))$.*

Then for every $(\lambda,\mu)\in U(\overline{\lambda})\times U(\overline{\mu})$, the solution of (PVEP) is unique, $x(\lambda,\mu)$, and this function satisfies the Hölder condition

$$\begin{aligned} d_X(x(\lambda_1,\mu_1),x(\lambda_2,\mu_2)) &\le \left(\frac{4\gamma_e\ell' h}{k}\right)^{1/\beta} d^{\alpha/\beta}(\lambda_1,\lambda_2) \\ &\quad+\left(\frac{\gamma_e\widetilde{m}}{k}\right)^{1/(\beta-\theta)} d^{\gamma/(\beta-\theta)}(\mu_1,\mu_2). \end{aligned} \tag{33}$$

Remark 28. When $Y=\mathbb{R}$, $C=\mathbb{R}_+$, and $e=1$, $\gamma_e=1$, Theorem 26 (Theorem 27, resp.) reduces to Theorem 3.1 (Theorem 3.2, resp.) of [25]. So we have generalized [25, Theorems 3.1 and 3.2] to vector-valued setting. These results are new in the literature, and the proof is totally based on the technique of nonlinear scalarization.

Remark 29. (a) The monotonicity of $F(\cdot,\cdot,\mu)$ (condition (ii′) of Theorem 27) is stronger than the pseudomonotonicity (condition (ii) of Theorem 26). Under this case, we see that the assumption on the Lipschitz property of $F(\cdot,y,\mu)$ in (PVEP) is superfluous. (b) If $K(U(\overline{\lambda}))$ in condition (v) of Theorem 26 (Theorem 27, resp.) is bounded, then without loss of generality we can take $\theta=0$ in assumption (v), since $d_X(x,y)\le w$ for some $w>0$, $\forall x,y\in K(U(\overline{\lambda}))$. Thus, the condition "$\beta>\theta$" in Theorem 26 (Theorem 27, resp.) can be omitted.

3.2. A Set-Valued Case. In this subsection, let X, W, Z be linear normed spaces, $A\subset X$ nonempty, and $\Lambda\subset W$, $\Omega\subset Z$ nonempty and convex subsets. Let $K:\Lambda\rightrightarrows A$ have nonempty bounded convex values. Let Y be a linear normed space, and let $C\subset Y$ be a pointed, closed, and convex cone with $\operatorname{int}C\neq\emptyset$. Let $F:X\times X\times\Omega\to Y$ be a vector-valued mapping.

For $(\lambda,\mu)\in\Lambda\times\Omega$, $\varepsilon\ge 0$ and fixed $e\in\operatorname{int}C$, in this subsection we mainly consider the following parametric vector approximate equilibrium problem (PVAEP) of finding $\overline{x}\in K(\lambda)$ such that

$$F(y,\overline{x},\mu)\in\varepsilon e-C,\quad\forall y\in K(\lambda). \tag{34}$$

Obviously, it is a special case of the model of finding $\overline{x}\in K(\lambda)$ such that

$$F(y,\overline{x},\mu)\notin\varepsilon e+\operatorname{int}C,\quad\forall y\in K(\lambda), \tag{35}$$

which is the Minty-type dual problem to the Stampacchia-type primal problem (e.g., [41, Section 4]) of finding $\overline{x}\in K(\lambda)$ such that

$$F(\overline{x},y,\mu)\notin-\varepsilon e-\operatorname{int}C,\quad\forall y\in K(\lambda). \tag{36}$$

Stability for parametric variational problems of the Minty type has not received much attention so far. Very recently, Lalitha and Bhatia [38] have studied upper and lower semicontinuity of the solutions as well as the approximate solutions to a parametric quasivariational inequality of the Minty type. Chen and Li [30] have established upper Hölder continuity of the solutions to Minty-type parametric vector quasiequilibrium problems. In this subsection, by using nonlinear scalarization technique, we will study a special Minty-type parametric vector approximate equilibrium problem (PVAEP).

To (PVAEP) combining with Proposition 2(ii), we could introduce the following equivalent scalarization equilibrium problem (SEP) of finding $\overline{x}\in K(\lambda)$ such that

$$\xi_e(F(y,\overline{x},\mu))-\varepsilon\le 0,\quad\forall y\in K(\lambda). \tag{37}$$

Set $S_e(\varepsilon,\lambda,\mu):=\{x\in K(\lambda)\mid\xi_e(F(y,x,\mu))-\varepsilon\le 0,\forall y\in K(\lambda)\}$.

In this subsection, for fixed $e\in\operatorname{int}C$, we assume that $S_e(\varepsilon,\lambda,\mu)\neq\emptyset$ for small positive ε and (λ,μ) in a neighborhood of the considered point (λ_0,μ_0). In general, $S_e(\varepsilon,\lambda,\mu)$ may not be a singleton.

As pointed out in [15], we will show how to establish the Hölder continuity of set-valued solution mappings to parametric vector equilibrium problems without using any priori information of the solution sets, by employing nonlinear scalarization approach. In [17], we have given a positive answer to this subject. Now we give another answer herein.

For $A,B\subset X$, the Pompeiu-Hausdorff distance between A and B is defined as

$$H(A,B):=\max\left\{\sup_{a\in A}d(a,B),\ \sup_{b\in B}d(A,b)\right\}, \tag{38}$$

where $d(a,B):=\inf_{b\in B}d(a,b)$ and $d(A,b):=\inf_{a\in A}d(a,b)$.

Theorem 30. *For problem (SEP) (i.e., (PVAEP)), let $S_e(\varepsilon,\lambda,\mu)$ be nonempty for small $\varepsilon>0$ and (λ,μ) in a neighborhood of the considered point (λ_0,μ_0). Assume that the following conditions hold.*

(i) *$K(\cdot)$ is $l\cdot\alpha$-Hölder continuous at λ_0; that is, there is a neighborhood N of λ_0 such that, for all $\lambda_1,\lambda_2\in N$, $K(\lambda_1)\subseteq K(\lambda_2)+l\|\lambda_1-\lambda_2\|^{\alpha}\mathbb{B}_X$.*

(ii) *There is a neighborhood U of μ_0 such that, for each $y\in K(N)$ and $\mu\in U$, $F(y,\cdot,\mu)$ is C-convex on $K(N)$; that is, for every $x_1,x_2\in K(\lambda)$ and $t\in[0,1]$, $tF(y,x_1,\mu)+(1-t)F(y,x_2,\mu)\in F(y,tx_1+(1-t)x_2,\mu)+C$.*

(iii) *For $x,y\in K(N)$, $F(y,x,\cdot)$ is $\overline{h}\cdot\beta$-Hölder continuous on U.*

(iv) *For $\mu\in U$ and $x\in K(N)$, $F(\cdot,x,\mu)$ is $\overline{q}\cdot\delta$-Hölder continuous on $K(N)$.*

Then, for any $\overline{\varepsilon}>0$, S_e satisfies the following Hölder property on $[\overline{\varepsilon},+\infty[\times N\times U$:

$$\begin{aligned} &H(S_e(\varepsilon_1,\lambda_1,\mu_1),S_e(\varepsilon_2,\lambda_2,\mu_2)) \\ &\quad\le\overline{k}_1|\varepsilon_1-\varepsilon_2|+\overline{k}_2\|\mu_1-\mu_2\|^{\beta}+\overline{k}_3\|\lambda_1-\lambda_2\|^{\alpha\delta}, \end{aligned} \tag{39}$$

where $\overline{k}_1,\overline{k}_2,\overline{k}_3>0$ and depends on $\overline{\varepsilon},l,\alpha,\overline{h},\beta,\overline{q},\delta,\gamma_e$, and so forth. Herein $\gamma_e:=\sup_{s\in C^e}\|s\|\in[1/\|e\|,+\infty[$ (or $\gamma_e=1/\tau_{C,e}^{\max}$) is the Lipschitz constant of ξ_e on Y.

Proof. To apply [31, Theorem 2.1] by exchanging the roles of x and y, we need to check the convexity and the Hölder continuity of $\xi_e \circ F$. First, by virtue of the globally Lipschitz property of ξ_e, on one hand, for $x, y \in K(N)$, $\forall \mu_1, \mu_2 \in U$,

$$\begin{aligned} &\left|\xi_e\left(F\left(y,x,\mu_1\right)\right)-\xi_e\left(F\left(y,x,\mu_2\right)\right)\right| \\ &\quad \le \gamma_e\left\|F\left(y,x,\mu_1\right)-F\left(y,x,\mu_2\right)\right\| \le \gamma_e\overline{h}\left\|\mu_1-\mu_2\right\|^{\beta}. \end{aligned} \tag{40}$$

On the other hand, for $\mu \in U$ and $x \in K(N)$, $\forall y_1, y_2 \in K(N)$,

$$\begin{aligned} &\left|\xi_e\left(F\left(y_1,x,\mu\right)\right)-\xi_e\left(F\left(y_2,x,\mu\right)\right)\right| \\ &\quad \le \gamma_e\left\|F\left(y_1,x,\mu\right)-F\left(y_2,x,\mu\right)\right\| \le \gamma_e\overline{q}\left\|y_1-y_2\right\|^{\delta}. \end{aligned} \tag{41}$$

Letting $f := -\xi_e \circ F$, whence the Hölder constants of [31, Theorem 2.1] are fulfilled with $h := \gamma_e\overline{h}$ and $q := \gamma_e\overline{q}$.

Second, using the monotonicity and sublinearity of ξ_e, for $y \in K(N)$ and $\mu \in U$, $\forall x_1, x_2 \in K(\lambda)$ and $t \in [0,1]$,

$$\begin{aligned} &\xi_e\left(F\left(y,tx_1+(1-t)x_2,\mu\right)\right) \\ &\quad \le \xi_e\left(tF\left(y,x_1,\mu\right)+(1-t)F\left(y,x_2,\mu\right)\right) \\ &\quad \le t\xi_e\left(F\left(y,x_1,\mu\right)\right)+(1-t)\xi_e\left(F\left(y,x_2,\mu\right)\right), \end{aligned} \tag{42}$$

which implies that $\xi_e \circ F$ is convex with respect to the second argument, so $f := -\xi_e \circ F$ is concave.

Thus, all conditions of [31, Theorem 2.1] by exchanging the roles of x and y therein are satisfied, and hence the conclusion follows. □

Remark 31. When $Y = \mathbb{R}$, $C = \mathbb{R}_+$, and $e = 1$, $S_e(\varepsilon,\lambda,\mu) = \{x \in K(\lambda) \mid F(y,x,\mu) - \varepsilon \le 0,\ \forall y \in K(\lambda)\}$, Theorem 30 becomes that of Minty type corresponding to [31, Theorem 2.1]. Theorem 30 is new in the literature. In addition, the proof approach via nonlinear scalarization is different from the ones used in previous works [26–30] for set-valued solution mappings.

Similar to the discussion of [31, Corollary 2.3], we readily get the following result when $K(\lambda) \equiv K$, $\forall \lambda \in \Lambda$.

Corollary 32. *For problem (SEP), assume that $K(\lambda) = K$, a nonempty bounded convex subset of A and $S_e(\varepsilon,\mu)$ is nonempty for small $\varepsilon > 0$ and μ in a neighborhood of the considered point μ_0. Assume further that*

(i) *there is a neighborhood U of μ_0 such that, for each $y \in K$ and $\mu \in U$, $F(y,\cdot,\mu)$ is C-convex on K;*

(ii) *for $x, y \in K$, $F(y,x,\cdot)$ is $\overline{h}\cdot\beta$-Hölder continuous on U.*

Then, for any $\overline{\varepsilon} > 0$, S_e satisfies the following Hölder property on $[\overline{\varepsilon},+\infty[\times U$:

$$H\left(S_e\left(\varepsilon_1,\mu_1\right),S_e\left(\varepsilon_2,\mu_2\right)\right) \le \overline{k}_1\left|\varepsilon_1-\varepsilon_2\right|+\overline{k}_2\left\|\mu_1-\mu_2\right\|^{\beta}, \tag{43}$$

where $\overline{k}_1, \overline{k}_2 > 0$ and depends on $\overline{\varepsilon}, \overline{h}, \beta, \gamma_e$, and so forth. Herein γ_e is the Lipschitz constant of ξ_e on Y.

4. Conclusions

In this paper, motivated by the work of Tammer and Zălinescu [8], we deduce the globally Lipschitz property of ξ_q in linear normed spaces via the primal space approach. This is different from the dual space approach adopted by our previous works [15, 16]. The equivalence between them is established, and primal-dual interpretations of both expressions for globally Lipschitz constants are explained. Furthermore, exact characterizations to the globally Lipschitz property for general Gerstewitz function φ_{-v} are discussed. We mention that the expression for calculating Lipschitz constant is given when the ordering cone is polyhedral. As simple applications, Hölder continuity of solutions for parametric vector equilibrium problems is also showed. Besides this, the globally Lipschitz property of ξ_q and φ_{-v} seems to have many potential applications, for example, the stability and sensitivity analysis of vector equilibrium problems [15–17] and optimality conditions for vector optimization problems [8, 14]. For more applications, we will exploit in future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. Jahn, "Scalarization in vector optimization," *Mathematical Programming*, vol. 29, no. 2, pp. 203–218, 1984.

[2] J. Jahn, *Vector Optimization-Theory, Applications, and Extensions*, Springer, Berlin, Germany, 2nd edition, 2004.

[3] A. Zaffaroni, "Degrees of efficiency and degrees of minimality," *SIAM Journal on Control and Optimization*, vol. 42, no. 3, pp. 1071–1086, 2003.

[4] G. Eichfelder, "An adaptive scalarization method in multiobjective optimization," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1694–1718, 2009.

[5] F. Flores-Bazán and E. Hernández, "A unified vector optimization problem: complete scalarizations and applications," *Optimization*, vol. 60, no. 12, pp. 1399–1419, 2011.

[6] J.-B. Hiriart-Urruty, "Tangent cones, generalized gradients and mathematical programming in Banach spaces," *Mathematics of Operations Research*, vol. 4, no. 1, pp. 79–97, 1979.

[7] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer, New York, NY, USA, 2003.

[8] C. Tammer and C. Zălinescu, "Lipschitz properties of the scalarization function and applications," *Optimization*, vol. 59, no. 2, pp. 305–319, 2010.

[9] G.-y. Chen, X. Huang, and X. Yang, *Vector Optimization: Set-Valued and Variational Analysis*, Springer, Berlin, Germany, 2005.

[10] D. T. Luc, *Theory of Vector Optimization*, Springer, Berlin, Germany, 1989.

[11] C. Gerstewitz, "Nichtkonvexe Dualität in der Vektoroptimierung," *Leuna-Merseburg*, vol. 25, no. 3, pp. 357–364, 1983.

[12] A. Pascoletti and P. Serafini, "Scalarizing vector optimization problems," *Journal of Optimization Theory and Applications*, vol. 42, no. 4, pp. 499–524, 1984.

[13] C. Gerth and P. Weidner, "Nonconvex separation theorems and some applications in vector optimization," *Journal of Optimization Theory and Applications*, vol. 67, no. 2, pp. 297–320, 1990.

[14] M. Durea and C. Tammer, "Fuzzy necessary optimality conditions for vector optimization problems," *Optimization*, vol. 58, no. 4, pp. 449–467, 2009.

[15] C. R. Chen and M. H. Li, "Hölder continuity of solutions to parametric vector equilibrium problems with nonlinear scalarization," *Numerical Functional Analysis and Optimization*, vol. 35, no. 6, pp. 685–707, 2014.

[16] C. R. Chen, "Hölder continuity of the unique solution to parametric vector quasiequilibrium problems via nonlinear scalarization," *Positivity*, vol. 17, no. 1, pp. 133–150, 2013.

[17] M. H. Li, S. J. Li, and C. R. Chen, "Hölder-likeness and contingent derivative of solutions to parametric weak vector equilibrium problems," *Scientia Sinica. Series A, Mathematical, Physical, Astronomical and Technical Sciences*, vol. 43, pp. 61–74, 2013 (Chinese).

[18] N. M. Nam and C. Zălinescu, "Variational analysis of directional minimal time functions and applications to location problems," *Set-Valued and Variational Analysis*, vol. 21, no. 2, pp. 405–430, 2013.

[19] F. Giannessi, Ed., *Vector Variational Inequalities and Vector Equilibria: Mathematical Theories*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[20] Q. H. Ansari and J. C. Yao, Eds., *Recent Developments in Vector Optimization*, Springer, Berlin, Germany, 2012.

[21] L. Q. Anh and P. Q. Khanh, "On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 308–315, 2006.

[22] L. Q. Anh and P. Q. Khanh, "Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces," *Journal of Global Optimization*, vol. 37, no. 3, pp. 449–465, 2007.

[23] L. Q. Anh and P. Q. Khanh, "Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions," *Journal of Global Optimization*, vol. 42, no. 4, pp. 515–531, 2008.

[24] M. Bianchi and R. Pini, "Sensitivity for parametric vector equilibria," *Optimization*, vol. 55, no. 3, pp. 221–230, 2006.

[25] X. B. Li, X. J. Long, and J. Zeng, "Hölder continuity of the solution set of the Ky Fan inequality," *Journal of Optimization Theory and Applications*, vol. 158, no. 2, pp. 397–409, 2013.

[26] S. J. Li, X. B. Li, and K. L. Teo, "The Hölder continuity of solutions to generalized vector equilibrium problems," *European Journal of Operational Research*, vol. 199, no. 2, pp. 334–338, 2009.

[27] S. J. Li and X. B. Li, "Hölder continuity of solutions to parametric weak generalized Ky Fan inequality," *Journal of Optimization Theory and Applications*, vol. 149, no. 3, pp. 540–553, 2011.

[28] S. J. Li, C. R. Chen, X. B. Li, and K. L. Teo, "Hölder continuity and upper estimates of solutions to vector quasiequilibrium problems," *European Journal of Operational Research*, vol. 210, no. 2, pp. 148–157, 2011.

[29] C. R. Chen, S. J. Li, J. Zeng, and X. B. Li, "Error analysis of approximate solutions to parametric vector quasiequilibrium problems," *Optimization Letters*, vol. 5, no. 1, pp. 85–98, 2011.

[30] C. Chen and S. Li, "Upper Hölder estimates of solutions to parametric primal and dual vector quasi-equilibria," *Journal of Industrial and Management Optimization*, vol. 8, no. 3, pp. 691–703, 2012.

[31] L. Q. Anh, P. Q. Khanh, and T. N. Tam, "On Hölder continuity of approximate solutions to parametric equilibrium problems," *Nonlinear Analysis: Theory, Methods & Applications A: Theory and Methods*, vol. 75, no. 4, pp. 2293–2303, 2012.

[32] X. B. Li, S. J. Li, and C. R. Chen, "Lipschitz continuity of an approximate solution mapping to equilibrium problems," *Taiwanese Journal of Mathematics*, vol. 16, no. 3, pp. 1027–1040, 2012.

[33] N. H. Chieu, J.-C. Yao, and N. D. Yen, "Relationships between Robinson metric regularity and Lipschitz-like behavior of implicit multifunctions," *Nonlinear Analysis: Theory, Methods & Applications A: Theory and Methods*, vol. 72, no. 9-10, pp. 3594–3601, 2010.

[34] N. D. Yen, J.-C. Yao, and B. T. Kien, "Covering properties at positive-order rates of multifunctions and some related topics," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 467–478, 2008.

[35] C. R. Chen and S. J. Li, "Semicontinuity results on parametric vector variational inequalities with polyhedral constraint sets," *Journal of Optimization Theory and Applications*, vol. 158, no. 1, pp. 97–108, 2013.

[36] P. H. Sach and L. A. Tuan, "New scalarizing approach to the stability analysis in parametric generalized Ky Fan inequality problems," *Journal of Optimization Theory and Applications*, vol. 157, no. 2, pp. 347–364, 2013.

[37] P. H. Sach, "New nonlinear scalarization functions and applications," *Nonlinear Analysis: Theory, Methods & Applications A: Theory and Methods*, vol. 75, no. 4, pp. 2281–2292, 2012.

[38] C. S. Lalitha and G. Bhatia, "Stability of parametric quasivariational inequality of the Minty type," *Journal of Optimization Theory and Applications*, vol. 148, no. 2, pp. 281–300, 2011.

[39] C. Gutiérrez, B. Jiménez, and V. Novo, "New second-order directional derivative and optimality conditions in scalar and vector optimization," *Journal of Optimization Theory and Applications*, vol. 142, no. 1, pp. 85–106, 2009.

[40] Y. Sawaragi, H. Nakayama, and T. Tanino, *Theory of Multiobjective Optimization*, Academic Press, Orlando, Fla, USA, 1985.

[41] X. B. Li and S. J. Li, "Continuity of approximate solution mappings for parametric equilibrium problems," *Journal of Global Optimization*, vol. 51, no. 3, pp. 541–548, 2011.