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Research Article

Improvement of the Asymptotic Properties of Zero Dynamics for Sampled-Data Systems in the Case of a Time Delay

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It is well known that the existence of unstable zero dynamics is recognized as a major barrier in many control systems, and deeply limits the achievable control performance. When a continuous-time system with relative degree greater than or equal to three is discretized using a zero-order hold (ZOH), at least one of the zero dynamics of the resulting sampled-data model is obviously unstable for sufficiently small sampling periods, irrespective of whether they involve time delay or not. Thus, attention is here focused on continuous-time systems with time delay and relative degree two. This paper analyzes the asymptotic behavior of zero dynamics for the sampled-data models corresponding to the continuous-time systems mentioned above, and further gives an approximate expression of the zero dynamics in the form of a power series expansion up to the third order term of sampling period. Meanwhile, the stability of the zero dynamics is discussed for sufficiently small sampling periods and a new stability condition is also derived. The ideas presented here generalize well-known results from the delay-free control system to time-delay case.

1. Introduction

It is well known that unstable zero dynamics limit the control performance that can be achieved. Some techniques based on zero dynamics cancellation for control system design are hard to be applied when a plant has unstable zero dynamics [1, 2]. When a continuous-time system is discretized by a sampler and a zero-order hold (ZOH), stable poles are transformed into the unit circle. The transformations of zero dynamics, however, are much more complicated and the stability of zero dynamics is not preserved in the discretization process in some cases [1]. It is generally impossible to derive a closed-form expression that relates the continuous-time zero dynamics with the discrete-time ones because the zero dynamics of discrete-time systems depend on sampling period T. Therefore, the analysis of zero dynamics has a great deal of interest [1–7].

Perhaps the first attempt to study discrete system zero dynamics was given by Åström and coworkers [1], who described the asymptotic behavior of the discrete-time zero dynamics for fast sampling rate when the original continuous-time plant is discretized with ZOH. More research for discrete zero dynamics has been shown in the past decade, and these known results present the asymptotic characterization of zero dynamics for the discretized systems without time delay [8–15]. However, time delay of controlled systems inherently exists in many mechanical engineering applications and therefore must be integrated into system models. In fact, it is inevitable that the influence of time delay on digital control system must be considered because it also occurs frequently in information transmission between elements or systems, transport of controls and sensors, data computation, and so forth. Hence, it is important to investigate the asymptotic properties of zero dynamics in the sampled-data models corresponding to the continuous-time plants with time delay for the digital control system design.

Anyway, one would reasonably expect similar results in the case of delay-free systems to hold for time-delay plants. However, the situation for the time-delay case is more complex than for delay-free models. Indeed, to the best of our

knowledge, an explicit characterization of the zero dynamics for time-delay systems has previously remained unresolved, although an implicit characterization has been given in [16–18].

On the other hand, from the view point of the stability of the discrete zero dynamics, attention is here focused on continuous-time systems with relative degree equal to two. Furthermore, when the relative degree of a continuous-time transfer function with time delay is two, discrete zero dynamics, especially sampling zero dynamics, are located just on the unit circle, that is, in the marginal case of the stability when the sampling period tends to zero. Thus, the asymptotic behavior of the sampling zero dynamics is an interesting issue because it is stable for sufficiently small sampling periods if it approaches the unit circle from inside as the sampling period tends to zero.

Nevertheless, many problems remain unsolved in spite of the constant efforts. It is still interesting to find out the asymptotic properties on the zero dynamics of the sampled-data models with time delay and to derive new stability criteria of the zero dynamics though the impact of the stability of zero dynamics in sampled-data time-delay models may not have been strong in control engineering applications. In this paper, we present an approximate sampled-data model for a continuous-time-delay system, which is accurate to some order in the sampling period. We also show how a particular strategy can be used to approximate the system output and state variable and their derivatives in such a way as to obtain an approximate expression of discrete zero dynamics by Taylor expansion with respect to a sampling period up to the third order term. An insightful interpretation of the proposed results can be made in terms of an explicit characterization of these zero dynamics. Moreover, the zero dynamics turn out to be identical to those found in the delay-free case. Thus, the current paper extends the well-known notion of zero dynamics from the delay-free case to time-delay system.

The layout of the paper is structured as follows. In Section 2, we review the results of time delay and basic concepts. Section 3 presents the main result of this paper, namely, the asymptotic behavior of zero dynamics in the case of time delay and relative degree two. The numerical simulation is represented in Section 4. Finally, conclusions are presented in Section 5.

2. Preliminaries

In the present study, we consider a class of the following single-input single-output nth-order continuous-time control systems with time delay and state-space representations of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))\,u(t-D), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t)$ is the vector of the states evolving in an open subset $M \subset R^n$, $u \in R$ is the input variable, and D is the system constant time delay that directly affects the input. It is assumed that the vector fields $f(\cdot)$ and $g(\cdot)$ and the output function $h(\cdot)$ are analytic.

We are interested in the sampled-data model for the linear systems when the input is a piecewise constant signal generated by a ZOH. Thus, for a sampling period T,

$$u(t) = u(kT) \equiv u_k, \quad kT \le t < kT + T. \tag{2}$$

Furthermore, we suppose the time delay D and mesh T are related as follows:

$$D = qT + \gamma, \tag{3}$$

where $q \in 0, 1, 2, \ldots$ and $0 < \gamma < T$. Equivalently, the time delay D is customarily represented as an integer multiple of the sampling period plus a fractional part of T [19, 20]. Under the ZOH assumption and the above notation, it is rather straightforward to verify that the "delayed" input variable attains the following two distinct values within the sampling interval [21]:

$$u(t-D) = \begin{cases} u(kT - qT - T) \equiv u_1, & kT \le t < kT + \gamma, \\ u(kT - qT) \equiv u_2, & kT + \gamma \le t < kT + T. \end{cases} \tag{4}$$

When the relative degree of a continuous-time system with time delay is two, we consider the continuous-time transfer function of a control system (1):

$$\begin{aligned} G(s) &= K\frac{N(s)}{D(s)}e^{-Ds}, \quad K \ne 0, \\ N(s) &= s^{\lambda} + b_{\lambda-1}s^{\lambda-1} + b_{\lambda-2}s^{\lambda-2} + \cdots + b_0, \\ &\qquad \lambda = n-2, \\ D(s) &= s^{n} + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0, \end{aligned} \tag{5}$$

where a_i, $i = 0, 1, \ldots, n-1$ and b_j, $j = 0, 1, \ldots, \lambda - 1$ are real coefficients and D is the time delay. As it can be easily inferred from (5), the input variable $u(t-D)$ remains constant within the two subintervals: $[kT, kT+\gamma)$, $[kT+\gamma, kT+T)$. One readily obtains

$$\begin{aligned} x(kT+\gamma) &= \exp(A\gamma)\,x(kT) + u_1 \\ &\quad \times \int_{kT}^{kT+\gamma} \exp(A(kT+\gamma-\tau))\,b\,d\tau, \\ x(kT+T) &= \exp(A(T-\gamma))\,x(kT+\gamma) + u_2 \\ &\quad \times \int_{kT+\gamma}^{kT+T} \exp(A(kT+T-\tau))\,b\,d\tau \end{aligned} \tag{6}$$

from which

$$\begin{aligned} x(kT+T) &= \exp(AT)\,x(kT) + u_1 \int_{T-\gamma}^{T} \exp(A\tau)\,b\,d\tau \\ &\quad + u_2 \int_{0}^{T-\gamma} \exp(A\tau)\,b\,d\tau \\ &\approx \sum_{i=0}^{r-1} \frac{T^i}{i!} y_k^{(i)} + \sum_{j=r}^{\infty} \frac{T^j}{j!} c_m^j(\alpha)\, y_k^{(j)} \end{aligned} \tag{7}$$

in which the final equation of (7) is supported from the Appendix, and notice here that $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$ and further, where

$$c_m^p(\alpha) = \sum_{j=1}^{m} \left(\Lambda_{1j}^p - \Lambda_{2j}^p\right)\alpha_j,$$
$$\Lambda_{1j} = 1 - \frac{j-1}{m}, \qquad \Lambda_{2j} = 1 - \frac{j}{m}. \tag{8}$$

Remark 1. Expression (7) represents the sampled-data representation of the original continuous-time system (5) with time delay D. Note that the value of the state vector at $(k+1)T$ is a linear combination of the states evaluated at kT and the past values of the input variable u at $kT - qT - T$ and $kT - qT$.

The paper treats systems with relative degree two because at least one of the zero dynamics is unstable when the relative degree is greater than or equal to three though it is slightly a limitation.

First, the following assumptions are introduced.

Assumption 1. The unique equilibrium point lies on the origin.

Assumption 2. The continuous-time linear system (1) has the uniform relative degree r $(\leq n)$ and is minimum phase in the open subset M, where the state x evolves.

These assumptions ensure that there is a coordinate transformation such that we express the system in its normal form. The normal form of (5) with relative degree two, $\lambda = n-2$, is represented with an input $\tilde{u}$ and an output y as [22, 23]

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &\quad \times \left(K\tilde{u} - w - d_0\xi_1 - d_1\xi_2\right), \\ \dot{\eta} &= P\eta + q\xi_1, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\xi,\end{aligned} \tag{9}$$

where $\tilde{u} = u(t - D)$

$$\begin{aligned}&\xi = \begin{bmatrix}\xi_1 & \xi_2\end{bmatrix}^T, \qquad \eta = \begin{bmatrix}\eta_1 & \cdots & \eta_{n-2}\end{bmatrix}^T, \\ &w = c^T\eta, \qquad c = \begin{bmatrix}c_0 & c_1 & \cdots & c_{n-3}\end{bmatrix}^T, \\ &P = \begin{bmatrix} 0 & 1 & & O \\ & & \ddots & \\ & O & & 1 \\ -b_0 & \cdots & -b_{n-4} & -b_{n-3} \end{bmatrix}, \\ &q = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\end{aligned} \tag{10}$$

and the scalars d_i $(i = 0, 1)$ and c_i $(i = 0, \ldots, n-3)$ are obtained from

$$\begin{aligned}D(s) &= Q(s)N(s) + R(s), \\ Q(s) &= s^2 + d_1 s + d_0, \\ R(s) &= c_{n-3}s^{n-3} + \cdots + c_0,\end{aligned} \tag{11}$$

where

$$\begin{aligned}d_0 &= a_{n-2} - b_{n-4} - b_{n-3}d_1, \\ d_1 &= a_{n-1} - b_{n-3}, \\ c_i &= a_i - b_{i-2} - b_{i-1}d_1 - b_i d_0, \quad i = 0, \ldots, n-3.\end{aligned} \tag{12}$$

When a ZOH is used and the relations $\tilde{u} = \ddot{\tilde{u}} = \cdots = 0$ are taken notice of input-output relations, the normal form (9) yields the derivatives of the output

$$\begin{aligned}\ddot{y} &= K\tilde{u} - d_0\xi_1 - d_1\xi_2 - c^T\eta, \\ y^{(3)} &= -d_1K\tilde{u} + \left(d_0d_1 - c^Tq\right)\xi_1 + \left(d_1^2 - d_0\right)\xi_2 \\ &\quad + \left(d_1c^T - c^TP\right)\eta, \\ y^{(4)} &= \left(d_1^2 - d_0\right)K\tilde{u} + \left(d_0^2 - d_0d_1^2 + d_1c^Tq - c^TPq\right)\xi_1 \\ &\quad + \left(-c^Tq + 2d_0d_1 - d_1^3\right)\xi_2 \\ &\quad + \left\{-\left(d_1^2 - d_0\right)c^T + d_1c^TP - c^TP^2\right\}\eta, \\ y^{(5)} &= \left(-c^Tq + 2d_0d_1 - d_1^3\right)K\tilde{u} \\ &\quad + \left(2c^Tqd_0 - 2d_0^2d_1 + d_0d_1^3 - d_1^2c^Tq \right. \\ &\quad \left. + d_1c^TPq - c^TP^2q\right)\xi_1 \\ &\quad + \left(d_0^2 - 3d_0d_1^2 + 2d_1c^Tq - c^TPq + d_1^4\right)\xi_2 \\ &\quad + \left\{c^Tqc^T - 2d_0d_1c^T + d_1^3c^T - \left(d_1^2 - d_0\right)c^TP \right. \\ &\quad \left. + d_1c^TP^2 - c^TP^3\right\}\eta.\end{aligned} \tag{13}$$

Further, the derivatives of η are also represented as

$$\begin{aligned}\dot{\eta} &= P\eta + q\xi_1, \\ \ddot{\eta} &= Pq\xi_1 + q\xi_2 + P^2\eta, \\ \eta^{(3)} &= qK\tilde{u} + \left(P^2q - qd_0\right)\xi_1 + \left(Pq - qd_1\right)\xi_2 \\ &\quad + \left(P^3 - qc^T\right)\eta, \\ \eta^{(4)} &= \left(Pq - qd_1\right)K\tilde{u} \\ &\quad + \left(-Pqd_0 + qd_0d_1 + P^3q - qc^Tq\right)\xi_1 \\ &\quad + \left(-qd_0 + P^2q - Pqd_1 + qd_1^2\right)\xi_2 \\ &\quad + \left(-Pqc^T + qd_1c^T + P^4 - qc^TP\right)\eta.\end{aligned} \tag{14}$$

Hence, substitute (13)–(14) into the right hand side of

$$\begin{aligned} y_{k+1} &= \sum_{i=0}^{1}\frac{T^i}{i!}y_k^{(i)} + \sum_{j=2}^{5}\frac{T^j}{j!}c_m^j(\alpha)\,y_k^{(j)} + O\left(T^6\right), \\ \dot{y}_{k+1} &= \sum_{i=0}^{1}\frac{T^i}{i!}y_k^{(i+1)} + \sum_{j=2}^{4}\frac{T^j}{j!}c_m^j(\alpha)\,y_k^{(j+1)} + O\left(T^5\right), \\ \eta_{k+1} &= \sum_{i=0}^{4}\frac{T^i}{i!}\,\eta_k^{(i)} + O\left(T^5\right) \end{aligned} \tag{15}$$

and define the state variables $x_k = [y_k, \dot{y}_k, \eta_k^T]^T$, then discrete-time state equations are obtained.

Now, the zero dynamics of the discrete-time system (15) are analyzed using the explicit expressions of $y_k, \dot{y}_k, \ldots, y_k^{(5)}$ and $\eta_k, \ldots, \eta_k^{(4)}$ as follows:

$$\begin{aligned} y_{k+1} &= \sum_{i=0}^{1}\frac{T^i}{i!}y_k^{(i)} + \sum_{j=2}^{5}\frac{T^j}{j!}c_m^j(\alpha)\,y_k^{(j)} + O\left(T^6\right) \\ &= \Bigg(1 - \frac{d_0}{2}c_m^2(\alpha)T^2 + \frac{d_0d_1 - c^Tq}{6}c_m^3(\alpha)T^3 \\ &\quad + \frac{d_0^2 - d_0d_1^2 + d_1c^Tq - c^TPq}{24}c_m^4(\alpha)T^4 \\ &\quad + \frac{2c^Tqd_0 - 2d_0^2d_1 - d_1^2c^Tq}{120}c_m^5(\alpha)T^5 \\ &\quad + \frac{d_1c^TPq - c^TP^2q + d_0d_1^3}{120}c_m^5(\alpha)T^5\Bigg)y_k \\ &\quad + \Bigg(T - \frac{d_1}{2}c_m^2(\alpha)T^2 + \frac{d_1^2 - d_0}{6}c_m^3(\alpha)T^3 \\ &\quad + \frac{2d_0d_1 - c^Tq - d_1^3}{24}c_m^4(\alpha)T^4 + \frac{d_0^2}{120}c_m^5(\alpha)T^5 \\ &\quad + \frac{2d_1c^Tq - c^TPq - 3d_0d_1^2 + d_1^4}{120}c_m^5(\alpha)T^5\Bigg)\dot{y}_k \\ &\quad + \Bigg(\frac{c_m^2(\alpha)T^2}{2} - \frac{d_1}{6}c_m^3(\alpha)T^3 + \frac{d_1^2 - d_0}{24}c_m^4(\alpha) \\ &\quad \times T^4 + \frac{2d_0d_1 - c^Tq - d_1^3}{120}c_m^5(\alpha)T^5\Bigg)K\tilde{u}_k \\ &\quad + \Bigg\{-\frac{c^T}{2}c_m^2(\alpha)T^2 + \frac{d_1c^T - c^TP}{6}c_m^3(\alpha)T^3 \\ &\quad + \frac{-\left(d_1^2 - d_0\right)c^T + d_1c^TP - c^TP^2}{24}c_m^4(\alpha)T^4 \\ &\quad + \frac{c^Tqc^T - 2d_0d_1c^T + d_1^3c^T}{120}c_m^5(\alpha)T^5 \\ &\quad + \frac{-\left(d_1^2 - d_0\right)c^TP - c^TP^3}{120}c_m^5(\alpha)T^5 \\ &\quad + \frac{d_1c^TP^2}{120}c_m^5(\alpha)T^5\Bigg\}\eta_k + O\left(T^6\right), \end{aligned}$$

$$\begin{aligned} \dot{y}_{k+1} &= \sum_{i=0}^{1}\frac{T^i}{i!}y_k^{(i+1)} + \sum_{j=2}^{4}\frac{T^j}{j!}c_m^j(\alpha)\,y_k^{(j+1)} + O\left(T^5\right) \\ &= \Bigg(-d_0T + \frac{d_0d_1 - c^Tq}{2}c_m^2(\alpha)T^2 \\ &\quad + \frac{d_0^2 - d_0d_1^2 + d_1c^Tq - c^TPq}{6}c_m^3(\alpha)T^3 \\ &\quad + \frac{2c^Tqd_0 - 2d_0^2d_1 + d_0d_1^3 - d_1^2c^Tq}{24}c_m^4(\alpha)T^4 \\ &\quad + \frac{d_1c^TPq - c^TP^2q}{24}c_m^4(\alpha)T^4\Bigg)y_k \\ &\quad + \Bigg(1 - d_1T + \frac{d_1^2 - d_0}{2}c_m^2(\alpha)T^2 \\ &\quad + \frac{2d_0d_1 - c^Tq - d_1^3}{6}c_m^3(\alpha)T^3 + \frac{d_0^2}{24}c_m^4(\alpha)T^4 \\ &\quad + \frac{2d_1c^Tq - 3d_0d_1^2 - c^TPq + d_1^4}{24}c_m^4(\alpha)T^4\Bigg)\dot{y}_k \\ &\quad + \Bigg(T - \frac{d_1}{2}c_m^2(\alpha)T^2 + \frac{d_1^2 - d_0}{6}c_m^3(\alpha)T^3 \\ &\quad + \frac{-c^Tq + 2d_0d_1 - d_1^3}{24}c_m^4(\alpha)T^4\Bigg)K\tilde{u}_k \\ &\quad + \Bigg\{-c^TT + \frac{d_1c^T - c^TP}{2}c_m^2(\alpha)T^2 \\ &\quad + \frac{-\left(d_1^2 - d_0\right)c^T + d_1c^TP - c^TP^2}{6}c_m^3(\alpha)T^3 \\ &\quad + \frac{c^Tqc^T - 2d_0d_1c^T + d_1^3c^T - c^TP^3}{24}c_m^4(\alpha)T^4 \\ &\quad + \frac{d_1c^TP^2 - \left(d_1^2 - d_0\right)c^TP}{24}c_m^4(\alpha)T^4\Bigg\}\eta_k \\ &\quad + O\left(T^5\right), \end{aligned}$$

$$\begin{aligned} \eta_{k+1} &= \sum_{i=0}^{4}\frac{T^i}{i!}\,\eta_k^{(i)} + O\left(T^5\right) \\ &= \Bigg(qT + \frac{Pq}{2}T^2 + \frac{P^2q - qd_0}{6}T^3 \\ &\quad + \frac{P^3q - qc^Tq - Pqd_0 + qd_0d_1}{24}T^4\Bigg)y_k \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{q}{2}T^2+\frac{Pq-qd_1}{6}T^3\right.\\
&\qquad\left.+\frac{P^2q-qd_0-Pqd_1+qd_1^2}{24}T^4\right)\dot{y}_k\\
&+\left(\frac{q}{6}T^3+\frac{Pq-qd_1}{24}T^4\right)K\tilde{u}_k\\
&+\left(I+PT+\frac{P^2}{2}T^2+\frac{P^3-qc^T}{6}T^3\right.\\
&\qquad\left.+\frac{P^4-Pqc^T-qc^TP+qd_1c^T}{24}T^4\right)\eta_k\\
&+O\left(T^5\right).
\end{aligned}
\tag{16}
$$

3. Main Results

In this section, an approximate expression of zero dynamics of a discrete-time system with time delay and relative degree two is obtained from (15). The main result is given by Theorem 2.

Theorem 2. *The zero dynamics of a discrete-time system with time delay and relative degree two for a continuous-time transfer function* (5) *are approximately given for* $T \ll 1$ *by the roots of*

$$
\begin{aligned}
&(z+1)\left[-z+1-\frac{2}{c_m^2(\alpha)}+\frac{\widetilde{A}}{9\left[c_m^2(\alpha)\right]^2}T+\frac{\overline{A}}{18}\right.\\
&\qquad\left.\times\frac{T^2}{\left[c_m^2(\alpha)\right]^3}+\left(\frac{A_1}{108\left[c_m^2(\alpha)\right]^4}+\frac{A_2}{6\left[c_m^2(\alpha)\right]^2}\right)T^3\right]\\
&\times\left|(1-z)I+PT+\frac{P^2}{2}T^2+\frac{P^3}{6}T^3\right|=0,
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
\widetilde{A}&=6d_1c_m^2(\alpha)c_m^3(\alpha)-2d_1c_m^3(\alpha)-9d_1\left[c_m^2(\alpha)\right]^2,\\
\overline{A}&=3\left(d_1^2-d_0\right)c_m^2(\alpha)\\
&\quad\times\left\{c_m^4(\alpha)-4c_m^2(\alpha)c_m^3(\alpha)+3\left[c_m^2(\alpha)\right]^3\right\}\\
&\quad+8d_1c_m^3(\alpha)\left[3c_m^2(\alpha)-1\right],\\
A_1&=2d_1c_m^3(\alpha)\\
&\quad\times\left\{\left(d_1^2-d_0\right)c_m^2(\alpha)\right.\\
&\quad\times\left[2c_m^4(\alpha)-36c_m^2(\alpha)c_m^3(\alpha)+3c_m^2(\alpha)c_m^4(\alpha)\right]\\
&\quad\left.+24d_1c_m^3(\alpha)\left[3c_m^2(\alpha)-1\right]\right\},\\
A_2&=\left(2d_0d_1-c_{n-3}-d_1^3\right)\left\{c_m^4(\alpha)-\left[c_m^2(\alpha)\right]^2\right\}\\
&\quad-2d_1\left(d_1^2-d_0\right)\left[c_m^2(\alpha)\right]^2c_m^3(\alpha).
\end{aligned}
\tag{18}
$$

Proof. Zero dynamics of a discrete-time system with time delay and relative degree two (15), equivalently (16), are given by substituting $y_k = y_{k+1} = 0$ into (16) as follows:

$$
M\begin{bmatrix} Y_d\\ KU_{k-1}\\ KU_k\\ H\end{bmatrix}=0_n,
\tag{19}
$$

where Y_d, U_{k-1}, U_k, and H are the z-transforms of $\dot{y}_k$, u_1, u_2, and η_k, respectively, and the matrix M is defined by

$$
M=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14}^T\\ m_{21} & m_{22} & m_{23} & m_{24}^T\\ 0 & z & 1 & 0^T\\ m_{41} & m_{42} & m_{43} & M_{44}\end{bmatrix}
\tag{20}
$$

with

$$
\begin{aligned}
m_{11}&=T\overline{m}_{11}+O\left(T^5\right)\\
&=T\left(1-\frac{d_1}{2}c_m^2(\alpha)T+\frac{d_1^2-d_0}{6}c_m^3(\alpha)T^2\right.\\
&\qquad\left.+\frac{-c_{n-3}+2d_0d_1-d_1^3}{24}c_m^4(\alpha)T^3\right)\\
&\quad+O\left(T^5\right),\\
m_{12}&=T^2\overline{m}_{12}+O\left(T^6\right)\\
&=T^2\left(\frac{c_m^2(\alpha)}{2}-\frac{d_1}{6}c_m^3(\alpha)T+\frac{d_1^2-d_0}{24}c_m^4(\alpha)T^2\right.\\
&\qquad\left.+\frac{2d_0d_1-c_{n-3}-d_1^3}{120}c_m^5(\alpha)T^3\right)\\
&\quad+O\left(T^6\right),\\
m_{13}&=-T^2\overline{m}_{13}+O\left(T^6\right)\\
&=-T^2\left(\frac{c_m^2(\alpha)}{2}-\frac{d_1}{6}c_m^3(\alpha)T+\frac{d_1^2-d_0}{24}c_m^4(\alpha)T^2\right.\\
&\qquad\left.+\frac{2d_0d_1-c_{n-3}-d_1^3}{120}c_m^5(\alpha)T^3\right)\\
&\quad+O\left(T^6\right),
\end{aligned}
$$

$$
\begin{aligned}
m_{14}^T &= T\overline{m}_{14}^T + O\left(T^5\right) \\
&= T\left(-\frac{c^T}{2}c_m^2(\alpha)T + \frac{d_1c^T - c^TP}{6}c_m^3(\alpha)T^2 \right. \\
&\qquad \left. + \frac{\left(d_0 - d_1^2\right)c^T + d_1c^TP - c^TP^2}{24}c_m^4(\alpha)T^3\right) \\
&\quad + O\left(T^5\right), \\
m_{21} &= -z + 1 - d_1T + \frac{d_1^2 - d_0}{2}c_m^2(\alpha)T^2 \\
&\quad + \frac{2d_0d_1 - c_{n-3} - d_1^3}{6}c_m^3(\alpha)T^3 + O\left(T^4\right), \\
m_{22} &= T\left(1 - \frac{d_1}{2}c_m^2(\alpha)T + \frac{d_1^2 - d_0}{6}c_m^3(\alpha)T^2 \right. \\
&\qquad \left. + \frac{2d_0d_1 - c_{n-3} - d_1^3}{24}c_m^4(\alpha)T^3\right) + O\left(T^5\right), \\
m_{23} &= -T\left(1 - \frac{d_1}{2}c_m^2(\alpha)T + \frac{d_1^2 - d_0}{6}c_m^3(\alpha)T^2 \right. \\
&\qquad \left. + \frac{2d_0d_1 - c_{n-3} - d_1^3}{24}c_m^4(\alpha)T^3\right) \\
&\quad + O\left(T^5\right), \\
m_{24}^T &= -c^TT + \frac{d_1c^T - c^TP}{2}c_m^2(\alpha)T^2 \\
&\quad - \frac{d_1c^TP - \left(d_1^2 - d_0\right)c^T - c^TP^2}{6}c_m^3(\alpha)T^3 \\
&\quad + O\left(T^4\right), \\
m_{41} &= \frac{q}{2}T^2 + \frac{Pq - qd_1}{6}T^3 + O\left(T^4\right), \\
m_{42} &= T\left(\frac{q}{6}T^2 + \frac{Pq - qd_1}{24}T^3\right) + O\left(T^5\right), \\
m_{43} &= -T\left(\frac{q}{6}T^2 + \frac{Pq - qd_1}{24}T^3\right) + O\left(T^5\right), \\
M_{44} &= (-z + 1)I + PT + \frac{P^2}{2}T^2 \\
&\quad + \frac{P^3 - qc^T}{6}T^3 + O\left(T^4\right).
\end{aligned} \tag{21}
$$

Thus, the zero dynamics are derived from

$$
|M| = z\left|M_1\right| + \left|M_2\right| = 0, \tag{22}
$$

where

$$
\begin{aligned}
\left|M_1\right| &= \begin{vmatrix} m_{11} & m_{13} & m_{14}^T \\ m_{21} & m_{23} & m_{24}^T \\ m_{41} & m_{43} & M_{44} \end{vmatrix}, \\
\left|M_2\right| &= \begin{vmatrix} m_{11} & m_{12} & m_{14}^T \\ m_{21} & m_{22} & m_{24}^T \\ m_{41} & m_{42} & M_{44} \end{vmatrix}.
\end{aligned} \tag{23}
$$

Here, consider matrices M_a, $M_{1,a}$, and $M_{2,a}$ which are defined by neglecting the higher-order terms $O(\cdot)$ with respect to T in the block matrices M, M_1 and M_2 because the interests lie in the case of $T \ll 1$.

Multiplying $M_{1,a}$ and $M_{2,a}$ by

$$
R = \begin{bmatrix} 1 & & & & \\ & \frac{1}{T} & & O & \\ & & 1 & & \\ & O & & \ddots & \\ & & & & 1 \end{bmatrix} \tag{24}
$$

from the right hand side and by

$$
L = \begin{bmatrix} \frac{1}{T} & 0_{n-1}^T \\ l_1 & \\ 0_{n-4} & I_{n-1} \\ l_2 & \\ l_3 & \end{bmatrix}, \tag{25}
$$

where

$$
\begin{aligned}
l_1 &= -\frac{1}{m_0}\left(1 - \frac{d_1}{2}c_m^2(\alpha)T + \frac{d_1^2 - d_0}{6}c_m^3(\alpha)T^2 \right. \\
&\qquad \left. - \frac{2d_0d_1 - c_{n-3} - d_1^3}{24}c_m^4(\alpha)T^3\right), \\
l_2 &= -\frac{T^4}{120m_0} \\
l_3 &= -\frac{1}{m_0}\left(\frac{T^3}{24} - \frac{a_{n-1}}{120}T^4\right), \\
m_0 &= \frac{c_m^2(\alpha)}{2} - \frac{d_1}{6}c_m^3(\alpha)T + \frac{d_1^2 - d_0}{24}c_m^4(\alpha)T^2 \\
&\quad + \frac{2d_0d_1 - c_{n-3} - d_1^3}{120}c_m^5(\alpha)T^3
\end{aligned} \tag{26}
$$

from the left hand side and further multiplying the result by

$$\overline{L} = \begin{bmatrix} 1 & 0 & \\ & & O_{2\times(n-2)} \\ 0 & 1 & \\ 0_{n-4} & 0_{n-4} & \\ 0 & \widetilde{l}_1 & I_{n-2} \\ 0 & \widetilde{l}_2 & \end{bmatrix},$$

$$\widetilde{l}_1 = \frac{T^3}{12m_1},$$

$$\widetilde{l}_2 = \frac{1}{m_1}\left[\frac{T^2}{6} + \left(\frac{d_1}{18} - \frac{a_{n-1}}{12}\right)T^3\right],$$

$$m_1 = -z - 1 + \frac{2}{c_m^2(\alpha)} + \frac{\widetilde{A}}{9[c_m^2(\alpha)]^2}T - \frac{\overline{A}}{18[c_m^2(\alpha)]^3}T^2 + \left\{\frac{A_1}{108[c_m^2(\alpha)]^4} + \frac{A_2}{6[c_m^2(\alpha)]^2}\right\}T^3, \tag{27}$$

where

$$\begin{aligned}
\widetilde{A} &= 6d_1c_m^2(\alpha)c_m^3(\alpha) - 2d_1c_m^3(\alpha) - 9d_1\left[c_m^2(\alpha)\right]^2, \\
\overline{A} &= 3\left(d_1^2 - d_0\right)c_m^2(\alpha) \\
&\quad \times \left\{c_m^4(\alpha) - 4c_m^2(\alpha)c_m^3(\alpha) + 3\left[c_m^2(\alpha)\right]^3\right\} \\
&\quad + 8d_1c_m^3(\alpha)\left[3c_m^2(\alpha) - 1\right], \\
A_1 &= 2d_1c_m^3(\alpha) \\
&\quad \times \left\{\left(d_1^2 - d_0\right)c_m^2(\alpha)\right. \\
&\quad \times \left[2c_m^4(\alpha) - 36c_m^2(\alpha)c_m^3(\alpha) + 3c_m^2(\alpha)c_m^4(\alpha)\right] \\
&\quad \left. + 24d_1c_m^3(\alpha)\left[3c_m^2(\alpha) - 1\right]\right\}, \\
A_2 &= \left(2d_0d_1 - c_{n-3} - d_1^3\right)\left\{c_m^4(\alpha) - \left[c_m^2(\alpha)\right]^2\right\} \\
&\quad - 2d_1\left(d_1^2 - d_0\right)\left[c_m^2(\alpha)\right]^2c_m^3(\alpha)
\end{aligned} \tag{28}$$

from the left hand side yield

$$\overline{L}LM_aR = (z+1)\begin{bmatrix} \overline{m}_{11} & \overline{m}_{12} & m_{14}^T \\ \overline{m}_{21} & 0 & m_{24}^T \\ 0_{n-2} & 0_{n-2} & \overline{M}_{44} \end{bmatrix}, \tag{29}$$

where

$$\begin{aligned}
\overline{m}_{21} &= -z - 1 + \frac{2}{c_m^2(\alpha)} + \frac{\widetilde{A}}{9[c_m^2(\alpha)]^2}T - \frac{\overline{A}}{18[c_m^2(\alpha)]^3}T^2 \\
&\quad + \left\{\frac{A_1}{108[c_m^2(\alpha)]^4} + \frac{A_2}{6[c_m^2(\alpha)]^2}\right\}T^3 + O\left(T^4\right), \\
\overline{m}_{24}^T &= -\frac{c^TP}{6}T^2 + \frac{-3c^TP^2 + d_1c^TP}{36}T^3 + O\left(T^4\right), \\
\overline{M}_{44} &= (-z+1)I + PT + \frac{P^2}{2!}T^2 + \frac{P^3}{3!}T^3 + O\left(T^4\right).
\end{aligned} \tag{30}$$

Noting here that $|R| = |L| = 1/T$, $|\overline{L}| = 1$ leads to

$$\left|M_a\right| = -T^2(z+1)\overline{m}_{12}\overline{m}_{21}\left|\overline{M}_{44}\right|. \tag{31}$$

Hence, the approximate values of the zero dynamics of the discrete-time system with time delay and relative degree two are obtained as the roots of (17).

As a result, the proof is complete. □

Remark 3. The approximate values of the sampling zero dynamics are expressed as

$$\begin{aligned}
&z = -1, \\
&z = -1 + \frac{2}{c_m^2(\alpha)} + \frac{\widetilde{A}}{9[c_m^2(\alpha)]^2}T - \frac{\overline{A}}{18[c_m^2(\alpha)]^3}T^2 \\
&\quad + \left\{\frac{A_1}{108[c_m^2(\alpha)]^4} + \frac{A_2}{6[c_m^2(\alpha)]^2}\right\}T^3
\end{aligned} \tag{32}$$

and the approximate values of the intrinsic zero dynamics are derived from

$$\left|(1-z)I + PT + \frac{P^2}{2}T^2 + \frac{P^3}{6}T^3\right| = 0. \tag{33}$$

Remark 4. An insightful observation in Theorem 2 is that it clearly gives the approximate values with higher order of accuracy than the previous result [16, 17] in the form of a power term of T.

Remark 5. When the ZOH signal reconstruction device is used, notice here that the delay-free discrete-time systems can be considered as a particular case of the discretized plants with time delay when $m = 1$ and $\alpha_1 = \alpha_2$.

In particular, consider the simplest possible system of relative degree two, namely, $1/s^2$. Then we have the corresponding sampling zero dynamics in the case of a time delay

$$(z+1)\left(-z - 1 + \frac{2T^2}{D(2T-D)\alpha_1 + (T-D)\alpha_2}\right) = 0. \tag{34}$$

It is easy to obtain the following corollary and theorem from Theorem 2. The following corollary shows

the asymptotic behavior of the sampling zero dynamics when a continuous-time system has a time delay and relative degree two, and the sum of the zero dynamics of a continuous-time system is equal to the sum of the poles; that is, $d_1 = a_{n-1} - b_{n-3} = 0$. A new stability condition of the zero dynamics, which is shown in the following theorem, for discrete-time system with time delay and relative degree two (5) is also presented by analyzing (32).

Corollary 6. *Assume that the relative degree of a continuous-time system is two, and $d_1 = a_{n-1} - b_{n-3} = 0$. The sampling zero dynamics of a discrete-time system corresponding to a continuous-time transfer function with time delay* (5) *are expressed as*

$$\begin{aligned} z &= -1, \\ z &= -1 + \frac{2}{c_m^2(\alpha)} \\ &\quad + \frac{3d_0 c_m^2(\alpha)\left\{c_m^4(\alpha) - 4c_m^2(\alpha)c_m^3(\alpha) + 3\left[c_m^2(\alpha)\right]^3\right\}}{18\left[c_m^2(\alpha)\right]^3}T^2 \\ &\quad - \frac{c_{n-3}\left\{c_m^4(\alpha) - \left[c_m^2(\alpha)\right]^3\right\}}{6\left[c_m^2(\alpha)\right]^2}T^3 \\ &\quad + O\left(T^4\right). \end{aligned} \tag{35}$$

Theorem 7. *Consider a transfer function with time delay in the case of relative degree two* (5). *Then, for sufficiently small sampling periods, all the zero dynamics of the corresponding sampled-data model* (15) *are stable if all the zero dynamics of* (5) *are stable and*

$$\begin{aligned} &\frac{2T^2}{D(2T-D)\alpha_1 + (T-D)\alpha_2} > 0, \\ &\frac{T^2 - D(2T-D)\alpha_1 - (T-D)\alpha_2}{D(2T-D)\alpha_1 + (T-D)\alpha_2} < 0. \end{aligned} \tag{36}$$

Proof. First, we have

$$\begin{aligned} c_m^2(\alpha) &= \sum_{j=1}^{m} \frac{1}{m}\left(2 - \frac{2j}{m} + \frac{1}{m}\right)\alpha_j \\ &= \frac{1}{m}\left[\frac{2m-1}{m}\alpha_1 + \left(\frac{2m-3}{m} + \cdots + \frac{1}{m}\right)\alpha_2\right] \\ &= \frac{1}{m}\left[\frac{2m-1}{m}\alpha_1 + \frac{(m-1)^2}{m}\alpha_2\right] \\ &= \frac{D}{T}\left[\frac{2T-D}{T}\alpha_1 + \frac{(T-D)^2}{TD}\alpha_2\right]. \end{aligned} \tag{37}$$

Thus

$$\begin{aligned} |\Delta| &= -1 + \frac{2}{c_m^2(\alpha)} \\ &= \frac{(2D\alpha_1 + \alpha_2)T - D^2\alpha_1 - D\alpha_2 - 2T^2}{D(2T-D)\alpha_1 + (T-D)\alpha_2}. \end{aligned} \tag{38}$$

Next, it is obvious that all of the zero dynamics are stable, that is, located strictly inside the unit circle when satisfying $|\Delta| < 1$ $(-1 < \Delta < 1)$ by selecting the suitable parameters α_1, α_2, sampling period T, and time delay D. Simple straightforward calculation can verify that all the discretized zero dynamics are stable if (36) holds.

As a result, the proof is complete. □

Remark 8. When a continuous-time system with time delay reduces to a delay-free model, stability of the corresponding discrete-time zero dynamics can be decided by d_1, that is, the difference between the sum of the zero dynamics and that of the poles for a continuous-time plant. In particular, when $d_1 = 0$, then the corresponding discretized system has the stable zero dynamics if $c_{n-3} = a_{n-3} - b_{n-5} - b_{n-3}(a_{n-2} - b_{n-4}) < 0$.

Remark 9. A key interpretation is given in terms of an explicit characterization of the discretized zero dynamics of the obtained model. Of particular interest is that the contribution of this paper on the linear single-input single-output cases with time delay can be extended the multivariable time-delay systems and further will be generalized the nonlinear plants in the case of a time delay.

4. Numerical Simulation

In this section, we present an interesting example to show the stability of discrete zero dynamics with time delay and relative degree two. It has been also shown that the stability of zero dynamics, in the case of a time delay and relative degree two, will be improved by comparing with the previous results. Both kinds of the zero dynamics are calculated by applying MATLAB, and in the simulation figures, the solid line (blue) and dotted line (red) indicate the exact values and approximate values, respectively.

Consider a transfer function with time delay and relative degree two, and the sum of the zero dynamics is equal to the sum of the poles as follows:

$$G(s) = \frac{s+6}{(s+1)(s+2)(s+3)}e^{-Ds}. \tag{39}$$

It is obvious that

$$d_1 = 0, \qquad c_0 = a_0 - b_0 a_1 = -60. \tag{40}$$

The intrinsic zero dynamics z_1 and the sampling zero dynamics z_2, z_3 are expressed approximately as, respectively,

$$\begin{aligned} z_1 &\approx 1 - 6T + 18T^2 - 36T^3, \\ z_2 &\approx -1, \\ z_3 &\approx -1 + \frac{2}{c_m^2(\alpha)} + 10\left(1 - c_m^2(\alpha)\right)T^3. \end{aligned} \tag{41}$$

Table 1: Zero dynamics of the discrete-time model with time delay and relative degree two.

T	Approximate values (17)	Exact values
0.01	−0.499998000, 0.941764000	−0.499998001, 0.941764533
0.02	−0.499984000, 0.886912000	−0.499984021, 0.886920411
0.05	−0.499750000, 0.740500000	−0.499752042, 0.740816150
0.1	−0.498000000, 0.544000000	−0.498064368, 0.548763414
0.2	−0.484000000, 0.232000000	−0.485895895, 0.300398195

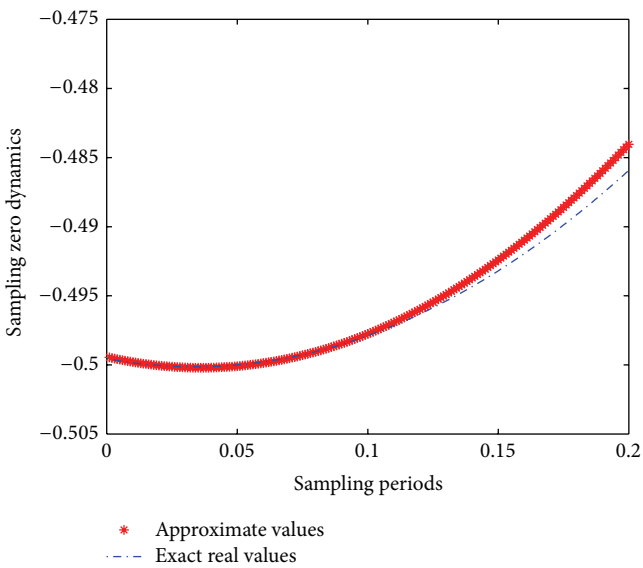

Figure 1: Sampling zero dynamics of the discrete-time model with time delay and relative degree two.

We can easily determine that the stability of the sampling zero dynamics z_3 can be satisfied by choosing design parameters α_1, α_2, T, and D of (36). Without loss of generality, let the time delay $D = 0.005$ for a sufficiently small sampling period T and select the suitable design parameters α_1 and α_2; then the approximate values (17) and the exact values of zero dynamics of the discrete-time system with time delay and relative degree two for the transfer function (39) are shown in Table 1 and corresponding figures, where the sampling zero dynamics and intrinsic zero dynamics are, respectively, shown in Figures 1 and 2.

As you can see in these table and figures, (17) gives good approximation and the sampling zero dynamics can lie inside the unit circle for a sufficiently small sampling period by satisfying condition (36).

5. Conclusions

This paper analyzes the asymptotic behavior of zero dynamics for a discrete-time system when a continuous-time system with a time delay is explicitly discretized in the case of a ZOH and relative degree two. Moreover, we also give an approximate expression of the zero dynamics for the discrete systems with time delay as power series expansions up to the third order term with respect to sampling periods. Further, in this case, the stability of discrete zero dynamics is discussed for sufficiently small sampling periods and a new stability condition is also derived. The idea of this paper is a further extension from the delay-free cases to the time-delay systems.

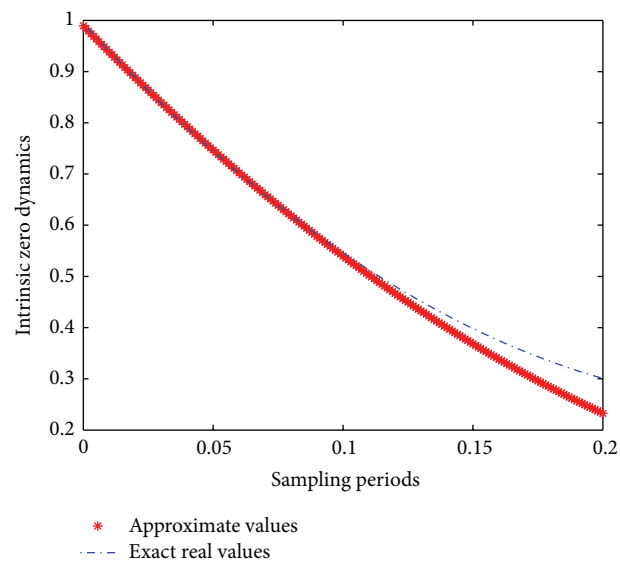

Figure 2: Intrinsic zero dynamics of the discrete-time model with time delay and relative degree two.

Appendix

Assumptions 1 and 2 ensure the existence of the normal form for the linear time-delay system. When the input signal $u(t)$ is generated by a ZOH, we first apply the Taylor formula with remainder to the system output $y(t)$ and its derivatives with respect to a sampling period up to the $r+2$ order term at any point t_0. However, owing to the existence of time delay, it is difficult to derive directly the corresponding discrete-time models. Therefore, we present an approximate sampled-data model for linear system with time delay by means of multiple step approach. Namely, apply the higher-order Taylor expansion formula in the sampling periods such as

$$\begin{aligned}\xi_{i+1,k+1} &= y_{k+1}^{(i)} \\ &\approx y_k^{(i)} + T y_k^{(i+1)} + \frac{T^2}{2!} y_k^{(i+2)} + \cdots \\ &\quad + \frac{T^{(r-i+2)}}{(r-i+2)!} y_k^{(r+2)}, \quad i = 0, \ldots, r-1.\end{aligned} \tag{A.1}$$

Without loss of generality, assume that $D < T$ and $T \approx mD$; then $D \approx T/m$ ($m > 1$, m is an integer). In the time interval $t \in [kT, kT + T/m]$, when the input is

a $u_{1,k}(\approx \alpha_1 u_k, kT \le u_k < (k+(1/m))T)$, one can obtain for sufficiently small sampling periods that

$$\begin{aligned} y_{k+(1/m)} &\approx y_k + \frac{T}{m}\dot{y}_k + \cdots + \frac{1}{r!}\left(\frac{T}{m}\right)^r y_k^{(r)} + \cdots \\ &\quad + \frac{1}{(r+2)!}\left(\frac{T}{m}\right)^{(r+2)} y_k^{(r+2)} + O\left(\left(\frac{T}{m}\right)^{r+3}\right) \\ &\vdots \\ y_{k+(1/m)}^{(r-1)} &\approx y_k^{(r-1)} + \cdots + \frac{1}{3!}\left(\frac{T}{m}\right)^3 y_k^{(r+2)} + O\left(\left(\frac{T}{m}\right)^{r+3}\right), \end{aligned} \tag{A.2}$$

where

$$y_k^{(i)} = b_k^{(i)} + a_k^{(i)}\alpha_1 u_k, \quad i = r, r+1, r+2. \tag{A.3}$$

Denote

$$\mathbf{Y}_{k+(1/m)} = \left[y_{k+(1/m)} \quad \dot{y}_{k+(1/m)} \quad \cdots \quad y_{k+(1/m)}^{(r-1)}\right]^T; \tag{A.4}$$

then

$$\begin{aligned} \mathbf{Y}_{k+(1/m)} &= A_{k,1/m}\mathbf{Y}_k + B_{k,1/m}\left(b_k^r + a_k^r\alpha_1 u_k\right) + \overline{B}_{k,1/m} \\ &\quad \times \left(b_k^{r+1} + a_k^{r+1}\alpha_1 u_k\right) + \widetilde{B}_{k,1/m} \\ &\quad \times \left(b_k^{r+2} + a_k^{r+2}\alpha_1 u_k\right) + O\left(\left(\frac{T}{m}\right)^{r+3}\right), \end{aligned} \tag{A.5}$$

where

$$A_{k,1/m} = \begin{bmatrix} 1 & \frac{T}{m} & \cdots & \frac{1}{(r-1)!}\left(\frac{T}{m}\right)^{r-1} \\ 0 & 1 & \cdots & \frac{1}{(r-2)!}\left(\frac{T}{m}\right)^{r-2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{T}{m} \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$B_{k,1/m} = \begin{bmatrix} \frac{1}{r!}\left(\frac{T}{m}\right)^r \\ \frac{1}{(r-1)!}\left(\frac{T}{m}\right)^{r-1} \\ \vdots \\ \frac{T}{m} \end{bmatrix},$$

$$\overline{B}_{k,1/m} = \begin{bmatrix} \frac{1}{(r+1)!}\left(\frac{T}{m}\right)^{r+1} \\ \frac{1}{r!}\left(\frac{T}{m}\right)^r \\ \vdots \\ \frac{1}{2!}\left(\frac{T}{m}\right)^2 \end{bmatrix},$$

$$\widetilde{B}_{k,1/m} = \begin{bmatrix} \frac{1}{(r+2)!}\left(\frac{T}{m}\right)^{r+2} \\ \frac{1}{(r+1)!}\left(\frac{T}{m}\right)^{r+1} \\ \vdots \\ \frac{1}{3!}\left(\frac{T}{m}\right)^3 \end{bmatrix}. \tag{A.6}$$

Similarly, in the other time interval $[kT + T/m, kT + 2T/m], \ldots, [kT + (m-1)T/m, kT + T]$, we can also give approximate asymptotic expressions of the outputs $\mathbf{Y}_{k+(2/m)}, \ldots, \mathbf{Y}_{k+1}$ as power series expansions with respect to a sufficiently small sampling period when the input is the constant value $u_{2,k}$ $(\approx \alpha_2 u_k)$. Deleting $\mathbf{Y}_{k+(1/m)}, \ldots, \mathbf{Y}_{k+((m-1)/m)}$ leads to the approximate expressions of $\mathbf{Y}_{k+1}$:

$$\begin{aligned} \mathbf{Y}_{k+1} &= A_k^m\mathbf{Y}_k + \left(A_k^{m-1} + \cdots + A_k + I_r\right) \\ &\quad \times \left(B_k b_k^r + \overline{B}_k b_k^{r+1} + \widetilde{B}_k b_k^{r+2}\right) \\ &\quad + \left(\alpha_1 A_k^{m-1} + \alpha_2 A_k^{m-2} + \cdots + \alpha_{m-1}A_k + \alpha_m I_r\right) \\ &\quad \times \left(B_k a_k^r + \overline{B}_k a_k^{r+1} + \widetilde{B}_k a_k^{r+2}\right) u_k + O\left(T^{r+3}\right) \\ &= \mathbf{\Phi}_r\mathbf{Y}_k + e b_k^r + \overline{e} b_k^{r+1} + \widetilde{e} b_k^{r+2} \\ &\quad + \left(\mathbf{d} a_k^r + \overline{\mathbf{d}} a_k^{r+1} + \widetilde{\mathbf{d}} a_k^{r+2}\right) u_k \\ &\quad + O\left(T^{r+3}\right), \end{aligned} \tag{A.7}$$

where

$$\begin{gathered} A_k = A_{k,1} = \cdots = A_{k,1/m}, \qquad B_k = B_{k,1} = \cdots = B_{k,1/m}, \\ \overline{B}_k = \overline{B}_{k,1} = \cdots = \overline{B}_{k,1/m}, \qquad \widetilde{B}_k = \widetilde{B}_{k,1} = \cdots = \widetilde{B}_{k,1/m}, \\ \left(A_k^{m-1} + \cdots + A_k + I_r\right) B_k = e, \\ \left(A_k^{m-1} + \cdots + A_k + I_r\right) \overline{B}_k = \overline{e}, \\ \left(A_k^{m-1} + \cdots + A_k + I_r\right) \widetilde{B}_k = \widetilde{e}, \\ \left(\alpha_1 A_k^{m-1} + \alpha_2 A_k^{m-2} + \cdots + \alpha_{m-1}A_k + \alpha_m I_r\right) B_k = \mathbf{d}, \\ \left(\alpha_1 A_k^{m-1} + \alpha_2 A_k^{m-2} + \cdots + \alpha_{m-1}A_k + \alpha_m I_r\right) \overline{B}_k = \overline{\mathbf{d}}, \end{gathered}$$

$$\left(\alpha_1 A_k^{m-1} + \alpha_2 A_k^{m-2} + \cdots + \alpha_{m-1} A_k + \alpha_m I_r\right) \widetilde{B}_k = \widetilde{\mathbf{d}},$$

$$\alpha_2 = \cdots = \alpha_{m-1} = \alpha_m,$$

$$\mathbf{\Phi}_r = \begin{bmatrix} 1 & T & \cdots & \dfrac{T^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \dfrac{T^{r-2}}{(r-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

$$\overline{e} = \begin{bmatrix} \overline{e}_1 \\ \overline{e}_2 \end{bmatrix}, \qquad \widetilde{e} = \begin{bmatrix} \widetilde{e}_1 \\ \widetilde{e}_2 \end{bmatrix}, \qquad e_1 = \frac{T^r}{r!},$$

$$\overline{e}_1 = \frac{T^{r+1}}{(r+1)!}, \qquad \widetilde{e}_1 = \frac{T^{r+2}}{(r+2)!},$$

$$e_2 = \begin{bmatrix} \dfrac{T^{r-1}}{(r-1)!} & \cdots & \dfrac{T^2}{2!} & T \end{bmatrix}^T,$$

$$\overline{e}_2 = \begin{bmatrix} \dfrac{T^r}{r!} & \cdots & \dfrac{T^3}{3!} & \dfrac{T^2}{2!} \end{bmatrix}^T,$$

$$\widetilde{e}_2 = \begin{bmatrix} \dfrac{T^{r+1}}{(r+1)!} & \cdots & \dfrac{T^4}{4!} & \dfrac{T^3}{3!} \end{bmatrix}^T,$$

$$\mathbf{d} = \begin{bmatrix} d \\ d' \end{bmatrix}, \qquad \overline{\mathbf{d}} = \begin{bmatrix} \overline{d} \\ \overline{d}' \end{bmatrix}, \qquad \widetilde{\mathbf{d}} = \begin{bmatrix} \widetilde{d} \\ \widetilde{d}' \end{bmatrix},$$

$$d' = \begin{bmatrix} d_1' & d_2' & \cdots & d_{r-1}' \end{bmatrix}^T,$$

$$\overline{d}' = \begin{bmatrix} \overline{d}_1' & \overline{d}_2' & \cdots & \overline{d}_{r-1}' \end{bmatrix}^T,$$

$$\widetilde{d}' = \begin{bmatrix} \widetilde{d}_1' & \widetilde{d}_2' & \cdots & \widetilde{d}_{r-1}' \end{bmatrix}^T,$$

$$d = \frac{T^r}{r!} c_m^r(\alpha), \qquad \overline{d} = \frac{T^{r+1}}{(r+1)!} c_m^{r+1}(\alpha),$$

$$\widetilde{d} = \frac{T^{r+2}}{(r+2)!} c_m^{r+2}(\alpha), \qquad d_k' = \frac{T^{r-k}}{(r-k)!} c_m^{r-k}(\alpha),$$

$$\overline{d}_l' = \frac{T^{r-l+1}}{(r-l+1)!} c_m^{r-l+1}(\alpha),$$

$$\widetilde{d}_\ell' = \frac{T^{r-\ell+2}}{(r-\ell+2)!} c_m^{r-\ell+2}(\alpha), \quad k, l, \ell = 1, 2, \ldots, r-1 \tag{A.8}$$

in which

$$c_m^q(\alpha) = \sum_{j=1}^{m} \left(\Lambda_{1j}^q - \Lambda_{2j}^q\right) \alpha_j,$$
$$\Lambda_{1j} = 1 - \frac{j-1}{m}, \qquad \Lambda_{2j} = 1 - \frac{j}{m}. \tag{A.9}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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