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Research Article

Nonstandard Methods in Measure Theory

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Ideas and techniques from standard and nonstandard theories of measure spaces and Banach spaces are brought together to give a new approach to the study of the extension of vector measures. Applications of our results lead to simple new proofs for theorems of classical measure theory. The novelty lies in the use of the principle of extension by continuity (for which we give a nonstandard proof) to obtain in an unified way some notable theorems which have been obtained by Fox, Brooks, Ohba, Diestel, and others. The methods of proof are quite different from those used by previous authors, and most of them are realized by means of nonstandard analysis.

Dedicated to Professor Solomon Marcus in honour of the 90th anniversary of his birthday

1. Introduction

Let Ω be a nonempty fixed set and ν a real-valued positive measure on a ring $\mathscr{R}$ of subsets of Ω; the measure is assumed to be countably additive in the sense that if (E_n) is a sequence of disjoint members of $\mathscr{R}$ and if $\bigcup_{n=1}^{\infty} E_n$ is also in $\mathscr{R}$, then $\nu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \nu(E_n)$. A fundamental problem in measure theory is that of finding conditions under which a countably additive measure on a ring $\mathscr{R}$ can be extended to a countably additive measure on a wider class of sets containing $\mathscr{R}$. This problem is essentially solved by the Caratheodory process of generating an outer measure ν^* and taking the family of ν^*-measurable sets (see [1]); then the original measure can be extended to a σ-ring Σ which contains the σ-ring $\Sigma(\mathscr{R})$ generated by $\mathscr{R}$.

Suppose instead that ν is no longer real-valued, but it is a set function on $\mathscr{R}$ taking values in a Banach space X. If ν is countably additive in the above sense, in what circumstances is it still possible to extend ν to $\Sigma(\mathscr{R})$? An obvious necessary condition is that ν should be bounded over $\mathscr{R}$; that is, $\sup\{\|\nu(E)\| : E \in \mathscr{R}\}$ should be finite; for if the extension ν_1 onto $\Sigma(\mathscr{R})$ exists, then ν_1 as a finite-valued measure on a sigma ring is well known to be bounded over its domain [2, III, 4.5], so that ν is a fortiori bounded over $\mathscr{R}$. So, if ν is a bounded set function on a ring R with values in a Banach space, what are the possibilities to obtain an extension?

One of the simplest methods is to consider the family (λ_{x^*}) of signed measure on $\mathscr{R}$ obtaining from each element x^* of the topological dual X^* by the following real-valued mapping on $\mathscr{R}$: $\lambda_{x^*}(E) = \langle \nu(E), x^* \rangle$, $E \in \mathscr{R}$. Such set functions λ_{x^*} are bounded and countably additive over $\mathscr{R}$ and as such can be subjected to the Jordan decomposition. Thus, the problem is reduced to the Caratheodory procedure, and the extension ν_1 of ν is defined in terms of the elements of X^* with the following properties: ν_1 takes values in the algebraic dual of X^* and is countably additive over $\mathscr{R}$ in the weak topology. So, the extension problem reduces therefore to finding circumstances in which the range of ν_1 is X (identifying X with its natural embedding into X^{**}). Whenever this is the case, for instance, in the case of reflexive spaces, the set function is countably additive in view of Pettis' theorem [2]. This result was obtained for the first time by Fox [3].

Since vector-valued measures are important tools in integral representation as well as disintegration of measures, many authors have considered the extension problem when

the range of ν is contained in a vector space X. In the literature there are two main approaches to proving theorems concerning the extension of vector measures.

The first approach is due to the properties which have to be satisfied by the measure which follows to be extended. If X is a Banach space, we can mention the solutions due to Găină [4] (in the case when ν has finite variation), Dinculeanu [5, 6] (if ν is regular and of bounded variation), Arsene and Strătilă [7] (when ν is bounded above in norm by a positive measure), Dinculeanu and Kluvanek [8] (in the case when ν is absolutely continuous with respect to a positive measure), and Fox [9] (ν satisfies a monotone-convergence condition).

The second approach relies on the conditions which have to be satisfied by the range of ν. In this category we have the results of Fox [3], Kluvanek [10], Ohba [11], or Gould [12]. Generalizing the notion of outer measure to vector-valued measures and imitating the ν^*-measurability procedure in order to obtain a Lebesgue extension of ν, Gould [12] showed that a necessary and sufficient condition for ν to have a Lebesgue extension is that the following property should hold for the image space X of ν.

(A) If (x_n) is a sequence in X whose norms have a positive lower bound, then there exists for arbitrary positive K a finite subsequence (x_{n_r}) such that $\|\sum_r x_{n_r}\| > K$.

This suggests a connection between weak completeness and property (A), and it is shown in [12] that all weakly complete spaces satisfy property (A). In Section 3 we present another proof of this result. It is easy to verify that l^∞ does not hold property (A). Thus, l^∞ is not weakly complete. More generally, Banach spaces which are infinite-dimensional function spaces with a supremum norm fail to satisfy property (A) and therefore are not weakly complete. For Hilbert spaces property (A) is satisfied. A direct proof of the fact that property (A) is satisfied by l^p $(1 \le p < \infty)$ is much harder.

For a masterful study of measures with values in a topological group we refer the reader to Sion [13] or Drewnowski [14–16]. When X is a commutative complete topological group and ν is of bounded variation, then there is a very nice extension theorem by Takahashi [17].

The starting point in nonstandard theory of measure spaces is a paper [18] by Loeb. He gave a way to construct new rich standard measure spaces from internal measure spaces. This construction has been used in recent years to establish new standard results in a variety of different areas. Some of these results can be found in [19] or [20].

Also, nonstandard analysis has proved to be a natural framework for studying vector measures and Banach spaces. The central construction in this approach is the notion of nonstandard hull introduced by Luxemburg [21]. This notion is not only a useful tool in studying vector measures and Banach spaces, but also a construction arising naturally throughout nonstandard analysis. For a deeper discussion of nonstandard hulls and their applications we refer the reader to the survey paper [22] of Henson and Moore.

Živaljević [23] has pursued the extension problem using the nonstandard hull of X. Osswald and Sun [24] treated the same problem from a different point of view; the extension of additive vector measures has been made using the internal control measures. Furthermore, the authors present a different approach of Loeb's [18] in order to construct a countably additive vector measure from internal, locally convex space-valued measure.

In this work we study the extension of vector valued set functions in the framework of nonstandard analysis.

The plan of this paper is as follows.

Section 2 is devoted to some preliminary results on standard vector measures. Using concurrent relations, we also obtain a result concerning the concentration of s-bounded vector measures on a specific set from the nonstandard extension of $\mathscr{R}$. Moreover, the principle of extension by continuity [2] will be proved using nonstandard techniques.

In Section 3 we give a nonstandard characterization of the absolute continuity and an extension theorem for vector measures. The proof still uses nonstandard arguments. Since reflexive spaces are weakly complete and the weakly complete spaces satisfy property (A), we can rederive the Fox's result [3]. Some results of Gould [12] will be reproved in a different manner. For this, we use a result of Diestel et al. (see [25] or [26]) on s-bounded measures. To obtain these results, Gould has used Pettis' theorem. Our approach does not use this result.

In Section 4 we address the issues of the existing control measures. We tackle this subject by using the extension of set functions and linking these extensions to control measures.

Section 5 deals with the extension of set functions with finite semivariation. The results in this section (for which we present a nonstandard proof) were originally obtained by Lewis [27].

The last section shows that the extension of set functions with finite variation is a particular case of the extension of set functions with finite semivariation.

We adopt the nonstandard framework of [28]. The nonstandard model used in this paper is assumed to be sufficiently saturated for our needs. In what follows, $\mathbb{N}_\infty$ denotes the set ${}^*\mathbb{N} \setminus \mathbb{N}$, where ${}^*\mathbb{N}$ is the extension of $\mathbb{N}$ in our model.

2. Preliminaries

In this section we collect some basic definitions, notations, and elementary results about standard vector measures. Also, the principle of extension by continuity will be proved using nonstandard techniques.

The terminology concerning families of sets, set functions, and so forth, will be, in general, that of [2] or [1]. Let $\mathbb{R}_+$ denote the nonnegative reals, and let $\mathbb{N}$ denote the set of positive integers. Sets are denoted as $A, B, C, \ldots$; $\emptyset$ means the empty set. Notation for set operations is that commonly used, in particular $A \Delta B$ means $(A \setminus B) \cup (B \setminus A)$. Everywhere in the sequel $\mathscr{R}$ denotes a ring of subsets of a nonempty fixed set Ω; the cases $\mathscr{R}$ should be a δ-ring or σ-ring will be explicitly specified. The complement (in Ω) of a set A is denoted by A^c. Symbols $\nearrow$ and $\searrow$ for sequences of sets or of reals have their usual meaning. A set function $\mu : \mathscr{R} \to [0, \infty]$ will be called a submeasure (subadditive measure in the terminology of Orlicz [29]) if 1^0 $\mu(\emptyset) = 0$, 2^0 $\mu(A \cup B) \le \mu(A) + \mu(B)$ whenever $A, B \in \mathscr{R}$ and $A \cap B = \emptyset$, 3^0 $\mu(A) \le \mu(B)$ if

$A, B \in \mathscr{R}$ and $A \subseteq B$. The ring $\mathscr{R}$ is an abelian group with respect to the symmetric difference operation $\triangle$ and each submeasure μ generates a semimetric on the group $(\mathscr{R}, \triangle)$ by the Frechet-Nikodym ecart $\rho(A, B) = \mu(A \triangle B)$. This semimetric is invariant in the sense that $\rho(A, B) = \rho(A \triangle C, B \triangle C)$ for sets $A, B, C \in \mathscr{R}$. Therefore, any submeasure μ generates a topology on the group $(\mathscr{R}, \triangle)$, and a base of neighborhoods is given by the family of sets $N(A_0, \epsilon) = \{A \in \mathscr{R} : \mu(A \triangle A_0) < \epsilon\}$. In the semimetric space $(\mathscr{R}, \rho)$ the set operations are continuous [5] (we also present a nonstandard proof of this result). Requiring from a topology in a ring to possess this property one obtains the topological ring of sets, a natural generalization of the so-called spaces of measurable sets, introduced by M. Fréchet and O. Nikodym, in which a distance between two sets is defined as the measure of their symmetric difference.

Let $(X, \|\cdot\|)$ be a Banach space, and let ν be a set function from $\mathscr{R}$ to X. We say that ν is a finitely additive vector measure, or simply a vector measure, if whenever E_1 and E_2 are disjoint members of $\mathscr{R}$ then $\nu(E_1 \cup E_2) = \nu(E_1) + \nu(E_2)$. If, in addition, $\nu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \nu(E_n)$ in the norm topology of X for all disjoint sequences (E_n) of members of $\mathscr{R}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathscr{R}$, then ν is termed a countably additive vector measure or simply, ν is countably additive. We use the terminology from [30] and call a set function ν from $\mathscr{R}$ to X strongly bounded (often abbreviated s-bounded) if $\nu(E_n) \to 0$, whenever (E_n) is a disjoint sequence. We say that ν is order continuous ($o.c$) if, for each sequence $(E_n) \subset \mathscr{R}$ such that $E_n \searrow \emptyset$, we have $\nu(E_n) \to 0$. We will denote by $\tau(\mathscr{R})$ the class of all subsets E of Ω which have the property that $A \cap E \in \mathscr{R}$ for every $A \in \mathscr{R}$. Then, $\mathscr{R}$ forms an ideal in $\tau(\mathscr{R})$, and $\tau(\mathscr{R})$ is an algebra. We will denote by $\mathscr{P}(\Omega)$ the power set of Ω, and for any $\mathscr{C} \subseteq \mathscr{P}(\Omega)$ we put $\widetilde{\mathscr{C}} = \{\bigcup_{n=1}^{\infty} E_n : E_n \in \mathscr{C}, n = 1, 2, \ldots\}$.

If ν is a vector measure from $\mathscr{R}$ to X, we call $(\Omega, \mathscr{R}, \nu)$ a vector measure space. The inner quasi-variation $\widetilde{\nu}(A)$ of an arbitrary subset A of Ω is defined by $\widetilde{\nu}(A) = \sup\{\|\nu(E)\| : E \in \mathscr{R}, E \subseteq A\}$; a set A is ν-bounded if $\widetilde{\nu}(A)$ is finite [12]. If $\widetilde{\nu}(\Omega)$ is finite, then ν will be called a bounded vector measure. Clearly, the inner quasi-variation is a submeasure on $\mathscr{R}$.

Abstraction of the condition of strong boundedness on a ring is the concept of the Rickart submeasure. Thus, a sequence $(A_n) \subset \mathscr{R}$ is called dominated if there exists a set $B \in \mathscr{R}$ such that $A_n \subseteq B$, for $n = 1, 2, \ldots$. The submeasure ν is said to be Rickart on the ring $\mathscr{R}$ if for each dominated, disjoint sequence $(A_n) \subset \mathscr{R}$, we have $\lim_n \nu(A_n) = 0$. Note that every finite additive submeasure on the ring $\mathscr{R}$ is Rickart.

An extensive research of topological rings of sets generated by Rickart families of submeasures, with applications to vector measures, was initiated by Oberle [31] and developed by Bogdanowicz and Oberle [32]. The topological point of view was realized by Drewnowski [14–16].

On the other hand, essential properties of finite or countably additive vector measures are reflected on the properties of corresponding submeasures. This enables us to use submeasures as a convenient tool in various questions concerning vector measures. For instance, Walker [33] has used the corresponding submeasures to study uniform sigma additivity or equicontinuity.

Throughout this section we assume that ν is countably additive and s-bounded. Then, we know that ν is a bounded vector measure, and $\widetilde{\nu}$ is s-bounded. Furthermore, for any sequences $E_n \searrow \emptyset$ of members of $\mathscr{P}(\Omega)$ we have $\widetilde{\nu}(E_n) \searrow 0$, so the countable additivity of ν implies order continuity of the submeasure $\widetilde{\nu}$ [14–16, 26, 34]. For any set $A \subseteq \Omega$, $\Gamma(A) = \{B \in \widetilde{\tau(\mathscr{R})} : A \subseteq B\}$ is a directed set, where $B_1 \leq B_2$ if and only if $B_1 \supseteq B_2$ for $B_1, B_2 \in \Gamma(A)$. Then, the generalized sequence $\{\widetilde{\nu}(B) : B \in \Gamma(A)\}$ is a Cauchy net. By the completeness of $\mathbb{R}$ we put for any subset A of Ω the outer quasi-variation of A given by $\widehat{\nu}(A) = \lim_{B \in \Gamma(A)} \widetilde{\nu}(B)$. Consequently, there exists a unique set function $\widehat{\nu} : \mathscr{P}(\Omega) \to \mathbb{R}_+$ such that for every set $A \in \widetilde{\tau(\mathscr{R})}$ we have $\widehat{\nu}(A) = \widetilde{\nu}(A)$. We may refer to $\widetilde{\nu}$ and $\widehat{\nu}$ as the inner and the outer measures generated by ν.

Since ν is bounded we can define on $\mathscr{P}(\Omega)$ the ecart function $\rho(A, B) = \widehat{\nu}(A \triangle B)$, where $A \triangle B$ is the symmetric difference of A and B. Then $(\mathscr{P}(\Omega), \rho)$ is a semimetric space. We denote by $\overline{\mathscr{R}}$ the closure of $\mathscr{R}$ in the space $(\mathscr{P}(\Omega), \rho)$; if (T, τ) is a topological space and $A \subseteq T$, the closure of A in (T, τ) is denoted by $\overline{A}$. For A a subset of $\mathbb{R}$ and n a positive integer, let $nA := \{\sum_{i=1}^{n} x_i : x_i \in A \text{ for } i = 1, \ldots, n\}$. If U is a neighbourhood of 0 in $\mathbb{R}$ we put $\widehat{U} = \{(A, B) : A, B \in \mathscr{P}(\Omega), \widehat{\nu}(A \triangle B) \in U\}$.

The below proposition is straightforward, so we omit its proof.

Definition 1. A ring $\mathscr{R}$ is called ν-dense if for any subset A of Ω and for any positive real number ε, there is $E \in \mathscr{R}$, $E \subseteq A$ such that $\widetilde{\nu}(A \setminus E) < \varepsilon$.

Definition 2. The sets A, B of ${}^*\mathscr{R}$ will be called equivalent $(A \approx B)$ if $\rho(A, B) \approx 0$, where ρ is the ecart function $\rho(A, B) = \widehat{\nu}(A \triangle B)$.

Proposition 3. *Let $\nu : \mathscr{R} \to X$ be a vector measure. If ν is s-bounded, then $\mathscr{R}$ is ν-dense.*

Lemma 4. (*i*) *Let $A \subseteq B$ subsets of Ω. Then $\widetilde{\nu}(A) \leq \widetilde{\nu}(B)$ and $\widehat{\nu}(A) \leq \widehat{\nu}(B)$.*

(*ii*) *For A subset of Ω we have $\widetilde{\nu}(A) \leq \widehat{\nu}(A)$ and $\widehat{\nu}(A) = \widetilde{\nu}(A)$ on $\widetilde{\tau(\mathscr{R})}$.*

(*iii*) *Let $A \in \widetilde{\tau(\mathscr{R})}$ and B subset of Ω. Then $\widetilde{\nu}(A \cup B) \leq \widetilde{\nu}(A) + \widetilde{\nu}(B)$.*

(*iv*) *If $(E_n) \subset \mathscr{P}(\Omega)$ and $(E_n) \searrow \emptyset$ then $\widetilde{\nu}(E_n) \searrow 0$.*

(*v*) *If $(E_n) \subset \widetilde{\tau(\mathscr{R})}$ and $(E_n) \nearrow E$ then $\widetilde{\nu}(E_n) \nearrow \widetilde{\nu}(E)$.*

(*vi*) *If $(E_n) \subset \widetilde{\tau(\mathscr{R})}$ then $\widetilde{\nu}(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \widetilde{\nu}(E_n)$.*

Lemma 5. (*i*) *If $(E_n) \subset \mathscr{P}(\Omega)$ then $\widehat{\nu}(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \widehat{\nu}(E_n)$.*

(*ii*) *If $(E_n) \subset \mathscr{P}(\Omega)$, $(E_n) \nearrow E$ then $\widehat{\nu}(E_n) \nearrow \widehat{\nu}(E)$.*

The above properties of the inner and the outer measures generated by ν are well known. Now we are going to prove a result which is needed in the sequel.

Theorem 6. (*i*) *$\overline{\mathscr{R}}$ is a sigma algebra and $\widetilde{\tau(\mathscr{R})} \subset \overline{\mathscr{R}}$.*

(ii) If $(A_n)_n \subseteq \overline{\mathscr{R}}$ and $A = \bigcup_{n=1}^{\infty} A_n$ then $\rho(A, A_n) = \widehat{\nu}(A \triangle A_n) \searrow 0$.

(iii) If $(A_n)_n \subseteq \overline{\mathscr{R}}$ and $(A_n)_n \searrow \emptyset$ then $\widehat{\nu}(A_n) \searrow 0$.

Proof. (i) Let $A \in \overline{\mathscr{R}}$. There is a set $B \in {}^*\mathscr{R}$ such that $A \approx B$. Then $A^c \approx B^c$. Since $\mathscr{R}$ is ν-dense, there is $E \in {}^*\mathscr{R}$ such that $E \subseteq B^c$ and $\widetilde{\nu}(B^c \setminus E) \approx 0$. But $\widetilde{\nu}(B^c \setminus E) = \widehat{\nu}(B^c \setminus E)$, because of $B^c \setminus E \in {}^*\tau(\mathscr{R})$.

It is easy to see that $A^c \triangle E \subseteq (A^c \triangle B^c) \cup (B^c \setminus E)$. Hence $A^c \approx E$, so $A^c \in \overline{\mathscr{R}}$.

Let now $A_1, A_2 \in \overline{\mathscr{R}}$. There are $B_1, B_2 \in {}^*\mathscr{R}$ such that $A_1 \approx B_1$, $A_2 \approx B_2$. We have $A_1 \cup A_2 \approx B_1 \cup B_2$, $B_1 \cup B_2 \in {}^*\mathscr{R}$, so $A_1 \cup A_2 \in \overline{\mathscr{R}}$. Therefore. $\overline{\mathscr{R}}$ is algebra.

Suppose $(A_n)_n$ is a sequence of elements of $\overline{\mathscr{R}}$ and set $A = \bigcup_{n=1}^{\infty} A_n$. It remains to prove that $A \in \overline{\mathscr{R}}$. Since $\overline{\mathscr{R}}$ is algebra we can take $(A_n) \nearrow$. Choose U and V neighbourhoods of zero in $\mathbb{R}$ such that $\overline{V} + \overline{V} \subseteq U$.

Let $(V_n)_n$ be a sequence of neighbourhoods of zero in $\mathbb{R}$ so that $\sum_{k=1}^{n} V_k \subseteq V$ for all n. For every n we can choose $E_n \in \mathscr{R}$ such that $(A_n, E_n) \in \widehat{V_n}$. If $E = \bigcup_{n=1}^{\infty} E_n$, a trivial verification shows that $A \triangle E \subseteq \bigcup_{n=1}^{\infty} A_n \triangle E_n$. By Lemma 5(i) we have

$$\widehat{\nu}(A \triangle E) \leq \sum_{k=1}^{\infty} \widehat{\nu}(A_n \triangle E_n), \tag{1}$$

whence $\widehat{\nu}(A \triangle E) \in \overline{V}$. Setting $F_n = \bigcup_{k=1}^{n} E_k$, we have $F_n \in \mathscr{R}$, $(F_n) \nearrow$ and $A_n \triangle F_n \subseteq \bigcup_{k=1}^{n} A_k \triangle E_k$. Lemma 5(i) implies

$$\widehat{\nu}(A_n \triangle F_n) \leq \sum_{k=1}^{n} \widehat{\nu}(A_k \triangle E_k), \tag{2}$$

so $\widehat{\nu}(A_n \triangle F_n) \in V$. Moreover, $E \setminus F_n \in \widetilde{\tau(\mathscr{R})}$, and by Lemma 4(ii)

$$\widehat{\nu}(E \setminus F_n) = \widetilde{\nu}(E \setminus F_n) \searrow 0, \tag{3}$$

according to $(F_n) \nearrow$. It follows that $\widehat{\nu}(E \triangle F_n) \in V$ for n large enough. We observe that $A \triangle F_n \subseteq (A \triangle E) \cup (E \setminus F_n)$ for each n and, on account of Lemma 5(i), we conclude that

$$\widehat{\nu}(A \triangle F_n) \leq \widehat{\nu}(A \triangle E) + \widehat{\nu}(E \triangle F_n). \tag{4}$$

Thus, for n large enough, $\widehat{\nu}(A \triangle F_n) \in \overline{V} + \overline{V} \subseteq U$, so $A \in \overline{\mathscr{R}}$.

Let $A \in \widetilde{\tau(\mathscr{R})}$ and fix $\varepsilon > 0$. Proposition 3 now gives $E \in \mathscr{R}$ so that $E \subseteq A$ and $\widetilde{\nu}(A \setminus E) < \varepsilon$. According to Lemma 4(ii), since $A \setminus E \in \widetilde{\tau(\mathscr{R})}$ we get

$$\widehat{\nu}(A \setminus E) = \widetilde{\nu}(A \setminus E) < \varepsilon. \tag{5}$$

Thus, $A \in \overline{\mathscr{R}}$.

(ii) If $(A_n)_n \subseteq \overline{\mathscr{R}}$ and $A = \bigcup_{n=1}^{\infty} A_n$, we verify immediately that

$$A \triangle A_n \subseteq (A \triangle F_n) \bigcup (A_n \triangle F_n). \tag{6}$$

Lemma 5(i) implies

$$\widehat{\nu}(A \triangle A_n) \leq \widehat{\nu}(A \triangle F_n) + \widehat{\nu}(A_n \triangle F_n). \tag{7}$$

From this we deduce, for n large enough, that $\widehat{\nu}(A \triangle A_n) \in U + V \subseteq U + U = 2U$, so $\rho(A, A_n) \searrow 0$ as $n \to \infty$.

(iii) From $(A_n)_n \searrow \emptyset$ we see that $(A_n^c)_n \nearrow \Omega$. According to (i) and (ii), $(A_n^c)_n$ is a sequence of elements of $\overline{\mathscr{R}}$ and $\rho(\Omega, A_n^c) \searrow 0$ as $n \to \infty$. We remark at once that $\rho(\Omega, A_n^c) = \widehat{\nu}(A_n)$, and the proof is complete. □

Because the main tool in our approach is the principle of extension by continuity, we give a nonstandard proof of it. For a standard proof we refer the reader to [2].

Theorem 7 (principle of extension by continuity). *Let X and Y be metric spaces, and let Y be complete. If A is a dense subset of X and $f : A \to Y$ is uniformly continuous, then f has a unique uniformly continuous extension $g : X \to Y$.*

Proof. Let $x \in X$, then there is a $y \in {}^*A$ with $y \approx x$. Moreover, there is a sequence of points $(a_n)_n$ of A with $a_n \to x$. For each $\theta \in \mathbb{N}_\infty$ we have $a_\theta \approx x$. But f is uniformly continuous on A, so $a_\theta \approx y$ implies $f(a_\theta) \approx f(y)$. For ε a positive real number we put $V_\varepsilon = \{n \in {}^*\mathbb{N} : d_Y(f(a_n), f(y)) < \varepsilon\}$. Then V_ε is internal by the definition principle, and each $\theta \in \mathbb{N}_\infty$ belongs to V_ε. In particular, for some $n \in \mathbb{N}$ we have $n \in V_\varepsilon$. Hence $f(y)$ is a prenearstandard point, and so $f(y)$ is nearstandard. Let $g(x) = {}^\circ f(y)$, where ${}^\circ$ is the standard part map. Then g is obviously well defined and extends f. To verify that g is uniformly continuous, let ε be an arbitrary positive real number. The assumption implies that there is a positive real number δ such that

$$\left[d_X(y_1, y_2) < \delta \Longrightarrow d_Y(f(y_1), f(y_2)) < \frac{\varepsilon}{2}\right], \quad (\forall y_1, y_2 \in A). \tag{8}$$

By the transfer principle,

$$\left[d_X(y_1, y_2) < \delta \Longrightarrow d_Y(f(y_1), f(y_2)) < \frac{\varepsilon}{2}\right], \quad (\forall y_1, y_2 \in {}^*A). \tag{9}$$

Let $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta/2$. There are $y_1, y_2 \in {}^*A$ such that $y_1 \approx x_1$, $y_2 \approx x_2$. Then $d_X(y_1, y_2) \leq d_X(y_1, x_1) + d_X(x_1, x_2) + d_X(x_2, y_2)$, so $d_X(y_1, y_2) < \delta$. Hence $d_Y(f(y_1), f(y_2)) < \varepsilon/2$. By the definition of g we have $g(x_1) \approx f(y_1)$, $g(x_2) \approx f(y_2)$. Since we also have

$$\begin{aligned} d_Y(g(x_1), g(x_2)) &\leq d_Y(g(x_1), f(y_1)) + d_Y(f(y_1), f(y_2)) \\ &\quad + d_Y(f(y_2), g(x_2)), \end{aligned} \tag{10}$$

we get that $d_Y(g(x_1), g(x_2)) < \varepsilon$, whence g is uniformly continuous.

Let g_1 be another function with the same properties, and $x \in X$. Then there is $y \in {}^*A$ with $y \approx x$. Since g, g_1 are uniformly continuous we have $g(y) \approx g(x)$ and $g_1(y) \approx g_1(x)$. By the transfer principle $g(y) = g_1(y)$, so that $g(x) \approx g_1(x)$. As $g(x)$ and $g_1(x)$ are standard, it follows that $g(x) = g_1(x)$, and the proof is complete. □

The section closes with an outcome about the concentration of s-bounded measures on a set of ${}^*\mathscr{R}$. By $s.a(\mathscr{R}, X)$ we denote the set of all s-bounded additive measures from $\mathscr{R}$ to X.

Proposition 8. *There exists a set A of ${}^*\mathscr{R}$ so that for all B of ${}^*\mathscr{R}$, B disjoint of A and for every ν belonging to $s.a(\mathscr{R}, X)$, we have $\nu(B) \approx 0$.*

Proof. Let Ξ be a relation on $(s.a(\mathscr{R}, X) \times \mathbb{R}_+) \times \mathscr{R}$ defined by $((\nu, \epsilon), E) \in \Xi$ if and only if for all $F \in \mathscr{R}$, F disjoint of E we have $\|\nu(F)\| < \epsilon$. We see at once that $\mathrm{dom}(\Xi)$ is $s.a(\mathscr{R}, X) \times \mathbb{R}_+$, which is clear from Proposition 3. We verify that Ξ is a concurrent relation. Indeed, if $(\nu_1, \epsilon_1), \ldots, (\nu_n, \epsilon_n) \in \mathrm{dom}(\Xi)$, there exists $E_1, \ldots, E_n \in \mathscr{R}$ such that for all $F \in \mathscr{R}$, F disjoint of E_i we have $\|\nu_i(F)\| < \epsilon$, $i = 1, \ldots, n$. Setting $E = \bigcup_{i=1}^n E_i$, we have $E \in \mathscr{R}$ and $((\nu_i, \epsilon_i), E) \in \Xi$ for all $i = 1, \ldots, n$, which is our assertion. By the concurrence theorem [35], there is a set $A \in {}^*\mathscr{R}$ such that $((\nu, \epsilon), A) \in^* \Xi$ for all $(\nu, \epsilon) \in \mathrm{dom}(\Xi)$. Fix $\nu \in s.a(\mathscr{R}, X)$. Then, for all $B \in {}^*\mathscr{R}$, B disjoint of A we have $\|\nu(B)\| < \epsilon$. Let us regard ϵ as ran and the proof is complete. □

3. Extension of a Vector Valued Measure

We adopt here the main framework of nonstandard analysis from [28]. We also give the definition, nonstandard formulation, and some of the basic properties of the absolutely continuous concept. In this section μ denotes a submeasure from $\mathscr{R}$ to $\mathbb{R}_+$ and ρ stands for the Frechet-Nikodym ecart $\rho(A, B) = \mu(A \triangle B)$ associated with μ. We have seen in Section 2 that $(\mathscr{R}, \rho)$ is a semimetric space. We recall that the sets A, B of ${}^*\mathscr{R}$ are equivalent ($A \approx B$) if $\rho(A, B) \approx 0$.

Definition 9. A vector measure ν is called absolutely continuous with respect to μ, or simply μ-continuous, if for any positive real number ε there is a positive real number δ, such that for any $A \in \mathscr{R}$, $\|\nu(A)\| < \varepsilon$ if $\mu(A) < \delta$. In this case we say that μ is a control submeasure of ν, and we denote that by $\nu \ll \mu$.

A nonstandard formulation of absolutely continuous may be stated as follows.

Lemma 10. *A vector measure ν on $(\Omega, \mathscr{R})$ is absolutely continuous with respect to μ, if and only if for all $A \in {}^*\mathscr{R}$ with $\mu(A) \approx 0$ we have $\nu(A)$ in the monad of zero in X.*

Proof. Assume μ is a control submeasure of ν and let ε be an arbitrary positive real number. If A is some element in the internal algebra ${}^*\mathscr{R}$ with $\mu(A) \approx 0$, we have by the transfer principle that $\|\nu(A)\| < \varepsilon$. Since ε is arbitrary, we obtain $\nu(A) \approx 0$.

To prove the converse, fix $A \in {}^*\mathscr{R}$ with $\mu(A) \approx 0$, and for a given positive real number ε, let δ be an infinitesimal such that $\mu(A) < \delta$. Then, for any $B \in {}^*\mathscr{R}$ with $\mu(B) < \delta$ we get $\|\nu(B)\| < \varepsilon$. Thus, we have shown

$$(\exists \delta \in {}^*\mathbb{R}_+)(\forall B \in {}^*\mathscr{R})\left[\mu(B) < \delta \Longrightarrow \|\nu(B)\| < \varepsilon\right]. \quad (11)$$

Now apply the transfer principle to obtain the desired condition for absolute continuity. □

Remark 11. As we have mentioned the set operations are continuous [5] in the semimetric space $(\mathscr{R}, \rho)$. Here a nonstandard proof is given.

Lemma 12. *The maps $f, g, h : \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ defined by the equalities*

$$f(A, B) = A \bigcup B, \qquad g(A, B) = A \bigcap B, \qquad h(A, B) = A \setminus B \quad (12)$$

are uniformly continuous.

Proof. We denote by $\circ$ one of the operations $\cap, \cup, \setminus$. For every set A_1, A_2, B_1, B_2 of ${}^*\mathscr{R}$ we have $(A_1 \circ B_1) \triangle (A_2 \circ B_2) \subseteq (A_1 \triangle A_2) \cup (B_1 \triangle B_2)$. Hence,

$$\rho(A_1 \circ B_1, A_2 \circ B_2) \le \rho(A_1, A_2) + \rho(B_1, B_2). \quad (13)$$

If $A_1 \approx A_2$, $B_1 \approx B_2$, then $\rho(A_1, A_2) \approx 0$, $\rho(B_1, B_2) \approx 0$. Thus, we conclude $\rho(A_1 \circ B_1, A_2 \circ B_2) \approx 0$, and the set operations are uniformly continuous. □

Lemma 13. *A vector measure ν on $(\Omega, \mathscr{R})$ is absolutely continuous with respect to μ, if and only if ν is uniformly continuous on $(\mathscr{R}, \rho)$.*

Proof. Suppose $A, B \in {}^*\mathscr{R}$ and $A \approx B$. By Lemma 10 $\nu(A \setminus B) \approx 0$ and $\nu(B \setminus A) \approx 0$. But ν is a vector measure, so $\nu(A) = \nu(A \setminus B) + \nu(A \cap B) \approx \nu(A \cap B)$ and $\nu(B) = \nu(B \setminus A) + \nu(A \cap B) \approx \nu(A \cap B)$. Therefore, $\nu(A) \approx \nu(B)$, which means that ν is uniformly continuous.

Conversely, let $A \in {}^*\mathscr{R}$ with $\mu(A) \approx 0$. In this case $A \approx \emptyset$, and by uniform continuity of ν we have $\nu(A)$ in the monad of zero in X. Then, the result follows by Lemma 10. □

Theorem 14. *Let $\mathscr{A}$ be a ring of subsets of Ω such that $\mathscr{A} \subseteq \mathscr{R}$ and let ν be a vector measure on $(\Omega, \mathscr{A})$. Suppose $\mathscr{A}$ is dense in $(\mathscr{R}, \rho)$ and ν is absolutely continuous with respect to μ. Then ν has a unique vector measure extension $\nu_1 : \mathscr{R} \to X$. Furthermore, this extension is uniformly continuous on $(\Omega, \mathscr{R})$.*

Proof. On account of Theorem 7, there is a unique uniformly continuous map $\nu_1 : \mathscr{R} \to X$ which extends ν. It remains to prove that whenever E_1 and E_2 are disjoint members of $\mathscr{R}$ then $\nu_1(E_1 \cup E_2) = \nu_1(E_1) + \nu_1(E_2)$. By assumption, we can choose $F_1, F_2 \in {}^*\mathscr{A}$ such that $E_1 \approx F_1$, $E_2 \approx F_2$. According to Lemma 12, we have $E_1 \cup E_2 \approx F_1 \cup F_2$, $E_1 \setminus E_2 \approx F_1 \setminus F_2$. Uniform continuity of ν_1 implies $\nu_1(E_1 \cup E_2) \approx \nu_1(F_1 \cup F_2)$, $\nu_1(E_1 \setminus E_2) \approx \nu_1(F_1 \setminus F_2)$, and $\nu_1(E_2) \approx \nu_1(F_2)$. The transfer principle leads to ν that is finitely additive on ${}^*\mathscr{A}$, and

$$\begin{aligned} \nu_1\left(E_1 \bigcup E_2\right) \approx \nu\left(F_1 \bigcup F_2\right) &= \nu(F_1 \setminus F_2) + \nu(F_2) \\ &= \nu_1(F_1 \setminus F_2) + \nu_1(F_2), \quad (14) \\ \nu_1(F_1 \setminus F_2) + \nu_1(F_2) &\approx \nu_1(E_1 \setminus E_2) + \nu_1(E_2). \end{aligned}$$

Now

$$\nu_1(E_1 \setminus E_2) + \nu_1(E_2) = \nu_1(E_1) + \nu_1(E_2), \tag{15}$$

which is due to the fact that E_1 and E_2 are disjoint.

We conclude from (14) and (15) that

$$\nu_1\left(E_1 \bigcup E_2\right) \approx \nu_1(E_1) + \nu_1(E_2), \tag{16}$$

which clearly forces

$$\nu_1\left(E_1 \bigcup E_2\right) = \nu_1(E_1) + \nu_1(E_2). \tag{17}$$

This completes the proof. □

Theorem 15. *Let $\nu : \mathscr{R} \to X$ be a s-bounded and countably additive vector measure. Then there exists a unique countably additive vector measure $\nu_1 : \overline{\mathscr{R}} \to X$ extending ν.*

Proof. We claim that ν is absolutely continuous with respect to $\widetilde{\nu}$. Indeed, if $A \in {}^*\mathscr{R}$ and $\widetilde{\nu}(A) \approx 0$ the transfer principle leads to $\widetilde{\nu}(A) \approx 0$. Since $\|\nu(A)\| \leq \widetilde{\nu}(A)$ we have $\nu(A)$ in the monad of zero in X, which is our assertion. Theorem 14 shows that there exists a vector measure $\nu_1 : \overline{\mathscr{R}} \to X$ absolutely continuous with respect to $\widetilde{\nu}$ which extends ν. What is left to prove is that ν_1 is countably additive. Consider $(A_n)_n$ a sequence of elements of $\overline{\mathscr{R}}$ such that $(A_n)_n \searrow \emptyset$. Therefore, $\widetilde{\nu}(A_n) \searrow 0$ by Theorem 6(iii). Then for $\theta \in \mathbb{N}_\infty$, $\widetilde{\nu}(A_\theta) \approx 0$. Therefore, applying Theorem 14 and Lemma 10, we have $\nu_1(A_\theta)$ in the monad of zero in X. Then $\nu_1(A_n) \to 0$ as $n \to \infty$, which completes the proof. □

The following corollary can be found in [11, Theorem 1] or [36].

Corollary 16. *Let $\mathscr{R}$ be a ring and $\sigma(\mathscr{R})$ the σ-ring generated by $\mathscr{R}$. A countably additive vector measure $\nu : \mathscr{R} \to X$ can be extended uniquely to a countably additive vector measure $\nu_1 : \sigma(\mathscr{R}) \to X$ if and only if ν is s-bounded.*

The next two results were proved by Gould [12] and they are important consequences of Corollary 16. He has shown that in every weakly complete Banach space property (A) holds [12, Theorem 3.1]. Even though our construction is adapted from [12], Proposition 17 yields an elegant proof of this result.

Proposition 17. *Let X be a weakly complete Banach space and ν a bounded set function from $\mathscr{R}$ to X. If ν is a countably additive vector measure, then ν has a unique countably additive extension $\nu_1 : \sigma(\mathscr{R}) \to X$.*

Proof. We begin by proving that ν is s-bounded. If ν were not s-bounded, we would have X that contains a subspace isomorphic to c_0 (see, for instance, [26, page 20]). Since every closed linear subspace of weakly complete Banach space is weakly complete and c_0 is not weakly complete [2, page 339], we obtain a contradiction. Now it suffices to apply Corollary 16, and the proof is complete. □

Remark 18. The reader should observe the similarity between this proposition and some of the criteria, established by Gould [12, Theorem 2.5, page 688], for the extension of set functions with values in weakly complete Banach spaces. It is worth noting that in the work of Gould [12] it suffices to require that ν be locally bounded over $\mathscr{R}$ and the extension is made onto a family Σ_1 which is a $\sigma(\mathscr{R})$-hereditary ring containing $\mathscr{R}$ (if S is a given family of sets, a ring $\mathscr{R}$ is said to be S-hereditary if every member of S which is a subset of some member of $\mathscr{R}$ is also a member of $\mathscr{R}$). Our requirements are stronger, but the final conclusion is somewhat more general. Moreover, we do not use Pettis' theorem for the proof.

We are now ready to prove [12, Theorem 3.1, page 689].

Proposition 19. *If X is a weakly complete Banach space then property (A) holds.*

Proof. Suppose that the proposition was false. Then we could find positive δ, K, and a sequence (x_n) in X so that $\|x_n\| \geq \delta$ for all n and $\|\sum_r x_{n_r}\| \leq K$ for every finite subsequence (x_{n_r}) of (x_n). Let $\mathscr{R}$ be the ring of finite sets of positive integers, and let ν denote the set function taking each finite set (n_r) into the vector $\sum_r x_{n_r}$. Clearly ν is a bounded vector measure from $\mathscr{R}$ to X. Furthermore, ν is countably additive, since there are no infinite disjoint nonempty sequences (E_n) in $\mathscr{R}$ whose union is in $\mathscr{R}$. Proposition 17 yields that there is a countably additive extension ν_1 onto $\sigma(\mathscr{R})$, which extends ν. In particular, the set $\mathbb{N}$ of all the positive integers belongs to $\sigma(\mathscr{R})$. Hence, $\nu_1(\mathbb{N}) = \sum_r x_r$, and the convergence of this series contradicts the hypothesis that $\|x_n\| \geq \delta$ for all n. □

For Banach spaces reflexivity and semireflexivity are equivalent, and either implies weak completeness. Thus, the rederivation of Fox's theorem [3] is a consequence of Proposition 17 (see also [11, Corollary 2, page 65]).

Corollary 20. *A bounded countably additive vector measure on a field, taking its values in a reflexive Banach space, extends uniquely to a countably additive vector measure on the generated sigma-field.*

Theorem 21. *Let X be a Banach space for which property (A) holds. Assume $\nu : \mathscr{R} \to X$ is a countably additive vector measure. Then ν has a countably additive extension $\nu_1 : \sigma(\mathscr{R}) \to X$ if and only if ν is bounded.*

Proof.

Necessity. Assume that ν has a countably additive extension $\nu_1 : \sigma(\mathscr{R}) \to X$. Then ν_1 as a finite-valued measure on a σ-ring is bounded over its domain [2, III, 4.5], so ν is bounded over $\mathscr{R}$.

To prove the sufficiency for such ν we verify that ν is s-bounded. Indeed, if ν were not s-bounded, we would have a sequence $\{E_n\}$ of disjoint members of $\mathscr{R}$ and $\delta > 0$ such that $\|\nu(E_n)\| \geq \delta$ for all n. By property (A) for all positive K there is a finite subsequence $\nu(E_{n_r})$ such that $\|\sum_r \nu(E_{n_r})\| =$

$\|\nu(\bigcup_r E_{n_r})\| \geq K$. This contradicts that ν is bounded, and the theorem follows. □

4. Existence of Control Measure

In [8], the authors show that the Bartle-Dunford-Schwartz theorem [37] does not work if we replace the σ-ring by a ring and ask whether the result remains valid for δ-rings. The theorem states that for every countably additive measure ν defined on sigma algebra there exists a positive control measure μ such that $\mu(E) \to 0$ if and only if $\|\nu\|(E) \to 0$, where $\|\nu\|$ is the semivariation of ν. By a counterexample, it is shown in [38] that this result could not work even if the measure is defined on δ-rings. So, we have to impose additional conditions for obtaining this goal. In [38], one also shows that the theorem works if the space X is separable. Now, if we pass to the conditions imposed on the measure ν and no the space X in which it has values, we will prove the following two Brooks' results [36].

Theorem 22. *Let $\nu : \mathscr{R} \to X$ be a countably additive vector measure. Then ν is s-bounded if and only if there exists a positive countably additive bounded set function μ defined on $\mathscr{R}$ such that*

$$\lim_{\mu(E)\to 0} \nu(E) = 0. \tag{18}$$

To prove this theorem Brooks uses Orlicz-Pettis theorem and two results of Porcelli [39] about some embedding theorems and their implications in weak convergence, respectively, compactness in the space of finitely additive measure. He also uses a result of Leader [40] from the theory of L^p spaces for finitely additive measures. Brooks and Dinculeanu in [41], by extending a result of Dieudonné [42], prove the assertion for finitely additive and locally strongly additive measures. Traynor [43] gave an elementary proof of this result for strongly additive measures. Using this result, he shows that for strongly additive and countably additive measures on algebra, the existence of a finite control measure is equivalent to the relatively weak compactness of range of measures, which is equivalent to the existence of a countably additive extension on the sigma algebra generated by $\mathscr{R}$ [11, 26].

There is some interest in the extension measure theoretic approach given here. The proof that we will give uses the extension of ν to σ-ring generated by $\mathscr{R}$, and then we apply the Bartle-Dunford-Schwartz theorem. Thus, we avoid some deep results in vector measures and unconditionally convergent series.

Proof of Theorem 22. First assume that ν is countably additive and s-bounded. Theorem 15 gives a countably additive vector measure $\nu_1 : \overline{\mathscr{R}} \to X$, which extends ν. We know that $\overline{\mathscr{R}}$ is σ-algebra and $\mathscr{R} \subseteq \overline{\mathscr{R}}$ (see Theorem 6(i)). By the Bartle-Dunford-Schwartz theorem [37], there exists a positive countably additive bounded set function ψ on $\overline{\mathscr{R}}$ such that

$$\lim_{\psi(E)\to 0} \nu_1(E) = 0. \tag{19}$$

Define $\mu(A) = \psi(A)$ if $A \in \mathscr{R}$, and

$$\lim_{\mu(E)\to 0} \nu(E) = 0 \tag{20}$$

as claimed.

For the converse, let (E_n) be a sequence of pairwise disjoint members of $\mathscr{R}$. Since μ is bounded and countably additive there exists $M \geq 0$ such that for all n

$$\sum_{k=1}^{n} \mu(A_k) \leq M, \tag{21}$$

so $\sum_{k=1}^{\infty} \mu(A_k)$ is convergent. Then $\mu(A_k) \to 0$ as $k \to \infty$, and

$$\lim_{\mu(E)\to 0} \nu(E) = 0 \tag{22}$$

implies $\nu(A_k) \to 0$ as $k \to \infty$, which completes the proof. □

Corollary 23. *Let $\nu : \mathscr{R} \to X$ be countably additive. Then there is a countably additive bounded set function μ defined on $\mathscr{R}$ which is a control measure for ν if and only if ν is s-bounded.*

The next result can be found in [11, Theorem 2] or in [26, Theorem 2, page 27].

Corollary 24. *Let $\mathscr{R}$ be a ring and $\sigma(\mathscr{R})$ the σ-ring generated by $\mathscr{R}$. Every countably additive vector measure $\nu : \mathscr{R} \to X$ can be extended uniquely to a vector measure $\nu_1 : \sigma(\mathscr{R}) \to X$ if and only if one of the following conditions is satisfied:*

(i) *there exists a positive bounded measure μ on $\mathscr{R}$ such that*

$$\lim_{\mu(E)\to 0} \nu(E) = 0, \quad E \in \mathscr{R}; \tag{23}$$

(ii) *ν is s-bounded;*

(iii) *the range of the vector measure ν is relatively weakly compact.*

Proof. The equivalence of (i) and (ii) is given by Theorem 22.

To prove (ii) ⇒ (iii), we note by Corollary 16 that there is a countably additive vector measure $\nu_1 : \sigma(\mathscr{R}) \to X$, which extends ν. Applying the Bartle-Dunford-Schwartz theorem [37, Theorem 2.9], the set $\{\nu_1(A) : A \in \sigma(\mathscr{R})\}$ is relatively weakly compact, and so is the set $\{\nu(A) : A \in \mathscr{R}\}$.

The implication (iii) ⇒ (ii) is proved by Kluvánek in [44, Theorem 5.3]. □

5. Extension of Set Function with Finite Semivariation

Lewis in [27] used Caratheodory's method about the extension of set functions to perform the extension of set functions with finite semivariation. While the circumstances are somewhat similar to the extension of set functions with finite variation, the techniques employed from there have carried over to that situation studied by Dinculeanu [5]. In this

section we give a nonstandard proof of the central result of Lewis. This in turn is applied in Section 6 to achieve the extension of set functions with finite variation, so we can unify the extension of set function with finite semivariation with the extension of set function with finite variation.

Let $\mathscr{R}$ be a ring of subsets of a universal space Ω. For E, F Banach spaces we denote by $L(E,F)$ the Banach space of all bounded linear operators $f : E \to F$, and let $\nu : \mathscr{R} \to L(E,F)$ be a set function with finite semivariation. That is, if $A \in \mathscr{R}$ we assume that

$$\sup \left\| \sum \nu(A_i) x_i \right\| \tag{24}$$

is finite, where we take the supremum over all finite subdivisions (A_i) of A which consist of elements of $\mathscr{R}$ and all elements x_i of unit norm in E. Let $H(\mathscr{R})$ be the hereditary σ-ring generated by $\mathscr{R}$. We use the semivariation $\|\nu\|$ of ν to define an outer measure ν^* on $H(\mathscr{R})$ in the obvious way. Thus, $\nu^*(A)$ for $A \in H(\mathscr{R})$ is

$$\inf \sum \|\nu\|(A_i), \tag{25}$$

where the infimum is taken over all countable $\mathscr{R}$-coverings of A. Clearly ν^* is an outer measure on $H(\mathscr{R})$. Let $\Omega(\nu)$ be the set of all elements A in $H(\mathscr{R})$ so that if $B \in H(\mathscr{R})$, we have

$$\nu^*(B) = \nu^*\left(B \bigcap A\right) + \nu^*\left(B \bigcap A^c\right). \tag{26}$$

By virtue of a well-known result, $\Omega(\nu)$ is a sigma ring [5]. $\mathscr{R}(\nu)$ will be the largest class of subsets of Ω so that $\Omega(\nu)$ forms an ideal in $\mathscr{R}(\nu)$; that is, $A \in \mathscr{R}(\nu)$ if for each $B \in \Omega(\nu)$ we have $B \cap A \in \Omega(\nu)$. For $A \in \mathscr{R}(\nu)$ define

$$\mu^*(A) = \sup \nu^*(B), \tag{27}$$

where the supremum is taken over all $B \in \Omega(\nu)$ such $B \subseteq A$. $\Sigma(\nu)$ will be the δ-ring of all elements in $\mathscr{R}(\nu)$ with finite μ^* measure. The main result which will be proved by nonstandard means is the extension theorem of ν to a uniquely set function ν_1 defined on $\Sigma(\nu)$ with values in $L(E,F)$. A standard proof can be found in [27].

Definition 25. We say that a finitely additive set function $\nu : \mathscr{R} \to L(E,F)$ is variationally semiregular provided that if (A_n) is a decreasing sequence of sets in $\mathscr{R}$ whose intersection is empty and $\|\nu\|(A_1) < \infty$, then $\lim_{n\to\infty}\|\nu\|(A_n) = 0$. Halmos [1] says that $\|\nu\|$ is continuous from above at $\emptyset$.

For A, B in $\Sigma(\nu)$, define $\rho(A,B)$ to be $\mu^*(A \triangle B)$. Then ρ defines a semimetric on $\Sigma(\nu)$. In [27] the following two results are proved which we mention without proof.

Lemma 26. *Suppose that $\nu : \mathscr{R} \to L(E,F)$ is variationally semiregular with finite semivariation on $\mathscr{R}$. Let ν^* be the outer measure on $H(\mathscr{R})$. Then $\|\nu\| = \nu^*$ on $\mathscr{R}$.*

Lemma 27. *If $\mathscr{R} \subseteq \Omega(\nu)$, then $\mathscr{R}$ is ρ-dense in $\Sigma(\nu)$.*

We now move on to the nonstandard proof of our problem.

Theorem 28. *Let $\nu : \mathscr{R} \to L(E,F)$ be variationally semiregular. If ν is with finite semivariation on $\mathscr{R}$ and $\mathscr{R} \subseteq \Omega(\nu)$, then there exists a unique extension $\nu_1 : \Sigma(\nu) \to L(E,F)$ of ν such that ν_1 satisfies the following conditions:*

(a) *ν_1 is countably additive on $\Sigma(\nu)$;*

(b) *$\|\nu_1\| = \mu^*$ on $\Sigma(\nu)$;*

(c) *ν_1 has finite semivariation $\|\nu_1\|$;*

(d) *$\|\nu_1\|$ extends $\|\nu\|$.*

Proof. (a) Let $\lambda = \mu^*_{|\Sigma(\nu)}$. Obviously, λ is finite and countably additive on $\Sigma(\nu)$. From this we obtain that if $(A_n)_n$ is a sequence of members of $\Sigma(\nu)$ such that $A_n \searrow \emptyset$, then $\lambda(A_\theta) \approx 0$ for all $\theta \in \mathbb{N}_\infty$. By Lemma 27 the assumptions of Theorem 14 are satisfied, so there exists a unique finite additive extension $\nu_1 : \Sigma(\nu) \to L(E,F)$ of ν which is absolutely continuous with respect to λ. Then ν_1 is uniformly continuous on $\Sigma(\nu)$ (Lemma 13). Lemma 10 now leads to $\nu_1(A_\theta) \approx 0$ for all $\theta \in \mathbb{N}_\infty$, which is our claim.

(b) Let $A \in \Sigma(\nu)$. According to Lemma 27 there exists $B \in {}^*\mathscr{R}$ so that $B \approx A$. It is well known that if ν_1 is absolutely continuous with respect to λ, then $\|\nu_1\|$ is absolutely continuous with respect to λ. We conclude from Lemma 13 that $\|\nu_1\|$ is uniformly continuous, hence that $\|\nu_1\|(A) \approx \|\nu_1\|(B)$. Since ν and ν_1 agree on $\mathscr{R}$, $\|\nu_1\|$ and $\|\nu\|$ agree on $\mathscr{R}$. By using the transfer principle they agree on ${}^*\mathscr{R}$, so that $\|\nu_1\|(B) = \|\nu\|(B)$. We also have that $\|\nu\|$ and ν^* agree on $\mathscr{R}$ (Lemma 26), so a repeated application of the transfer principle enables us to write $\|\nu\| = \nu^*$ on ${}^*\mathscr{R}$. We get that ν^* and μ^* agree on ${}^*\mathscr{R}$, because they agree on $\Omega(\nu)$ and $\mathscr{R} \subseteq \Omega(\nu)$ (apply again the transfer principle). It follows that $\nu^*(B) = \mu^*(B)$ and finally that $\|\nu_1\|(A) \approx \mu^*(B)$. Now $B \approx A$ implies $\mu^*(B) \approx \mu^*(A)$, so $\|\nu_1\|(A) \approx \mu^*(A)$; with both parts being standard this clearly forces $\|\nu_1\| = \mu^*$ on $\Sigma(\nu)$.

(c) On account of (b) we have that ν_1 has finite semivariation on $\Sigma(\nu)$.

(d) Since μ^* is countably additive on $\Sigma(\nu)$, (b) we obtain that $\|\nu_1\|$ is countably additive on $\Sigma(\nu)$. As ν^* is countably additive on $\Omega(\nu)$ and $\mathscr{R} \subseteq \Omega(\nu)$ we have by Lemma 26 that $\|\nu\| = \nu^* = \mu^*$ on $\mathscr{R}$. Thus, $\|\nu\|$ is countably additive on $\mathscr{R}$. Therefore, $\|\nu_1\|$ extends $\|\nu\|$, both being countably additive on $\Sigma(\nu)$, respectively, and on $\mathscr{R}$. □

6. Extension of Set Function with Finite Variation

Finally, we deal with the extension of set functions with finite variation. As mentioned in Section 1, we are concerned in this section with the study of how the extension of set functions with finite semivariation implies the extension of set functions with finite variation. We use the same notations as in the previous section, and we will reduce the problem to the previous case. It is known that for any finite additive set function $\nu : \mathscr{R} \to X$ with $\nu(\emptyset) = 0$, we can choose the spaces E and F so that the semivariation of ν relative to these spaces is equal to the variation of ν. If X is a normed space, we have a well-known connection between semivariation and variation [5].

Proposition 29. *The semivariation of the set function $\nu : \mathscr{R} \to X \subset L(E, \mathbb{C})$, relatively to the spaces E and $\mathbb{C}$, is equal to the variation of ν.*

This result will be used in the next theorem. But first we make some additional observations related to the precedence theorems and lemmas. Note that if we choose E so that X is embedded in $L(E, \mathbb{C})$, not only the variation and the semivariation are equal but also $\mathscr{R} \subseteq \Omega(\nu)$.

Lemma 27 of Section 5 shows a density property of the ring R in the δ-ring $\Sigma(\nu)$ for the topology induced by the semimetric $\rho(A, B)$. We now expand sigma additive vector measures with finite variation from the ring $\mathscr{R}$ to δ-ring $\Sigma(\nu)$ which is wider than the δ-ring $\Sigma(\mathscr{R})$ generated by $\mathscr{R}$. Moreover, if the function is with finite variation and countably additive it is automatically variationally semiregular. Then, Theorem 28 leads to the following result [5].

Theorem 30. *Let X be a Banach space, $\mathscr{R}$ a ring of sets, and $\nu : \mathscr{R} \to X$ a countably additive measure with finite variation. Then ν can be extended uniquely to a vector measure $\nu_1 : \Sigma(\nu) \to X$ such that*

(a) *ν_1 is countably additive on $\Sigma(\nu)$;*

(b) *$|\nu_1| = \mu^*$ on $\Sigma(\nu)$;*

(c) *ν_1 has finite variation;*

(d) *$|\nu_1|$ extends $|\nu|$.*

Remark 31. In the construction presented by Lewis [27] the author uses a technique similar to that of the extension of set functions with finite variation. Basically, this technique carries over to that of Caratheodory process. Here, noting that variation and semivariation are equal in some particular cases, we could get the extension of set functions with finite variation as a particular case of the extension of set functions with finite semivariation. In addition to these results, nonstandard proofs of these classical measure theory results are found to be more intuitive and easier than the standard proofs.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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