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*Research Article*

# Exponential Stability of Periodic Solutions for Inertial Type BAM Cohen-Grossberg Neural Networks

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The existence and exponential stability of periodic solutions for inertial type BAM Cohen-Grossberg neural networks are investigated. First, by properly choosing variable substitution, the system is transformed to first order differential equation. Second, some sufficient conditions that ensure the existence and exponential stability of periodic solutions for the system are obtained by constructing suitable Lyapunov functional and using differential mean value theorem and inequality technique. Finally, two examples are given to illustrate the effectiveness of the results.

## 1. Introduction

The Cohen-Grossberg-type BAM neural networks model is initially proposed by Cohen and Grossberg [1], has their promising potential for the tasks of parallel computation, associative memory, and has great ability to solve difficult optimization problems. Thus, the analysis of the dynamical behaviors of bidirectional associative memory neural networks and Cohen-Grossberg neural networks is important and necessary. In recent years, many researchers have studied the stability and other dynamical behaviors of the Cohen-Grossberg-type BAM neural networks; see [2–10].

On the other hand, some authors studied neural networks, added the inertia, and obtained some results. For example, Li et al. [11] added the inertia to a delay differential equation which can be described by

$$\ddot{x} = a\dot{x} - bx + cf\left(x - hx\left(t - \tau\right)\right) \tag{1}$$

and obtained obvious chaotic behavior. Liu et al. [12, 13] found chaotic behavior of the inertial two-neuron system with time through numerical simulation and gave that the system will lose its stability when the time delay is increased and will rise a quasiperiodic motion and chaos under the interaction of the periodic excitation. Wheeler and Schieve [14] added the inertia to a continuous-time Hopfield effective-neuron system which is shown to exhibit chaos. They explain that the chaos is confirmed by Lyapunov exponents, power spectra, and phase space plotsthis system is described by

$$\begin{aligned} \ddot{x}_1 &= -a_{11}\dot{x}_1 - a_{12}x_1 + a_{13}\tanh\left(x_1\right) + a_{14}\tanh\left(x_2\right), \\ \ddot{x}_2 &= -b_{11}\dot{x}_2 - b_{12}x_2 + b_{13}\tanh\left(x_1\right) + b_{14}\tanh\left(x_2\right). \end{aligned} \tag{2}$$

Babcock and Westervelt [15] studied the electronic neural networks with added inertia and found that when the neuron couplings are of an inertial nature, the dynamics can be complex, in contrast to the simpler behavior displayed when they of the standard resistor-capacitor variety. For various values of the neuron gain and the quality factor of the couplings, they find ringing about the stationary points, instability and spontaneous oscillation, intertwined basins of attraction, and chaotic response to a harmonic drive. Ge and Xu [16] considered an inertial four-neuron delayed bidirectional associative memory model. Weak resonant double Hopf bifurcations are completely analyzed in the parameter space of the coupling weight and the coupling delay by the perturbation-incremental scheme. Others, Liu et al. [17, 18], investigated the Hopf bifurcation and dynamics of an inertial two-neuron system or in a single inertial neuron mode. Zhao et al. [19] investigated the stability and the bifurcation of a class of inertial neural networks. The authors Ke and Miao [20, 21] investigated stability of equilibrium point and periodic solutions in inertial BAM neural networks with

time delays, respectively. From the above, the inertia can be considered a useful tool that is added to help in the generation of chaos in neural systems. Horikawa and Kitajima [22] investigated a kinematical description of traveling waves of the oscillations in neural networks with inertia. When the inertia is below a critical value and the state of each neuron is overdamped, properties of the networks are the same as those without inertia. The duration of the transient oscillations increases with inertia, and the increasing rate of the logarithm of the duration becomes more than double. When the inertia exceeds a critical value and the state of each neuron becomes underdamped, properties of the networks qualitatively change. The periodic solution is stabilized through the pitchfork bifurcation as inertia increases. More bifurcations occur so that various periodic solutions are generated, and the stability of the periodic solutions changes alternately. Ke and Miao [23] investigated the stability of inertial Cohen-Grossberg-type neural networks with time delays. To the best of our knowledge, the question on the periodic solutions of inertial type BAM Cohen-Grossberg neural networks with time delays is still open. To provide the theoretical basis of practical application, this paper is devoted to present a sufficient criterion to ensure the existence and exponential stability of periodic solutions for inertial type BAM Cohen-Grossberg neural networks with time delays.

We consider the following inertial type BAM Cohen-Grossberg neural networks with time delays:

$$\begin{aligned}\frac{d^2u_i(t)}{dt^2} &= -\alpha_i\frac{du_i(t)}{dt} - a_i(u_i(t)) \\ &\quad\times\left[b_i(u_i(t)) - \sum_{j=1}^{m} c_{ij}f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right)\right. \\ &\quad\left. -I_i(t)\right], \\ \frac{d^2v_j(t)}{dt^2} &= -\beta_j\frac{dv_j(t)}{dt} - d_j(v_j(t)) \\ &\quad\times\left[e_j(v_j(t)) - \sum_{i=1}^{n} h_{ji}g_i\left(u_i\left(t-\sigma_{ji}(t)\right)\right)\right. \\ &\quad\left. -J_j(t)\right],\end{aligned} \tag{3}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where the second derivative is called an inertial term of system (3); $\alpha_i$, $\beta_j > 0$ are constants; $u_i(t)$ and $v_j(t)$ are the states of the $i$th neuron from the neural field $F_U$ and the $j$th neuron from the neural field $F_V$ at the time $t$, respectively; $f_j$, $g_i$ denote the activation functions of $j$th neuron from $F_V$ and the $i$th neuron from $F_U$, respectively; $c_{ij}$ weights the strength of the $i$th neuron on the $j$th neuron at the time $t-\tau_{ij}(t)$; $h_{ji}$ weights the strength of the $j$th neuron on the $i$th neuron at the time $t-\sigma_{ji}(t)$; $0 \le \tau_{ij}(t) \le \tau$ and $0 \le \sigma_{ji}(t) \le \sigma$; $I_i(t)$, $J_j(t)$ denote the external inputs on the $i$th neuron from $F_U$ and the $j$th neuron from $F_V$ at the time $t$, respectively; $a_i(u_i(t))$ and $d_j(v_j(t))$ represent amplification functions; $b_i(u_i(t))$ and $e_j(v_j(t))$ are appropriately behaved functions such that the solutions of model (3) remain bounded.

The initial conditions of system (3) are given by

$$\begin{aligned} u_i(s) &= \varphi_{ui}(s), \quad \frac{du_i(s)}{dt} = \psi_{ui}(s), \quad -\tau \le s \le 0, \\ v_j(s) &= \varphi_{vj}(s), \quad \frac{dv_j(s)}{dt} = \psi_{vj}(s), \quad -\sigma \le s \le 0, \end{aligned} \tag{4}$$

where $\varphi_{ui}(s)$, $\psi_{ui}(s)$, $\varphi_{vj}(s)$, and $\psi_{vj}(s)$ are bounded and continuous functions.

This paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, the sufficient conditions are derived which ensure the existence and exponential stability of periodic solutions for inertial Cohen-Grossberg-type BAM neural networks. In Section 4, two illustrative examples are given to show the effectiveness of the proposed theory.

## 2. Preliminaries

Throughout this paper, we make the following assumptions.

$(H_1)$ For each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, the functions $a_i(u)$, $d_j(v)$, $b_i(u)$, and $e_j(v)$ are differentiable and satisfy

$$\begin{aligned} &0 < \underline{a}_i \le a_i(u) \le \overline{a}_i, \qquad \left|a_i'(u)\right| \le A_i, \\ &0 < \underline{d}_j \le d_j(v) \le \overline{d}_j, \qquad \left|d_j'(v)\right| \le D_j, \\ &0 < \underline{b}_i \le b_i'(u) \le \overline{b}_i, \qquad 0 < \underline{e}_j \le e_j'(v) \le \overline{e}_j, \end{aligned} \tag{5}$$

for all $u, v \in R$.

$(H_2)$ For each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, the activation functions $f_j$, $g_i$ satisfy Lipschitz condition, and there exist constants $l_j > 0$, $\overline{f}_j > 0$, $k_i > 0$, and $\overline{g}_i > 0$, such that

$$\begin{aligned} &\left|f_j(v_1) - f_j(v_2)\right| \le l_j\left|v_1 - v_2\right|, \qquad \left|f_j(v)\right| \le \overline{f}_j, \\ &\qquad\qquad\qquad\qquad\qquad\qquad v_1, v_2, v \in R, \\ &\left|g_i(u_1) - g_i(u_2)\right| \le k_i\left|u_1 - u_2\right|, \qquad \left|g_i(u)\right| \le \overline{g}_i, \\ &\qquad\qquad\qquad\qquad\qquad\qquad u_1, u_2, u \in R, \end{aligned} \tag{6}$$

$(H_3)$ For each $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $I_i(t)$, $J_j(t)$ are continuously periodic functions defined on $t \in [0, \infty)$ with common period $\omega > 0$ and satisfy $0 < \underline{I}_i \le I_i(t) \le \overline{I}_i$, $0 < \underline{J}_j \le J_j(t) \le \overline{J}_j$.

$(H_4)$ Let $B_i(u_i) = a_i(u_i)b_i(u_i)$; there exist constants $T_i > 0$ and $K_i > 0$, such that

$$0 < T_i \le B_i'(u) \le K_i, \quad i = 1, 2, \ldots, n, \ u_i \in R. \tag{7}$$

$(H_5)$ Let $E_j(v_j) = d_j(v_j)e_j(v_j)$; there exist constants $T_j^* > 0$ and $K_j^* > 0$, such that

$$0 < T_j^* \le E_j'(v) \le K_j^*, \quad j = 1, 2, \ldots, m, \ v_j \in R. \tag{8}$$

$\tau_{ij}(t)$ and $\sigma_{ji}(t)$ are continuously differentiable periodic functions, and there exist constants $0 < \overline{\tau}_{ij} < 1$ and $0 < \overline{\sigma}_{ji} < 1$, such that

$$\tau'_{ij}(t) \le \overline{\tau}_{ij} < 1, \qquad \sigma'_{ji}(t) \le \overline{\sigma}_{ji} < 1, \tag{9}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $\tau'_{ij}(t)$ and $\sigma'_{ji}(t)$ express the derivative of $\tau_{ij}(t)$ and $\sigma_{ji}(t)$.

Introducing variable transformation

$$\begin{aligned} x_i(t) &= \frac{du_i(t)}{dt} + u_i(t), \quad i = 1, 2, \ldots, n, \\ y_j(t) &= \frac{dv_j(t)}{dt} + v_j(t), \quad j = 1, 2, \ldots, m, \end{aligned} \tag{10}$$

then (3) and (4) can be rewritten as

$$\begin{aligned} \frac{du_i(t)}{dt} &= -u_i(t) + x_i(t), \\ \frac{dx_i(t)}{dt} &= -(1-\alpha_i)u_i(t) - (\alpha_i - 1)x_i(t) - a_i(u_i(t)) \\ &\quad \times \Bigg[ b_i(u_i(t)) \\ &\quad - \sum_{j=1}^{m} c_{ij} f_j(v_j(t - \tau_{ij}(t))) - I_i(t) \Bigg], \\ \frac{dv_j(t)}{dt} &= -v_j(t) + y_j(t), \\ \frac{dy_j(t)}{dt} &= -(1-\beta_j)v_j(t) - (\beta_j - 1)y_j(t) - d_j(v_j(t)) \\ &\quad \times \Bigg[ e_j(v_j(t)) \\ &\quad - \sum_{i=1}^{n} h_{ji} g_i(u_i(t - \sigma_{ji}(t))) - J_j(t) \Bigg], \end{aligned} \tag{11}$$

$$\begin{aligned} u_i(s) &= \varphi_{ui}(s), \quad \frac{du_i(s)}{dt} = \psi_{ui}(s), \quad -\tau \le s \le 0, \\ x_i(s) &= \varphi_{ui}(s) + \psi_{ui}(s) \doteq \phi_{ui}(s), \quad -\tau \le s \le 0, \\ v_j(s) &= \varphi_{vj}(s), \quad \frac{dv_j(s)}{dt} = \psi_{vj}(s), \quad -\sigma \le s \le 0, \\ y_j(s) &= \varphi_{vj}(s) + \psi_{vj}(s) \doteq \chi_{vj}(s), \quad -\sigma \le s \le 0, \end{aligned} \tag{12}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

*Definition 1.* Let

$$\begin{aligned} \overline{u}(t) &= (\overline{u}_1(t), \overline{u}_2(t), \ldots, \overline{u}_n(t))^T, \\ \overline{v}(t) &= (\overline{v}_1(t), \overline{v}_2(t), \ldots, \overline{v}_m(t))^T \end{aligned} \tag{13}$$

be an $\omega$-periodic solution of system (3) with initial value

$$\begin{aligned} \overline{u}_i(s) &= \overline{\varphi}_{ui}(s), \quad \frac{d\overline{u}_i(s)}{dt} = \overline{\psi}_{ui}(s), \quad -\tau \le s \le 0, \\ \overline{v}_j(s) &= \overline{\varphi}_{vj}(s), \quad \frac{d\overline{v}_j(s)}{dt} = \overline{\psi}_{vj}(s), \quad -\sigma \le s \le 0, \end{aligned} \tag{IV1}$$

for every solution

$$\begin{aligned} u(t) &= (u_1(t), u_2(t), \ldots, u_n(t))^T, \\ v(t) &= (v_1(t), v_2(t), \ldots, v_m(t))^T \end{aligned} \tag{14}$$

of system (3) with any initial value

$$\begin{aligned} u_i(s) &= \varphi_{ui}(s), \quad \frac{du_i(s)}{dt} = \psi_{ui}(s), \quad -\tau \le s \le 0, \\ v_j(s) &= \varphi_{vj}(s), \quad \frac{dv_j(s)}{dt} = \psi_{vj}(s), \quad -\sigma \le s \le 0. \end{aligned} \tag{IV2}$$

If there exist constants $\delta > 0$ and $M > 0$, such that

$$\begin{aligned} &\sum_{i=1}^{n} (u_i(t) - \overline{u}_i(t))^2 + \sum_{j=1}^{m} (v_j(t) - \overline{v}_j(t))^2 \\ &\quad \le M e^{-\delta t} \left[ \|\varphi_{ui} - \overline{\varphi}_{ui}\|^2 + \|\varphi_{vj} - \overline{\varphi}_{vj}\|^2 \right], \end{aligned} \tag{15}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $t \ge 0$, then solutions $\overline{u}(t)$, $\overline{v}(t)$ are said to be exponentially stable, where

$$\begin{aligned} \|\varphi_{ui} - \overline{\varphi}_{ui}\|^2 &= \sup_{-\tau \le t \le 0} \sum_{i=1}^{n} |\varphi_{ui}(t) - \overline{\varphi}_{ui}(t)|^2, \\ \|\varphi_{vj} - \overline{\varphi}_{vj}\|^2 &= \sup_{-\sigma \le t \le 0} \sum_{j=1}^{m} |\varphi_{vj}(t) - \overline{\varphi}_{vj}(t)|^2. \end{aligned} \tag{16}$$

## 3. Main Results

In this section, we can derive some sufficient conditions which ensure the existence and exponential stability of periodic solutions for system (3).

**Theorem 2.** *For system* (3), *under the hypotheses* $(H_1)$–$(H_3)$, *then* $u_i(t)$, $u'_i(t)$, $v_j(t)$, *and* $v'_j(t)$ *are bounded,* $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and* $t \ge 0$.

*Proof.* If $u_i(t) > 0$, then we have

$$\frac{d|u_i(t)|}{dt} = \frac{du_i(t)}{dt}; \tag{17}$$

if $u_i(t) < 0$, then

$$\frac{d|u_i(t)|}{dt} = -\frac{du_i(t)}{dt}. \tag{18}$$

Hence, $d|u_i(t)|/dt = \operatorname{sgn}(u_i(t))(du_i(t)/dt)$. Similarly, we can get

$$\begin{aligned} \frac{d\left|v_i(t)\right|}{dt} &= \operatorname{sgn}\left(v_i(t)\right)\frac{dv_i(t)}{dt}, \\ \frac{d^2\left|u_i(t)\right|}{dt^2} &= \operatorname{sgn}\left(u_i(t)\right)\frac{d^2u_i(t)}{dt^2}, \\ \frac{d^2\left|v_i(t)\right|}{dt^2} &= \operatorname{sgn}\left(v_i(t)\right)\frac{d^2v_i(t)}{dt^2}. \end{aligned} \tag{19}$$

Since $b_i(u_i)$ are differentiable on $u_i$ $(i = 1, 2, \ldots, n)$, and then we have

$$b_i\left(u_i(t)\right) - b_i(0) = b_i'\left(u_i^*\right)u_i(t), \tag{20}$$

where $u_i^*$ lies between $u_i$ and 0.

It follows from (3) that

$$\begin{aligned} \frac{d^2\left|u_i(t)\right|}{dt^2} &= -\alpha_i\frac{d\left|u_i(t)\right|}{dt} - \operatorname{sgn}\left(u_i(t)\right)a_i\left(u_i(t)\right) \\ &\quad\times\left[b_i\left(u_i(t)\right) - \sum_{j=1}^{m}c_{ij}f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t)\right] \\ &= -\alpha_i\frac{d\left|u_i(t)\right|}{dt} - \operatorname{sgn}\left(u_i(t)\right)a_i\left(u_i(t)\right) \\ &\quad\times\left[b_i\left(u_i(t)\right) - b_i(0) + b_i(0)\right. \\ &\qquad\left. - \sum_{j=1}^{m}c_{ij}f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t)\right] \\ &= -\alpha_i\frac{d\left|u_i(t)\right|}{dt} - \operatorname{sgn}\left(u_i(t)\right)a_i\left(u_i(t)\right) \\ &\quad\times\left[b_i'\left(u_i^*\right)u_i(t) + b_i(0)\right. \\ &\qquad\left. - \sum_{j=1}^{m}c_{ij}f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t)\right] \\ &\le -\alpha_i\frac{d\left|u_i(t)\right|}{dt} - \underline{a}_i\underline{b}_i\left|u_i(t)\right| + \overline{a}_i \\ &\quad\times\left[\left|b_i(0)\right| + \sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right]. \end{aligned} \tag{21}$$

Similarly, we can obtain

$$\begin{aligned} \frac{d^2\left|v_j(t)\right|}{dt^2} &\le -\beta_j\frac{d\left|v_j(t)\right|}{dt} - \underline{d}_j\underline{e}_j\left|v_j(t)\right| \\ &\quad+ \overline{d}_j\left[\left|e_j(0)\right| + \sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right]. \end{aligned} \tag{22}$$

From (21), (22), we can obtain

$$\left|u_i(t)\right| \le C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + \frac{\overline{a}_i}{\underline{a}_i\underline{b}_i}\left[\left|b_i(0)\right| + \sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right], \tag{23}$$

where $\lambda_{1,2} = (-\alpha_i \pm \sqrt{\alpha_i^2 - 4\underline{a}_i\underline{b}_i})/2$ and $C_1$, $C_2$ are any real constants:

$$\left|v_j(t)\right| \le C_1^*e^{\lambda_1^* t} + C_2^*e^{\lambda_2^* t} + \frac{\overline{d}_j}{\underline{d}_j\underline{e}_j}\left[\left|e_j(0)\right| + \sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right], \tag{24}$$

where $\lambda_{1,2}^* = (-\beta_j \pm \sqrt{\beta_j^2 - 4\underline{d}_j\underline{e}_j})/2$ and $C_1^*$, $C_2^*$ are any real constants.

Since $\alpha_i > 0$, $\beta_j > 0$, we have $\operatorname{Re}(\lambda_1) < 0$, $\operatorname{Re}(\lambda_2) < 0$, $\operatorname{Re}(\lambda_1^*) < 0$, and $\operatorname{Re}(\lambda_2^*) < 0$, and formula (23) shows that all solutions $u_i(t)$ to (3) are bounded for $i = 1, 2, \ldots, n$, $t \ge 0$.

Formula (24) shows that all solutions $v_j(t)$ to (3) are bounded for $j = 1, 2, \ldots, m$, $t \ge 0$.

On the other hand, from (3) we also can obtain

$$\begin{aligned} \frac{du_i(t)}{dt} &= e^{-\alpha_i t}\frac{du_i(0)}{dt} - e^{-\alpha_i t}\int_0^t e^{\alpha_i s}a_i\left(u_i(s)\right) \\ &\quad\times\left[b_i\left(u_i(s)\right)\right. \\ &\qquad\left. - \sum_{j=1}^{m}c_{ij}f_j\left(v_j\left(s-\tau_{ij}(t)\right)\right) + I_i(s)\right]ds, \\ &\qquad i = 1, 2, \ldots, n, \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{dv_j(t)}{dt} &= e^{-\beta_j t}\frac{dv_j(0)}{dt} - e^{-\beta_j t}\int_0^t e^{\beta_i s}d_j\left(v_j(s)\right) \\ &\quad\times\left[e_j\left(v_j(s)\right) - \sum_{i=1}^{n}h_{ji}g_i\right. \\ &\qquad\left.\times\left(u_i\left(s-\sigma_{ji}(t)\right)\right) + J_j(s)\right]ds, \\ &\qquad j = 1, 2, \ldots, m. \end{aligned} \tag{26}$$

Since $u_i(t)$, $v_j(t)$ are bounded, we may assume that $|u_i(t)| \le R_i$, $|v_j(t)| \le R_j^*$, where $R_i > 0$, $R_j^* > 0$ are constants, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

From (25), we have

$$\left|\frac{du_i(t)}{dt}\right| \le \left|\psi_{ui}(0)\right| + \frac{\overline{a}_i}{\alpha_i}\left[\overline{b}_iR_i + \left|b_i(0)\right| + \sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right]. \tag{27}$$

Formula (27) shows that all solutions $u_i'(t)$ are bounded for $i = 1, 2, \ldots, n$, $t \ge 0$.

From (26), we have

$$\left|\frac{dv_j(t)}{dt}\right| \le \left|\psi_{vj}(0)\right| + \frac{\overline{d}_j}{\beta_j}\left[\overline{e}_j R_j^* + \left|e_j(0)\right| + \sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right]. \tag{28}$$

Formula (28) shows that all solutions $v_j'(t)$ are bounded for $j = 1, 2, \ldots, m$, $t \ge 0$. □

**Theorem 3.** *Under the hypotheses* $(H_1)$–$(H_5)$, *if* $\alpha_i - K_i > 0$, $\beta_j - K_j^* > 0$, *and*

$$\begin{gathered}
\alpha_i - T_i - 2 + A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right) + \sum_{j=1}^{m}\overline{d}_j\frac{\left|h_{ji}\right|}{1-\overline{\sigma}_{ji}}k_i < 0,\\
2 - \alpha_i - T_i + \overline{a}_i\sum_{j=1}^{m}\left|c_{ij}\right|l_j + A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right) < 0,\\
\beta_j - T_j^* - 2 + D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right) + \sum_{i=1}^{n}\overline{a}_i\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}}l_j < 0,\\
2 - \beta_j - T_j^* + \overline{d}_j\sum_{i=1}^{n}\left|h_{ji}\right|k_i + D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right) < 0,
\end{gathered} \tag{29}$$

*for* $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *then system* (3) *has one* $\omega$-*periodic solution, which is exponentially stable.*

*Proof.* If $u_i^*(t)$, $x_i^*(t)$, $v_i^*(t)$, and $y_i^*(t)$ are $\omega$-periodic solution of (11), which are exponentially stable, then we can obtain that $(u_i^*(t), v_i^*(t))$ are $\omega$-periodic solution of system (3), which is exponentially stable. In the following we only prove that (11) has one $\omega$-periodic solution, which is exponentially stable.

Let

$$\begin{gathered}
\overline{u}(t) = \left(\overline{u}_1(t), \overline{u}_2(t), \ldots, \overline{u}_n(t)\right)^T,\\
\overline{v}(t) = \left(\overline{v}_1(t), \overline{v}_2(t), \ldots, \overline{v}_m(t)\right)^T
\end{gathered} \tag{30}$$

be solution of system (3) with initial value $(IV1)$, and let

$$\begin{gathered}
u(t) = \left(u_1(t), u_2(t), \ldots, u_n(t)\right)^T,\\
v(t) = \left(v_1(t), v_2(t), \ldots, v_m(t)\right)^T
\end{gathered} \tag{31}$$

be solution of system (3) with any initial value $(IV2)$.

Let

$$\begin{gathered}
\overline{x}_i(t) = \frac{d\overline{u}_i(t)}{dt} + \overline{u}_i(t), \qquad \overline{y}_j(t) = \frac{d\overline{v}_j(t)}{dt} + \overline{v}_j(t),\\
z_i(t) = u_i(t) - \overline{u}_i(t), \qquad w_i(t) = x_i(t) - \overline{x}_i(t),\\
p_j(t) = v_j(t) - \overline{v}_j(t), \qquad q_j(t) = y_j(t) - \overline{y}_j(t),
\end{gathered} \tag{32}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

From (11), we can obtain

$$\begin{aligned}
\frac{dz_i(t)}{dt} &= -z_i(t) + w_i(t),\\
\frac{dw_i(t)}{dt} &= -\left(1-\alpha_i\right)z_i(t) - \left(\alpha_i - 1\right)w_i(t)\\
&\quad + a_i\left(u_i(t)\right)\left[\sum_{j=1}^{m}c_{ij}\left[f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right) - f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)\right]\right]\\
&\quad + \left(a_i\left(u_i(t)\right) - a_i\left(\overline{u}_i(t)\right)\right)\\
&\quad \times\left[\sum_{j=1}^{m}c_{ij}f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t)\right]\\
&\quad - \left[a_i\left(u_i(t)\right)b_i\left(u_i(t)\right) - a_i\left(\overline{u}_i(t)\right)b_i\left(\overline{u}_i(t)\right)\right],
\end{aligned} \tag{33}$$

for $i = 1, 2, \ldots, n$

$$\begin{aligned}
\frac{dp_j(t)}{dt} &= -p_j(t) + q_j(t),\\
\frac{dq_j(t)}{dt} &= -\left(1-\beta_j\right)p_j(t) - \left(\beta_j - 1\right)q_j(t)\\
&\quad + d_j\left(v_j(t)\right)\left[\sum_{i=1}^{n}h_{ji}\left[g_i\left(u_i\left(t-\sigma_{ji}(t)\right)\right) - g_i\left(\overline{u}_j\left(t-\sigma_{ji}(t)\right)\right)\right]\right]\\
&\quad + \left(d_j\left(v_j(t)\right) - d_j\left(\overline{v}_j(t)\right)\right)\\
&\quad \times\left[\sum_{i=1}^{n}h_{ji}g_i\left(\overline{u}_i\left(t-\sigma_{ji}(t)\right)\right) + J_j(t)\right]\\
&\quad - \left[d_j\left(v_j(t)\right)e_j\left(v_j(t)\right) - d_j\left(\overline{v}_j(t)\right)e_j\left(\overline{v}_j(t)\right)\right],
\end{aligned} \tag{34}$$

for $j = 1, 2, \ldots, m$.

Since functions $a_i(u)$ and $b_i(u)$ are differentiable, using differential mean value theorem, we have

$$\begin{aligned}
&a_i\left(u_i(t)\right) - a_i\left(\overline{u}_i(t)\right) = a_i'\left(\xi_i\right)z_i(t),\\
&a_i\left(u_i(t)\right)b_i\left(u_i(t)\right) - a_i\left(\overline{u}_i(t)\right)b_i\left(\overline{u}_i(t)\right)\\
&\quad = B_i\left(u_i\right) - B_i\left(\overline{u}_i\right) = B_i'\left(\overline{\xi}_i\right)z_i(t),
\end{aligned} \tag{35}$$

where $\xi_i$ and $\overline{\xi}_i$ lie between $u_i$ and $\overline{u}_i$.

Since $0 < T_i \le B_i'(u) \le K_i$, if $\alpha_i - K_i > 0$, then we have $\alpha_i - B_i'(\xi_i) \ge \alpha_i - K_i > 0$ and $0 < \alpha_i - B_i'(\xi_i) \le \alpha_i - T_i$.

From (33) we get

$$\begin{aligned}
\frac{1}{2}\frac{dz_i^2(t)}{dt} &= -z_i^2(t) + z_i(t)\,w_i(t),\\
\frac{1}{2}\frac{dw_i^2(t)}{dt} &= -(1-\alpha_i)\,z_i(t)\,w_i(t) - (\alpha_i - 1)\,w_i^2(t)\\
&\quad + a_i(u_i(t))\,w_i(t)\\
&\quad \times \left[\sum_{j=1}^{m} c_{ij}\left[f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right)\right.\right.\\
&\qquad \left.\left. - f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)\right]\right]\\
&\quad + \left(a_i(u_i(t)) - a_i(\overline{u}_i(t))\right)w_i(t)\\
&\quad \cdot \left[\sum_{j=1}^{m} c_{ij} f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t)\right]\\
&\quad - \left[a_i(u_i(t))\,b_i(u_i(t))\right.\\
&\qquad \left. - a_i(\overline{u}_i(t))\,b_i(\overline{u}_i(t))\right]w_i(t)\\
&= -(1-\alpha_i)\,z_i(t)\,w_i(t) - (\alpha_i - 1)\,w_i^2(t)\\
&\quad + a_i(u_i(t))\,w_i(t)\\
&\quad \times \left[\sum_{j=1}^{m} c_{ij}\left[f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right)\right.\right.\\
&\qquad \left.\left. - f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)\right]\right]\\
&\quad + \left(a_i'(\xi_i)\right)z_i(t)\,w_i(t)\\
&\quad \cdot \left[\sum_{j=1}^{m} c_{ij} f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right) + I_i(t)\right]\\
&\quad - B_i'\left(\overline{\xi}_i\right)z_i(t)\,w_i(t), \quad i = 1,2,\ldots,n.
\end{aligned} \tag{36}$$

From (36), we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(z_i^2(t) + w_i^2(t)\right)\\
&\quad \le -z_i^2(t) + \left[\alpha_i - B_i'\left(\overline{\xi}_i\right)\right]z_i(t)\,w_i(t) - (\alpha_i - 1)\,w_i^2(t)\\
&\qquad + \overline{a}_i\left|w_i(t)\right|\sum_{j=1}^{m}\left|c_{ij}\right| l_j \left|p_j\left(t-\tau_{ij}(t)\right)\right|\\
&\qquad + A_i\left|z_i(t)\right|\left|w_i(t)\right|\left[\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right]\\
&\quad \le -z_i^2(t) + \left[\alpha_i - B_i'\left(\overline{\xi}_i\right)\right]\frac{\left(z_i^2(t) + w_i^2(t)\right)}{2}\\
&\qquad - (\alpha_i - 1)\,w_i^2(t)\\
&\qquad + \overline{a}_i\sum_{j=1}^{m}\left|c_{ij}\right| l_j \frac{\left(w_i^2(t) + p_j^2\left(t-\tau_{ij}(t)\right)\right)}{2}\\
&\qquad + A_i\left[\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right]\frac{\left(z_i^2(t) + w_i^2(t)\right)}{2}\\
&\quad = \frac{1}{2}\left[\alpha_i - T_i\left(\overline{\xi}_i\right) - 2 + A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right)\right]z_i^2(t)\\
&\qquad + \frac{1}{2}\left[2 - \alpha_i - T_i\left(\overline{\xi}_i\right) + \overline{a}_i\sum_{j=1}^{m}\left|c_{ij}\right| l_j\right.\\
&\qquad\qquad \left. + A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right)\right]w_i^2(t)\\
&\qquad + \frac{\overline{a}_i}{2}\sum_{j=1}^{m}\left|c_{ij}\right| l_j p_j^2\left(t-\tau_{ij}(t)\right)
\end{aligned} \tag{37}$$

for $i = 1,2,\ldots,n$, $t > 0$.

Similar to the above derivation, from (34) we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(p_j^2(t) + q_j^2(t)\right)\\
&\quad \le \frac{1}{2}\left[\beta_j - T_j^*\left(\overline{\xi}_j\right) - 2 + D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right)\right]p_j^2(t)\\
&\qquad + \frac{1}{2}\left[2 - \beta_j - T_j^*\left(\overline{\xi}_j\right) + \overline{d}_j\sum_{i=1}^{n}\left|h_{ji}\right| k_i\right.\\
&\qquad\qquad \left. + D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right)\right]q_j^2(t)\\
&\qquad + \frac{\overline{d}_j}{2}\sum_{i=1}^{n}\left|h_{ji}\right| k_i z_i^2\left(t-\sigma_{ji}(t)\right)
\end{aligned} \tag{38}$$

for $j = 1,2,\ldots,m$, $t > 0$.

We consider the Lyapunov functional

$$\begin{aligned}
V(t) = \sum_{i=1}^{n}\left\{\frac{z_i^2(t) + w_i^2(t)}{2}e^{\varepsilon t}\right.\\
\left. + \frac{\overline{a}_i}{2}\sum_{j=1}^{m}\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}} l_j \int_{t-\tau_{ij}(t)}^{t} e^{\varepsilon(s+\tau_{ij}(s))}p_j^2(s)\,ds\right\}
\end{aligned}$$

$$+\sum_{j=1}^{m}\left\{\frac{p_j^2(t)+q_j^2(t)}{2}e^{\varepsilon t}+\frac{\overline{d}_j}{2}\sum_{i=1}^{n}\frac{|h_{ji}|}{1-\overline{\sigma}_{ji}}k_i\int_{t-\sigma_{ji}(t)}^{t}e^{\varepsilon(s+\sigma_{ji}(s))}z_i^2(s)\,ds\right\}, \tag{39}$$

where $\varepsilon > 0$ is a small number.

Calculating the upper right Dini-derivative $D^+V(t)$ of $V(t)$ along the solution of (33) and (34), using (37) and (38), we have

$$\begin{aligned}
D^+V(t) &= \sum_{i=1}^{n}\left\{\varepsilon\frac{z_i^2(t)+w_i^2(t)}{2}e^{\varepsilon t}+\frac{1}{2}\frac{d}{dt}\left(z_i^2(t)+w_i^2(t)\right)e^{\varepsilon t}\right.\\
&\quad+\frac{\overline{a}_i}{2}\sum_{j=1}^{m}\frac{|c_{ij}|}{1-\overline{\tau}_{ij}}l_j\left[p_j^2(t)e^{\varepsilon(t+\tau_{ij}(t))}\right.\\
&\qquad\left.\left.-\left(1-\tau_{ij}'(t)\right)p_j^2\left(t-\tau_{ij}(t)\right)e^{\varepsilon t}\right]\right\}\\
&\quad+\sum_{j=1}^{m}\left\{\varepsilon\frac{p_j^2(t)+q_j^2(t)}{2}e^{\varepsilon t}+\frac{1}{2}\frac{d}{dt}\left(p_j^2(t)+q_j^2(t)\right)e^{\varepsilon t}\right.\\
&\qquad+\frac{\overline{d}_j}{2}\sum_{i=1}^{n}\frac{|h_{ji}|}{1-\overline{\sigma}_{ji}}k_i\\
&\qquad\times\left[z_i^2(t)e^{\varepsilon(t+\sigma_{ji}(t))}-\left(1-\sigma_{ji}'(t)\right)z_i^2\right.\\
&\qquad\left.\left.\times\left(t-\sigma_{ji}(t)\right)e^{\varepsilon t}\right]\right\}\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\left\{\varepsilon\frac{z_i^2(t)+w_i^2(t)}{2}\right.\\
&\qquad+\frac{1}{2}\left[\alpha_i-T_i-2+A_i\left(\sum_{j=1}^{m}|c_{ij}|\overline{f}_j+\overline{I}_i\right)\right]z_i^2(t)\\
&\qquad+\frac{1}{2}\left[2-\alpha_i-T_i+\overline{a}_i\sum_{j=1}^{m}|c_{ij}|l_j\right.\\
&\qquad\left.+A_i\left(\sum_{j=1}^{m}|c_{ij}|\overline{f}_j+\overline{I}_i\right)\right]w_i^2(t)\\
&\qquad+\frac{\overline{a}_i}{2}\sum_{j=1}^{m}|c_{ij}|l_jp_j^2\left(t-\tau_{ij}(t)\right)+\frac{\overline{a}_i}{2}\sum_{j=1}^{m}\frac{|c_{ij}|}{1-\overline{\tau}_{ij}}l_j\\
&\qquad\left.\times\left[p_j^2(t)e^{\varepsilon\tau}-\left(1-\tau_{ij}'(t)\right)p_j^2\left(t-\tau_{ij}(t)\right)\right]\right\}\\
&\quad+e^{\varepsilon t}\sum_{j=1}^{m}\left\{\varepsilon\frac{p_j^2(t)+q_j^2(t)}{2}\right.\\
&\qquad+\frac{1}{2}\left[\beta_j-T_j^*-2\right.\\
&\qquad\left.+D_j\left(\sum_{i=1}^{n}|h_{ji}|\overline{g}_i+\overline{J}_j\right)\right]p_j^2(t)\\
&\qquad+\frac{1}{2}\left[2-\beta_j-T_j^*+\overline{d}_j\sum_{i=1}^{n}|h_{ji}|k_i\right.\\
&\qquad\left.+D_j\left(\sum_{i=1}^{n}|h_{ji}|\overline{g}_i+\overline{J}_j\right)\right]q_j^2(t)\\
&\qquad+\frac{\overline{d}_j}{2}\sum_{i=1}^{n}|h_{ji}|k_iz_i^2\left(t-\sigma_{ji}(t)\right)\\
&\qquad+\frac{\overline{d}_j}{2}\sum_{i=1}^{n}\frac{|h_{ji}|}{1-\overline{\sigma}_{ji}}k_i\\
&\qquad\left.\times\left[z_i^2(t)e^{\varepsilon\sigma}-\left(1-\sigma_{ji}'(t)\right)z_i^2\left(t-\sigma_{ji}(t)\right)\right]\right\}\\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\left\{\frac{1}{2}\left[\varepsilon+\alpha_i-T_i-2+A_i\left(\sum_{j=1}^{m}|c_{ij}|\overline{f}_j+\overline{I}_i\right)\right.\right.\\
&\qquad\left.+\sum_{j=1}^{m}\overline{d}_j\frac{|h_{ji}|}{1-\overline{\sigma}_{ji}}k_ie^{\varepsilon\sigma}\right]z_i^2(t)\\
&\qquad+\frac{1}{2}\left[\varepsilon+2-\alpha_i-T_i+\overline{a}_i\sum_{j=1}^{m}|c_{ij}|l_j\right.\\
&\qquad\left.\left.+A_i\left(\sum_{j=1}^{m}|c_{ij}|\overline{f}_j+\overline{I}_i\right)\right]w_i^2(t)\right\}\\
&\quad+e^{\varepsilon t}\sum_{j=1}^{m}\left\{\frac{1}{2}\left[\varepsilon+\beta_j-T_j^*-2+D_j\left(\sum_{i=1}^{n}|h_{ji}|\overline{g}_i+\overline{J}_j\right)\right.\right.\\
&\qquad\left.+\sum_{i=1}^{n}\overline{a}_i\frac{|c_{ij}|}{1-\overline{\tau}_{ij}}l_je^{\varepsilon\tau}\right]p_j^2(t)\\
&\qquad+\frac{1}{2}\left[\varepsilon+2-\beta_j-T_j^*+\overline{d}_j\sum_{i=1}^{n}|h_{ji}|k_i\right.\\
&\qquad\left.\left.+D_j\left(\sum_{i=1}^{n}|h_{ji}|\overline{g}_i+\overline{J}_j\right)\right]q_j^2(t)\right\}.
\end{aligned} \tag{40}$$

From condition of Theorem 3, we can choose a small $\varepsilon > 0$ such that

$$\begin{aligned}
&\varepsilon + \alpha_i - T_i - 2 + A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right) + \sum_{j=1}^{m}\overline{d}_j\frac{\left|h_{ji}\right|}{1-\overline{\sigma}_{ji}}k_i e^{\varepsilon\sigma} \\
&\quad \le 0, \\
&\varepsilon + 2 - \alpha_i - T_i + \overline{a}_i\sum_{j=1}^{m}\left|c_{ij}\right| l_j + A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j + \overline{I}_i\right) \\
&\quad \le 0, \\
&\varepsilon + \beta_j - T_j^* - 2 + D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right) + \sum_{i=1}^{n}\overline{a}_i\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}}l_j e^{\varepsilon\tau} \\
&\quad \le 0, \\
&\varepsilon + 2 - \beta_j - T_j^* + \overline{d}_j\sum_{i=1}^{n}\left|h_{ji}\right| k_i + D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i + \overline{J}_j\right) \le 0, \\
&\qquad i = 1,2,\ldots,n,\ \ j = 1,2,\ldots,m,\ \ t > 0.
\end{aligned} \tag{41}$$

From (40), we get $D^+V(t) \le 0$ and so $V(t) \le V(0)$, for all $t \ge 0$. From (39), we have

$$\begin{aligned}
V(t) &\ge \sum_{i=1}^{n}\frac{z_i^2(t) + w_i^2(t)}{2}e^{\varepsilon t} + \sum_{j=1}^{m}\frac{p_j^2(t) + q_j^2(t)}{2}e^{\varepsilon t} \\
&= \sum_{i=1}^{n}\frac{e^{\varepsilon t}}{2}\left[\left(u_i(t) - \overline{u}_i(t)\right)^2 + \left(x_i(t) - \overline{x}_i(t)\right)^2\right] \\
&\quad + \sum_{j=1}^{m}\frac{e^{\varepsilon t}}{2}\left[\left(v_j(t) - \overline{v}_j(t)\right)^2 + \left(y_j(t) - \overline{y}_j(t)\right)^2\right], \\
V(0) &= \sum_{i=1}^{n}\left\{\frac{z_i^2(0) + w_i^2(0)}{2}\right. \\
&\qquad \left. + \frac{\overline{a}_i}{2}\sum_{j=1}^{m}\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}}l_j\int_{-\tau_{ij}(t)}^{0}e^{\varepsilon(s+\tau_{ij}(s))}p_j^2(s)\,ds\right\} \\
&\quad + \sum_{j=1}^{m}\left\{\frac{p_j^2(0) + q_j^2(0)}{2}\right. \\
&\qquad + \frac{\overline{d}_j}{2}\sum_{i=1}^{n}\frac{\left|h_{ji}\right|}{1-\overline{\sigma}_{ji}}k_i \\
&\qquad \left. \times\int_{-\sigma_{ji}(t)}^{0}e^{\varepsilon(s+\sigma_{ji}(s))}z_i^2(s)\,ds\right\} \\
&= \sum_{i=1}^{n}\left\{\frac{\left(\varphi_{ui}(0) - \overline{\varphi}_{ui}(0)\right)^2}{2} + \frac{\left(\phi_{ui}(0) - \overline{\phi}_{ui}(0)\right)^2}{2}\right. \\
&\qquad + \frac{\overline{a}_i}{2}\sum_{j=1}^{m}\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}}l_j \\
&\qquad \left. \times\int_{-\tau_{ij}(t)}^{0}e^{\varepsilon(s+\tau_{ij}(s))}\left(\varphi_{vj}(s) - \overline{\varphi}_{vj}(s)\right)^2 ds\right\} \\
&\quad + \sum_{j=1}^{m}\left\{\frac{\left(\varphi_{vj}(0) - \overline{\varphi}_{vj}(0)\right)^2}{2} + \frac{\left(\chi_{vj}(0) - \overline{\chi}_{vj}(0)\right)^2}{2}\right. \\
&\qquad + \frac{\overline{d}_j}{2}\sum_{i=1}^{n}\frac{\left|h_{ji}\right|}{1-\overline{\sigma}_{ji}}k_i \\
&\qquad \left. \times\int_{-\sigma_{ji}(s)}^{0}e^{\varepsilon(s+\sigma_{ji}(t))}\left(\varphi_{ui}(s) - \overline{\varphi}_{ui}(s)\right)^2 ds\right\} \\
&\le \frac{\left\|\varphi_{ui} - \overline{\varphi}_{ui}\right\|^2}{2} + \frac{\left\|\phi_{ui} - \overline{\phi}_{ui}\right\|^2}{2} \\
&\quad + \tau\sum_{i=1}^{n}\frac{\overline{a}_i}{2}\max_{1\le j\le m}\left\{\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}}l_j\right\}e^{\varepsilon\tau}\left\|\varphi_{vj} - \overline{\varphi}_{vj}\right\|^2 \\
&\quad + \frac{\left\|\varphi_{vj} - \overline{\varphi}_{vj}\right\|^2}{2} + \frac{\left\|\chi_{vj} - \overline{\chi}_{vj}\right\|^2}{2} \\
&\quad + \sigma\sum_{j=1}^{m}\frac{\overline{d}_j}{2}\max_{1\le i\le n}\left\{\frac{\left|h_{ji}\right|}{1-\overline{\sigma}_{ji}}k_i\right\}e^{\varepsilon\sigma}\left\|\varphi_{ui} - \overline{\varphi}_{ui}\right\|^2 \\
&= \left[1 + \sigma\sum_{j=1}^{m}\overline{d}_j\max_{1\le i\le n}\left\{\frac{\left|h_{ji}\right|}{1-\overline{\sigma}_{ji}}k_i\right\}e^{\varepsilon\sigma}\right]\frac{\left\|\varphi_{ui} - \overline{\varphi}_{ui}\right\|^2}{2} \\
&\quad + \left[1 + \tau\sum_{i=1}^{n}\overline{a}_i\max_{1\le j\le m}\left\{\frac{\left|c_{ij}\right|}{1-\overline{\tau}_{ij}}l_j\right\}e^{\varepsilon\tau}\right]\frac{\left\|\varphi_{vj} - \overline{\varphi}_{vj}\right\|^2}{2} \\
&\quad + \frac{\left\|\phi_{ui} - \overline{\phi}_{ui}\right\|^2}{2} + \frac{\left\|\chi_{vj} - \overline{\chi}_{vj}\right\|^2}{2} \\
&\le M^*\left[\frac{\left\|\varphi_{ui} - \overline{\varphi}_{ui}\right\|^2}{2} + \frac{\left\|\varphi_{vj} - \overline{\varphi}_{vj}\right\|^2}{2}\right] \\
&\quad + \frac{\left\|\phi_{ui} - \overline{\phi}_{ui}\right\|^2}{2} + \frac{\left\|\chi_{vj} - \overline{\chi}_{vj}\right\|^2}{2} \\
&= \frac{M}{2}\left[\left\|\varphi_{ui} - \overline{\varphi}_{ui}\right\|^2 + \left\|\varphi_{vj} - \overline{\varphi}_{vj}\right\|^2\right],
\end{aligned} \tag{42}$$

where $M^* = \max\{1 + \sigma\sum_{j=1}^{m}\overline{d}_j\max_{1\le i\le n}\{(|h_{ji}|/(1-\overline{\sigma}_{ji}))k_i\}e^{\varepsilon\sigma}, 1+\tau\sum_{i=1}^{n}\overline{a}_i\max_{1\le j\le m}\{(|c_{ij}|/(1-\overline{\tau}_{ij}))l_j\}e^{\varepsilon\tau}\}$,

$$M = M^* + \frac{\left\|\phi_{ui}-\overline{\phi}_{ui}\right\|^2 + \left\|\chi_{vj}-\overline{\chi}_{vj}\right\|^2}{\left\|\varphi_{ui}-\overline{\varphi}_{ui}\right\|^2 + \left\|\varphi_{vj}-\overline{\varphi}_{vj}\right\|^2}. \tag{43}$$

Since $V(0) \ge V(t)$, from (42), we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\left[\left(u_i(t)-\overline{u}_i(t)\right)^2 + \left(x_i(t)-\overline{x}_i(t)\right)^2\right]\\&\quad+\sum_{j=1}^{m}\left[\left(v_j(t)-\overline{v}_j(t)\right)^2 + \left(y_j(t)-\overline{y}_j(t)\right)^2\right]\\&\quad< Me^{-\varepsilon t}\left[\left\|\varphi_{ui}-\overline{\varphi}_{ui}\right\|^2 + \left\|\varphi_{vj}-\overline{\varphi}_{vj}\right\|^2\right].\end{aligned} \tag{44}$$

From (44), we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\left(u_i(t)-\overline{u}_i(t)\right)^2 + \sum_{j=1}^{m}\left(v_j(t)-\overline{v}_j(t)\right)^2\\&\quad< Me^{-\varepsilon t}\left[\left\|\varphi_{ui}-\overline{\varphi}_{ui}\right\|^2 + \left\|\varphi_{vj}-\overline{\varphi}_{vj}\right\|^2\right].\end{aligned} \tag{45}$$

For $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, when $I_i(t)$, $J_j(t)$, $\tau_{ij}(t)$, and $\sigma_{ji}$ are continuously periodic functions defined on $t \in [0,\infty)$ with common period $\omega > 0$, if $u_i(t)$, $v_j(t)$ are the solutions of (3), then for any natural number $k$, $u_i(t+k\omega)$, $v_j(t+k\omega)$ are the solutions of (3). Thus, from (45), there exist constants $N > 0$ and $\delta > 0$, such that

$$\left|u_i(t+(k+1)\omega) - u_i(t+k\omega)\right| \le Ne^{-\delta(t+k\omega)}, \tag{46}$$

$$\left|v_j(t+(k+1)\omega) - v_j(t+k\omega)\right| \le Ne^{-\delta(t+k\omega)}, \tag{47}$$

for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $t > 0$.

It is noted that, for any natural number $p$,

$$\begin{aligned}u_i(t+(p+1)\omega) &= u_i(t)\\&\quad+\sum_{k=0}^{p}\left(u_i(t+(k+1)\omega) - u_i(t+k\omega)\right).\end{aligned} \tag{48}$$

Thus

$$\begin{aligned}\left|u_i(t+(p+1)\omega)\right| &\le \left|u_i(t)\right|\\&\quad+\sum_{k=0}^{p}\left|u_i(t+(k+1)\omega) - u_i(t+k\omega)\right|.\end{aligned} \tag{49}$$

Since $u_i(t)$ is bounded, it follows from (46) and (49) that $\{u(t+p\omega)\}$ uniformly converges to a continuous functions $u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_n^*(t))$ on any compact set of $R$.

Similarly, since $v_j(t)$ is bounded, from (47), $\{v(t+p\omega)\}$ uniformly converges to a continuous function $v^*(t) = (v_1^*(t), v_2^*(t), \ldots, v_n^*(t))$ on any compact set of $R$.

When $u_i(t)$, $u_i'(t)$, $v_j(t)$, and $v_j'(t)$ are bounded, $x_i(t) = u_i(t) + u_i'(t)$, $y_j(t) = v_j(t) + v_j'(t)$, and we can obtain that $x_i(t)$, $y_j(t)$ are bounded. Similarly, from (44), they can be proved that $\{x(t+p\omega)\}$, $\{y(t+p\omega)\}$ uniformly converge to continuous functions $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ and $y^*(t) = (y_1^*(t), y_2^*(t), \ldots, y_m^*(t))$ on any compact set of $R$, respectively.

Now we will show that $(u^{*T}(t), v^{*T}(t))^T$ is the $\omega$-periodic solution of system (3).

First, $u^*(t)$, $v^*(t)$ are $\omega$-periodic functions, since

$$\begin{aligned}u^*(t+\omega) &= \lim_{p\to\infty} u(t+(p+1)\omega) = u^*(t),\\v^*(t+\omega) &= \lim_{p\to\infty} v(t+(p+1)\omega) = v^*(t).\end{aligned} \tag{50}$$

Second, we prove that $(u^{*T}(t), v^{*T}(t))^T$ is a solution of system (3).

In fact, $I_i(t+p\omega) = I_i(t)$, $J_j(t+p\omega) = J_j(t)$, $\tau_{ij}(t+p\omega) = \tau_{ij}(t)$, $\sigma_{ji}(t+p\omega) = \sigma_{ji}(t)$, and

$$\begin{aligned}\frac{du_i(t+p\omega)}{dt} &= -u_i(t+p\omega) + x_i(t+p\omega),\\\frac{dx_i(t+p\omega)}{dt}&\\&= -(1-\alpha_i)u_i(t+p\omega) - (\alpha_i-1)x_i(t+p\omega)\\&\quad- a_i(u_i(t+p\omega))\\&\quad\times\Bigg[b_i(u_i(t+p\omega))\\&\qquad-\sum_{j=1}^{m}c_{ij}f_j\left(v_j\left(t+p\omega-\tau_{ij}(t)\right)\right) - I_i(t)\Bigg],\\\frac{dv_j(t+p\omega)}{dt} &= -v_j(t+p\omega) + y_j(t+p\omega),\\\frac{dy_j(t+p\omega)}{dt}&\\&= -\left(1-\beta_j\right)v_j(t+p\omega) - \left(\beta_j-1\right)y_j(t+p\omega)\\&\quad- d_j\left(v_j(t+p\omega)\right)\\&\quad\times\Bigg[e_j\left(v_j(t+p\omega)\right)\\&\qquad-\sum_{i=1}^{n}h_{ji}g_i\left(u_i\left(t+p\omega-\sigma_{ji}(t)\right)\right) - J_j(t)\Bigg],\end{aligned} \tag{51}$$

for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

Since $\{u(t+p\omega)\}$ and $\{v(t+p\omega)\}$ uniformly converge to continuous function

$$u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_n^*(t)),$$
$$v^*(t) = (v_1^*(t), v_2^*(t), \ldots, v_m^*(t)), \tag{52}$$

respectively, $\{x(t+p\omega)\}$ and $\{y(t+p\omega)\}$ uniformly converge to a continuous function

$$x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)),$$
$$y^*(t) = (y_1^*(t), y_2^*(t), \ldots, y_m^*(t)), \tag{53}$$

respectively. (51) implies that

$$\left\{\frac{du_i(t+p\omega)}{dt}\right\}, \quad \left\{\frac{dx_i(t+p\omega)}{dt}\right\},$$
$$\left\{\frac{dv_j(t+p\omega)}{dt}\right\}, \quad \left\{\frac{dy_j(t+p\omega)}{dt}\right\} \tag{54}$$

uniformly converge to continuous functions on any compact set of $R$, respectively. Thus, let $p \to \infty$; we obtain

$$\begin{aligned}
\frac{du_i^*(t)}{dt} &= -u_i^*(t) + x_i^*(t),\\
\frac{dx_i^*(t)}{dt} &= -(1-\alpha_i)u_i^*(t) - (\alpha_i - 1)x_i^*(t)\\
&\quad - a_i(u_i^*(t))\\
&\quad \times \left[b_i(u_i^*(t)) - \sum_{j=1}^{m} c_{ij} f_j\left(v_j^*\left(t-\tau_{ij}(t)\right)\right) - I_i(t)\right],\\
\frac{dv_j^*(t)}{dt} &= -v_j^*(t) + y_j^*(t),\\
\frac{dy_j^*(t)}{dt} &= -(1-\beta_j)v_j^*(t) - (\beta_j - 1)y_j^*(t)\\
&\quad - d_j(v_j^*(t))\\
&\quad \times \left[e_j(v_j^*(t)) - \sum_{i=1}^{n} h_{ji} g_i\left(u_i^*\left(t-\sigma_{ji}(t)\right)\right) - J_j(t)\right],
\end{aligned} \tag{55}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Thus, $(u^{*T}(t), v^{*T}(t))$ is a periodic solution of system (3). From (45), we obtain that system (3) has one $\omega$-periodic solution, which is exponentially stable. □

**Theorem 4.** *Under the hypotheses $(H_1)$–$(H_5)$, there is $\omega$-periodic solution of system (3), which is exponentially stable, if the following conditions hold:*

$$\begin{aligned}
&\alpha_i - 1 - T_i > 0, \qquad \beta_j - 1 - T_j^* > 0,\\
&-T_i + A_i \sum_{j=1}^{m} \overline{f}_j |c_{ij}| + A_i \overline{I}_i + \overline{a}_i \sum_{j=1}^{m} l_j |c_{ij}| < 0,\\
&-T_j^* + D_j \sum_{i=1}^{n} \overline{g}_i |h_{ji}| + D_j \overline{J}_j + \overline{d}_j \sum_{i=1}^{n} k_i |h_{ji}| < 0,
\end{aligned} \tag{56}$$

*for $i = 1, 2 \ldots, n$, $j = 1, 2 \ldots, m$.*

*Proof.* Let

$$\overline{u}(t) = (\overline{u}_1(t), \overline{u}_2(t), \ldots, \overline{u}_n(t))^T,$$
$$\overline{v}(t) = (\overline{v}_1(t), \overline{v}_2(t), \ldots, \overline{v}_m(t))^T \tag{57}$$

be solution of system (3) with initial value $(IV1)$, and let

$$u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T,$$
$$v(t) = (v_1(t), v_2(t), \ldots, v_m(t))^T \tag{58}$$

be solution of system (3) with any initial value $(IV2)$.

From (33), we can obtain

$$\begin{aligned}
\frac{d|z_i(t)|}{dt} &= \operatorname{sgn}(z_i(t))(-z_i(t) + w_i(t))\\
&\le -|z_i(t)| + |w_i(t)|, \quad i = 1, 2 \ldots, n,\\
\frac{d|w_i(t)|}{dt} &= \operatorname{sgn}(w_i(t)) \Bigg\{ -(1-\alpha_i) z_i(t) - (\alpha_i - 1) w_i(t)\\
&\quad + a_i(u_i(t))\\
&\quad \times \left[\sum_{j=1}^{m} c_{ij} \left[f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right) - f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)\right]\right]
\end{aligned} \tag{59}$$

$$
\begin{aligned}
&+\left(a_i\left(u_i(t)\right)-a_i\left(\overline{u}_i(t)\right)\right)\\
&\times\left[\sum_{j=1}^{m} c_{ij} f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)+I_i(t)\right]\\
&-\left[a_i\left(u_i(t)\right) b_i\left(u_i(t)\right)\right.\\
&\left.-a_i\left(\overline{u}_i(t)\right) b_i\left(\overline{u}_i(t)\right)\right]\Bigg\}\\
=\operatorname{sgn}&\left(w_i(t)\right)\Bigg\{-\left(1-\alpha_i\right) z_i(t)-\left(\alpha_i-1\right) w_i(t)\\
&+a_i\left(u_i(t)\right)\\
&\times\left[\sum_{j=1}^{m} c_{ij}\left[f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right)\right.\right.\\
&\left.\left.-f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)\right]\right]\\
&+a_i'\left(\xi_i\right) z_i(t)\\
&\times\left[\sum_{j=1}^{m} c_{ij} f_j\left(\overline{v}_j\left(t-\tau_{ij}(t)\right)\right)+I_i(t)\right]\\
&-B_i'\left(\overline{\xi}_i\right) z_i(t)\Bigg\}\\
\le&\left(\alpha_i-1-B_i'\left(\overline{\xi}_i\right)\right)\left|z_i(t)\right|-\left(\alpha_i-1\right)\left|w_i(t)\right|\\
&+\overline{a}_i\sum_{j=1}^{m}\left|c_{ij}\right| l_j\left|p_j\left(t-\tau_{ij}(t)\right)\right|\\
&+A_i\left[\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j+\overline{I}_i\right]\left|z_i(t)\right|\\
\le&\left[\alpha_i-1-T_i+A_i\left(\sum_{j=1}^{m}\left|c_{ij}\right|\overline{f}_j+\overline{I}_i\right)\right]\left|z_i(t)\right|\\
&-\left(\alpha_i-1\right)\left|w_i(t)\right|+\overline{a}_i\sum_{j=1}^{m}\left|c_{ij}\right| l_j\left|p_j\left(t-\tau_{ij}(t)\right)\right|,
\end{aligned}
\tag{60}
$$

for $i=1,2,\ldots,n$.

From (34), we can obtain

$$
\begin{aligned}
\frac{d\left|p_j(t)\right|}{dt}&=\operatorname{sgn}\left(p_j(t)\right)\left(-p_j(t)+q_j(t)\right)\\
&\le-\left|p_j(t)\right|+\left|q_j(t)\right|,\quad j=1,2,\ldots,m,
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&\frac{d\left|q_j(t)\right|}{dt}\\
&=\operatorname{sgn}\left(q_j(t)\right)\Bigg\{-\left(1-\beta_j\right) p_j(t)-\left(\beta_j-1\right) q_j(t)\\
&\quad+d_j\left(v_j(t)\right)\left[\sum_{i=1}^{n} h_{ji}\left[g_i\left(u_i\left(t-\sigma_{ji}(t)\right)\right)\right.\right.\\
&\quad\left.\left.-g_i\left(\overline{u}_j\left(t-\sigma_{ji}(t)\right)\right)\right]\right]\\
&\quad+\left(d_j\left(v_j(t)\right)-d_j\left(\overline{v}_j(t)\right)\right)\\
&\quad\cdot\left[\sum_{i=1}^{n} h_{ji} g_i\left(\overline{u}_i\left(t-\sigma_{ji}(t)\right)\right)+J_j(t)\right]\\
&\quad-\left[d_j\left(v_j(t)\right) e_j\left(v_j(t)\right)\right.\\
&\quad\left.-d_j\left(\overline{v}_j(t)\right) e_j\left(\overline{v}_j(t)\right)\right]\Bigg\}\\
&=\operatorname{sgn}\left(q_j(t)\right)\Bigg\{-\left(1-\beta_j\right) p_j(t)-\left(\beta_j-1\right) q_j(t)\\
&\quad+d_j\left(v_j(t)\right)\left[\sum_{i=1}^{n} h_{ji}\left[g_i\left(u_i\left(t-\sigma_{ji}(t)\right)\right)\right.\right.\\
&\quad\left.\left.-g_i\left(\overline{u}_j\left(t-\sigma_{ji}(t)\right)\right)\right]\right]\\
&\quad+d_j'\left(\xi_j^*\right) p_j(t)\left[\sum_{i=1}^{n} h_{ji} g_i\left(\overline{u}_i\left(t-\sigma_{ji}(t)\right)\right)\right.\\
&\quad\left.+J_j(t)\right]\\
&\quad-E_j'\left(\overline{\xi}_j^*\right) p_j(t)\Bigg\}\\
&\le\left(\beta_j-1-E_j'\left(\overline{\xi}_j^*\right)\right)\left|p_j(t)\right|-\left(\beta_j-1\right)\left|q_j(t)\right|\\
&\quad+\overline{d}_j\sum_{i=1}^{n}\left|h_{ji}\right| k_i\left|z_i\left(t-\sigma_{ji}(t)\right)\right|+D_j\left|p_j(t)\right|\\
&\quad\times\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i+\overline{J}_j\right)\\
&\le\left[\beta_j-1-T_j^*+D_j\left(\sum_{i=1}^{n}\left|h_{ji}\right|\overline{g}_i+\overline{J}_j\right)\right]\left|p_j(t)\right|\\
&\quad-\left(\beta_j-1\right)\left|q_j(t)\right|+\overline{d}_j\sum_{i=1}^{n}\left|h_{ji}\right| k_i\left|z_i\left(t-\sigma_{ji}(t)\right)\right|,
\end{aligned}
\tag{62}
$$

for $j=1,2,\ldots,m$.

From (60) and (62), we can obtain

$$\begin{aligned}|w_i(t)| \le{}& e^{(1-\alpha_i)t}|w_i(0)| \\ &+\left[\alpha_i - 1 - T_i + A_i\left(\sum_{j=1}^{m}|c_{ij}|\overline{f}_j + \overline{I}_i\right)\right] \\ &\times\int_0^t e^{(\alpha_i-1)(s-t)}|z_i(s)|\,ds \\ &+\overline{a}_i\sum_{j=1}^{m}|c_{ij}|l_j\int_0^t e^{(\alpha_i-1)(s-t)}\left|p_j\left(s-\tau_{ij}(t)\right)\right|ds,\end{aligned} \tag{63}$$

for $i = 1, 2, \ldots, n$.

$$\begin{aligned}|q_j(t)| \le{}& e^{(1-\beta_j)t}|q_j(0)| \\ &+\left[\beta_j - 1 - T_j^* + D_j\left(\sum_{i=1}^{n}|h_{ji}|\overline{g}_i + \overline{J}_j\right)\right] \\ &\times\int_0^t e^{(\beta_j-1)(s-t)}|p_j(s)|\,ds \\ &+\overline{d}_j\sum_{i=1}^{n}|h_{ji}|k_i\int_0^t e^{(\beta_j-1)(s-t)}\left|z_i\left(s-\sigma_{ji}(t)\right)\right|ds,\end{aligned} \tag{64}$$

for $j = 1, 2, \ldots, m$.

We consider the functions $g_i(\xi)$, $h_j(\eta)$ given by

$$\begin{aligned}g_i(\xi) &= \xi - T_i + A_i\sum_{j=1}^{m}\overline{f}_j|c_{ij}| + A_i\overline{I}_i + \overline{a}_i\sum_{j=1}^{m}l_j|c_{ij}|e^{\xi\tau}, \\ h_j(\eta) &= \eta - T_j^* + D_j\sum_{i=1}^{n}\overline{g}_i|h_{ji}| + D_j\overline{J}_j + \overline{d}_j\sum_{i=1}^{n}k_i|h_{ji}|e^{\eta\sigma},\end{aligned} \tag{65}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Obviously

$$\begin{aligned}&\frac{dg_i(\xi)}{d\xi} > 0, \quad \lim_{\xi\to+\infty} g_i(\xi) = +\infty, \quad g_i(0) < 0, \\ &\frac{dh_j(\eta)}{d\eta} > 0, \quad \lim_{\eta\to+\infty} h_j(\eta) = +\infty, \quad h_j(0) < 0,\end{aligned} \tag{66}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Therefore, there exist constants $\xi_i, \eta_j \in (0, +\infty)$, such that

$$\begin{aligned}g_i(\xi_i) &= 0, \quad i = 1, 2, \ldots, n, \\ h_j(\eta_j) &= 0, \quad j = 1, 2, \ldots, m.\end{aligned} \tag{67}$$

We choose $\overline{\xi} = \min\{\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_m\}$, then $\overline{\xi} > 0$; when $0 < \delta < \overline{\xi}$, we have

$$\begin{aligned}&\delta - T_i + A_i\sum_{j=1}^{m}\overline{f}_j|c_{ij}| + A_i\overline{I}_i + \overline{a}_i\sum_{j=1}^{m}l_j|c_{ij}|e^{\delta\tau} < 0, \\ &\delta - T_j^* + D_j\sum_{i=1}^{n}\overline{g}_i|h_{ji}| + D_j\overline{J}_j + \overline{d}_j\sum_{i=1}^{n}k_i|h_{ji}|e^{\delta\sigma} < 0,\end{aligned} \tag{68}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Since the initial values $\varphi_{ui}(s)$, $\psi_{ui}(s)$, $\varphi_{vj}(s)$, and $\psi_{vj}(s)$ are bounded and continuous functions, then there exist $N_1$, $N_2$, $N_3$, and $N_4 > 0$, such that $|\varphi_{ui}(t)| \le N_1$, $|\psi_{ui}(t)| \le N_2$, $|\varphi_{vj}(t)| \le N_3$, $|\psi_{vj}(4)| \le N_4$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $t \in [-\gamma, 0]$, $\gamma = \max\{\sigma, \tau\}$.

Let $L = \max\{N_1 + N_2, N_3 + N_4\}$; we will show that, for any sufficiently small constant $\varepsilon > 0$ and $0 < \delta < \overline{\xi}$,

$$\begin{aligned}&|z_i(t)| < (L+\varepsilon)e^{-\delta t}, \qquad |w_i(t)| < (L+\varepsilon)e^{-\delta t}, \\ &|p_j(t)| < (L+\varepsilon)e^{-\delta t}, \qquad |q_j(t)| < (L+\varepsilon)e^{-\delta t},\end{aligned} \tag{69}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $t \ge 0$.

Considering proof of contradiction, if (69) does not hold, there are 15 possible situations; here we only discuss the following four cases; that is,

$$\begin{aligned}&|z_k(t_1)| = (L+\varepsilon)e^{-\delta t_1}, \qquad \frac{d^+|z_k(t_1)|}{dt} \ge 0, \\ &|z_i(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1), \\ &|w_i(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1], \\ &|p_j(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1], \\ &|q_j(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1],\end{aligned} \tag{70}$$

$$\begin{aligned}&|w_k(t_1)| = (L+\varepsilon)e^{-\delta t_1}, \\ &|z_i(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1], \\ &|w_i(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1), \\ &|p_j(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1], \\ &|q_j(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1],\end{aligned} \tag{71}$$

$$\begin{aligned}&|p_l(t_1)| = (L+\varepsilon)e^{-\delta t_1}, \qquad \frac{d^+|p_l(t_1)|}{dt} \ge 0, \\ &|z_i(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1], \\ &|w_i(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1], \\ &|p_j(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1), \\ &|q_j(t)| < (L+\varepsilon)e^{-\delta t}, \quad t \in [0, t_1],\end{aligned} \tag{72}$$

or

$$\begin{aligned} |q_i(t_1)| &= (L+\varepsilon)e^{-\delta t_1}, \\ |z_i(t)| &< (L+\varepsilon)e^{-\delta t}, \quad t\in[0,t_1], \\ |w_i(t)| &< (L+\varepsilon)e^{-\delta t}, \quad t\in[0,t_1], \\ |p_j(t)| &< (L+\varepsilon)e^{-\delta t}, \quad t\in[0,t_1], \\ |q_j(t)| &< (L+\varepsilon)e^{-\delta t}, \quad t\in[0,t_1), \end{aligned} \tag{73}$$

where $k\in\{1,2,\ldots,n\}$, $l\in\{1,2,\ldots,m\}$, and $t_1\geq 0$, $i=1,2,\ldots,n$, and $j=1,2,\ldots,m$.

Therefore, by (59) and (70), we have

$$\begin{aligned} \frac{d^+|z_k(t_1)|}{dt} &\leq -|z_k(t_1)|+|w_k(t_1)| \\ &< -(L+\varepsilon)e^{-\delta t_1}+(L+\varepsilon)e^{-\delta t_1}=0, \end{aligned} \tag{74}$$

which is a contradiction with $d^+|z_k(t_1)|/dt\geq 0$.

By (63) and (71), we obtain

$$\begin{aligned} |w_k(t_1)| &= (L+\varepsilon)e^{-\delta t_1} \\ &\leq e^{(1-\alpha_k)t_1}|w_k(0)|+\left[\alpha_k-1-T_k+A_k\sum_{j=1}^m \overline{f}_j|c_{kj}|+A_k\overline{I}_k\right] \\ &\quad\times\int_0^{t_1}e^{(\alpha_k-1)(s-t_1)}|z_k(s)|\,ds+\overline{a}_k\sum_{j=1}^m l_j|c_{kj}| \\ &\quad\times\int_0^{t_1}e^{(\alpha_k-1)(s-t_1)}|p_j(s-\tau_{kj}(s))|\,ds \\ &\leq (L+\varepsilon)\left\{e^{(1-\alpha_k)t_1}+\left[\alpha_k-1-T_k+A_k\sum_{j=1}^m\overline{f}_j|c_{kj}|+A_k\overline{I}_k\right]\right. \\ &\qquad\times\int_0^{t_1}e^{(\alpha_k-1)(s-t_1)-\delta s}ds+\overline{a}_k\sum_{j=1}^m l_j|c_{kj}| \\ &\qquad\left.\times\int_0^{t_1}e^{(\alpha_k-1)(s-t_1)-\delta s+\delta\tau_{kj}(s)}ds\right\} \\ &\leq (L+\varepsilon)\left\{e^{(1-\alpha_k)t_1}+\left[\alpha_k-1-T_k+A_k\sum_{j=1}^m\overline{f}_j|c_{kj}|\right.\right. \\ &\qquad\left.+A_k\overline{I}_k+\overline{a}_k\sum_{j=1}^m l_j|c_{kj}|e^{\tau\delta}\right] \\ &\qquad\left.\times\frac{e^{-\delta t_1}-e^{(1-\alpha_k)t_1}}{\alpha_k-1-\delta}\right\}. \end{aligned} \tag{75}$$

Since

$$\begin{aligned} \alpha_k-1-\delta &> \alpha_k-1-T_k+A_k\sum_{j=1}^m\overline{f}_j|c_{kj}| \\ &\quad+A_k\overline{I}_k+\overline{a}_k\sum_{j=1}^m l_j|c_{kj}|e^{\tau\delta}>0, \end{aligned} \tag{76}$$

we have

$$\begin{aligned} &\frac{\left[\alpha_k-1-T_k+A_k\sum_{j=1}^m\overline{f}_j|c_{kj}|+A_k\overline{I}_k+\overline{a}_k\sum_{j=1}^m l_j|c_{kj}|e^{\tau\delta}\right]}{(\alpha_k-1-\delta)} \\ &\quad<1. \end{aligned} \tag{77}$$

From (75), we have

$$L+\varepsilon<L+\varepsilon, \tag{78}$$

which is a contradiction.

Therefore, by (61) and (75), we have

$$\begin{aligned} \frac{d^+|p_l(t_1)|}{dt} &\leq -|p_l(t_1)|+|q_l(t_1)| \\ &< -(L+\varepsilon)e^{-\delta t_1}+(L+\varepsilon)e^{-\delta t_1}=0, \end{aligned} \tag{79}$$

which is a contradiction with $d^+|p_l(t_1)|/dt\geq 0$.

By (70) and (80), we obtain

$$\begin{aligned} &|q_l(t_1)| \\ &= (L+\varepsilon)e^{-\delta t_1}\leq e^{(1-\beta_l)t_1}|q_l(0)| \\ &\quad+\left[\beta_l-1-T_l^*+D_l\sum_{i=1}^n\overline{g}_i|h_{li}|+D_l\overline{J}_l\right] \\ &\quad\times\int_0^{t_1}e^{(\beta_l-1)(s-t_1)}|p_l(s)|\,ds+\overline{d}_l\sum_{i=1}^n k_i|h_{li}| \\ &\quad\times\int_0^{t_1}e^{(\beta_l-1)(s-t_1)}|z_i(s-\sigma_{li}(s))|\,ds \\ &\leq (L+\varepsilon)\left\{e^{(1-\beta_l)t_1}+\left[\beta_l-1-T_l^*+D_l\sum_{i=1}^n\overline{g}_i|h_{li}|+D_l\overline{J}_l\right]\right. \\ &\qquad\times\int_0^{t_1}e^{(\beta_l-1)(s-t_1)-\delta s}ds+\overline{d}_l\sum_{i=1}^n k_i|h_{li}| \\ &\qquad\left.\times\int_0^{t_1}e^{(\beta_l-1)(s-t_1)-\delta s+\delta\sigma_{li}(s)}ds\right\} \end{aligned}$$

$$\leq (L+\varepsilon)\left\{e^{(1-\beta_l)t_1}+\left[\beta_l-1-T_l^*+D_l\sum_{i=1}^{n}\overline{g}_i\left|h_{li}\right|+D_l\overline{J}_l+\overline{d}_l\sum_{i=1}^{n}k_i\left|h_{li}\right|e^{\sigma\delta}\right]\times\frac{e^{-\delta t_1}-e^{(1-\beta_l)t_1}}{\beta_l-1-\delta}\right\}. \tag{80}$$

Since

$$\beta_l-1-\delta>\beta_l-1-T_l^*+D_l\sum_{i=1}^{n}\overline{g}_i\left|h_{li}\right|+D_l\overline{J}_l+\overline{d}_l\sum_{i=1}^{n}k_i\left|h_{li}\right|e^{\sigma\delta}>0, \tag{81}$$

we have

$$\frac{\left[\beta_l-1-T_l^*+D_l\sum_{i=1}^{n}\overline{g}_i\left|h_{li}\right|+D_l\overline{J}_l+\overline{d}_l\sum_{i=1}^{n}k_i\left|h_{li}\right|e^{\sigma\delta}\right]}{(\beta_l-1-\delta)}<1. \tag{82}$$

From (80), we have

$$L+\varepsilon<L+\varepsilon, \tag{83}$$

which is a contradiction.

Similarly, we can prove that the other 11 possible situations; thus (69) holds; let $\varepsilon\to 0$; we have

$$\begin{aligned}\left|z_i(t)\right|&\leq Le^{-\delta t}, & \left|w_i(t)\right|&\leq Le^{-\delta t},\\ \left|p_j(t)\right|&\leq Le^{-\delta t}, & \left|q_j(t)\right|&\leq Le^{-\delta t},\end{aligned} \tag{84}$$

for $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, and $t\geq 0$.

From (84), there exist constants $\delta>0$ and $M>0$ such that

$$\begin{aligned}&\sum_{i=1}^{n}\left(u_i(t)-\overline{u}_i(t)\right)^2+\sum_{j=1}^{m}\left(v_j(t)-\overline{v}_j(t)\right)^2\\ &<Me^{-\delta t}\left[\left\|\varphi_{ui}-\overline{\varphi}_{ui}\right\|^2+\left\|\varphi_{vj}-\overline{\varphi}_{vj}\right\|^2\right],\quad t>0.\end{aligned} \tag{85}$$

Using (85), similarly with the proof of Theorem 3, we know that system (3) has one $\omega$-periodic solution, which is exponentially stable. □

## 4. Numerical Examples

In this section, we give two examples for showing our results.

*Example 1.* We consider the following inertial Cohen-Grossberg-type BAM neural networks with time delays ($n=m=3$):

$$\begin{aligned}\frac{d^2u_i(t)}{dt^2}&=-\alpha_i\frac{du_i(t)}{dt}-a_i\left(u_i(t)\right)\times\left[b_i\left(u_i(t)\right)-\sum_{j=1}^{3}c_{ij}f_j\left(v_j\left(t-\tau_{ij}(t)\right)\right)-I_i(t)\right],\\ \frac{d^2v_j(t)}{dt^2}&=-\beta_j\frac{dv_j(t)}{dt}-d_j\left(v_j(t)\right)\times\left[e_j\left(v_j(t)\right)-\sum_{i=1}^{3}h_{ji}g_i\left(u_i\left(t-\sigma_{ji}(t)\right)\right)-J_j(t)\right],\end{aligned} \tag{86}$$

for $i,j=1,2,3$, where

$$\alpha_1=1.7,\quad \alpha_2=1.9,\quad \alpha_3=1.8,\quad \beta_1=2.3,$$
$$\beta_2=2,\quad \beta_3=2.2,\quad c_{11}=\frac{1}{16},\quad c_{12}=\frac{1}{16},$$
$$c_{13}=\frac{1}{24},\quad c_{21}=\frac{1}{8},\quad c_{22}=\frac{1}{8},\quad c_{23}=\frac{1}{32},$$
$$c_{31}=\frac{1}{32},\quad c_{32}=\frac{1}{32},\quad c_{33}=\frac{1}{64},\quad h_{11}=\frac{1}{32},$$
$$h_{12}=\frac{1}{32},\quad h_{13}=\frac{1}{64},\quad h_{21}=\frac{1}{32},\quad h_{22}=\frac{1}{32},$$
$$h_{23}=\frac{1}{32},\quad h_{31}=\frac{1}{64},\quad h_{32}=\frac{1}{64},\quad h_{33}=\frac{1}{32},$$
$$a_1(u_1)=2+\frac{1}{1+u_1^2},\quad a_2(u_2)=2-\frac{1}{1+u_2^2},$$
$$a_3(u_3)=1.6+\frac{2}{1+u_3^2},$$
$$b_1(u_1)=\frac{8}{15}u_1,\quad b_2(u_2)=\frac{1}{2}u_2,\quad b_3(u_3)=\frac{10}{20}u_3,$$
$$d_1(v_1)=1+\frac{1}{1+v_1^2},\quad d_2(v_2)=2-\frac{1}{1+v_2^2},$$
$$d_3(v_3)=2+\frac{1}{1+v_3^2},$$
$$e_1(v_1)=\frac{4}{7}v_1,\quad e_2(v_2)=\frac{1}{3}v_2,\quad e_3(v_3)=\frac{4}{15}v_3,$$
$$f_j(v_j)=\frac{e^{v_j}-e^{-v_j}}{8\left(e^{v_j}+e^{-v_j}\right)},\quad g_i(u_i)=\frac{e^{u_i}}{8\left(e^{u_i}+e^{-u_i}\right)},$$

$$\tau_{ij} = \frac{1+\sin t}{6}, \qquad \sigma_{ji} = \frac{1+\cos t}{6},$$
$$I_i(t) = \frac{1}{32}(2+\sin t), \quad J_j(t) = \frac{1}{32}(2+\cos t),$$
$$i, j = 1, 2, 3. \tag{87}$$

Obviously,

$$\begin{gathered}
2 \le a_1(u_1) \le 3, \qquad |a_1'(u_1)| \le 1, \qquad 1 \le a_2(u_2) \le 2, \\
|a_2'(u_2)| \le 1, \qquad 1.6 \le a_3(u_3) \le 3.6, \qquad |a_3'(u_3)| \le 2, \\
1 \le d_1(v_1) \le 2, \qquad |d_1'(v_1)| \le 1, \qquad 1 \le d_2(v_2) \le 2, \\
|d_2'(v_2)| \le 1, \qquad 2 \le d_3(v_3) \le 3, \qquad |d_3'(v_3)| \le 1, \\
b_1'(u_1) = \frac{8}{15}, \qquad b_2'(u_2) = \frac{1}{2}, \qquad b_3'(u_3) = \frac{10}{20}, \\
e_1'(v_1) = \frac{4}{7}, \qquad e_2'(v_2) = \frac{1}{3}, \qquad e_3'(v_3) = \frac{4}{15}, \\
|f_j(x) - f_j(y)| \le \frac{|x-y|}{8}, \qquad |f_j(x)| \le \frac{1}{8}, \\
j = 1, 2, 3, \quad x, y \in R, \\
|g_i(x) - g_i(y)| \le \frac{|x-y|}{16}, \qquad |g_i(x)| \le \frac{1}{8}, \\
i = 1, 2, 3, \quad x, y \in R, \\
0 \le \tau_{ij} \le \frac{1}{3}, \quad 0 \le \sigma_{ji} = \frac{1}{3}, \quad i = j = 1, 2, 3, \\
\tau_{ij}' \le \frac{1}{6}, \quad \sigma_{ji}' = \frac{1}{6}, \quad i = j = 1, 2, 3, \\
\frac{1}{32} < I_i(t) < \frac{3}{32}, \quad \frac{1}{32} < J_j(t) < \frac{3}{32}, \quad i, j = 1, 2, 3, \\
B_1(u_1) = a_1(u_1) b_1(u_1) = \frac{8}{15}\left(2u_1 + \frac{u_1}{1+u_1^2}\right), \\
1 \le B_1'(u_1) = \frac{8}{15}\left(2 + \frac{1-u_1^2}{(1+u_1^2)^2}\right) \le \frac{8}{5}, \\
B_2(u_2) = a_2(u_2) b_2(u_2) = \frac{1}{2}\left(2u_2 - \frac{u_2}{1+u_2^2}\right), \\
\frac{1}{2} \le B_2'(u_2) = \frac{1}{2}\left(2 + \frac{u_2^2-1}{(1+u_2^2)^2}\right) \le \frac{17}{16}, \\
B_3(u_3) = a_3(u_3) b_3(u_3) = \frac{10}{27}\left(1.6u_3 + \frac{2u_3}{1+u_3^2}\right), \\
\frac{1}{2} \le B_3'(u_3) = \frac{10}{27}\left(1.6 + 2\frac{1-u_3^2}{(1+u_3^2)^2}\right) \le \frac{4}{3}, \\
E_1(v_1) = d_1(v_1) e_1(v_1) = \frac{4}{7}\left(v_1 + \frac{v_1}{1+v_1^2}\right), \\
\frac{1}{2} \le E_1'(v_1) = \frac{4}{7}\left(1 + \frac{1-v_1^2}{(1+v_1^2)^2}\right) \le \frac{8}{7}, \\
E_2(v_2) = d_2(v_2) e_2(v_2) = \frac{1}{3}\left(2v_2 - \frac{v_2}{1+v_2^2}\right), \\
\frac{1}{3} \le E_2'(v_2) = \frac{1}{3}\left(2 + \frac{v_2^2-1}{(1+v_2^2)^2}\right) \le \frac{17}{24}, \\
E_3(v_3) = d_3(v_3) e_3(v_3) = \frac{4}{15}\left(2v_3 + \frac{v_3}{1+v_3^2}\right), \\
\frac{1}{2} \le E_3'(v_3) = \frac{4}{15}\left(2 + \frac{1-v_3^2}{(1+v_3^2)^2}\right) \le \frac{4}{5}.
\end{gathered} \tag{88}$$

By assumptions $(H_1)$–$(H_5)$, we select

$$\begin{gathered}
\underline{a}_1 = 2, \qquad \overline{a}_1 = 3, \qquad \underline{a}_2 = 1, \qquad \overline{a}_2 = 2, \qquad \underline{a}_3 = 1.6, \\
\overline{a}_3 = 3.6, \qquad A_1 = 1, \qquad A_2 = 1, \qquad A_3 = 2, \\
\underline{d}_1 = 1, \qquad \overline{d}_1 = 2, \qquad \underline{d}_2 = 1, \qquad \overline{d}_2 = 2, \\
\underline{d}_3 = 2, \qquad \overline{d}_3 = 3, \qquad D_1 = 1, \qquad D_2 = 1, \qquad D_3 = 1, \\
\underline{b}_1 = \overline{b}_1 = \frac{8}{15}, \qquad \underline{b}_2 = \overline{b}_2 = \frac{1}{2}, \qquad \underline{b}_3 = \overline{b}_3 = \frac{10}{27}, \\
\underline{e}_1 = \overline{e}_1 = \frac{4}{7}, \qquad \underline{e}_2 = \overline{e}_2 = \frac{1}{3}, \qquad \underline{e}_3 = \overline{e}_3 = \frac{4}{15}, \\
l_j = \overline{f}_j = \frac{1}{8}, \qquad k_i = \frac{1}{16}, \qquad \overline{g}_i = \frac{1}{8}, \qquad \underline{I}_i = \frac{1}{32}, \\
\overline{I}_i = \frac{3}{32}, \quad \underline{J}_j = \frac{1}{32}, \quad \overline{J}_j = \frac{3}{32}, \quad i, j = 1, 2, 3, \\
T_1 = 1, \qquad k_1 = \frac{8}{5}, \qquad T_2 = \frac{1}{2}, \qquad k_2 = \frac{17}{16}, \\
T_3 = \frac{1}{2}, \qquad k_3 = \frac{4}{3}, \qquad T_1^* = \frac{1}{2}, \qquad k_1^* = \frac{8}{7}, \\
T_2^* = \frac{1}{3}, \qquad k_2^* = \frac{17}{24}, \qquad T_3^* = \frac{1}{2}, \qquad k_3^* = \frac{4}{5}, \\
\tau = \frac{1}{3}, \qquad \sigma = \frac{1}{3}, \qquad \overline{\tau}_{ij} = \frac{1}{6}, \qquad \overline{\sigma}_{ji} = \frac{1}{3}.
\end{gathered} \tag{89}$$

Thus, hypotheses $(H_1)$–$(H_5)$ hold.

For numerical simulation, the following ten any intial values are given:

$$[\varphi_{u1}(0), \varphi_{u2}(0), \varphi_{u3}(0), \psi_{u1}(0), \psi_{u2}(0), \psi_{u3}(0),$$
$$\varphi_{v1}(0), \varphi_{v2}(0), \varphi_{v3}(0), \psi_{v1}(0), \psi_{v2}(0), \psi_{v3}(0)]$$

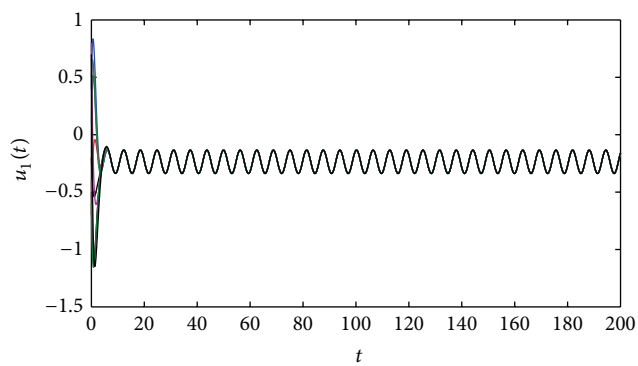


FIGURE 1: Transient response of state variables $u_1(t)$ of Example 1.

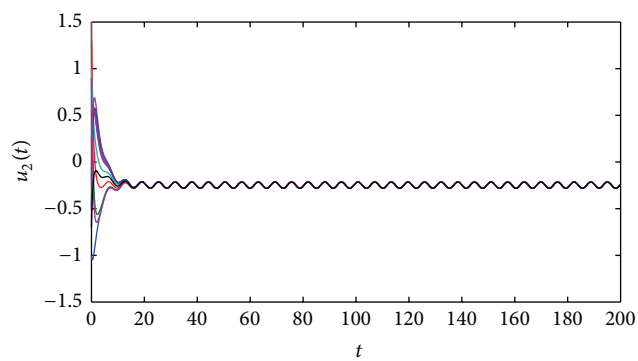


FIGURE 2: Transient response of state variables $u_2(t)$ of Example 1.

$$
\begin{aligned}
&= [0.4; -1; 1.5; 2; -1.3; 2; 0.5; 1.8; 3; 2; -3; 2]\,; \\
&[0.1; 1.3; 0.5; 1.5; -2; 1.5; 1; 1.3; 1; 1.5; -2; 1.5]\,; \\
&[-0.2; 1.5; -1; -1; -1; -1; -1.5; 1.5; -1; 1.5; -2; 1.5]\,; \\
&[-0.7; 1; 1.8; 0.5; 0; -1.3; 0.2; -1.3; 2; 1.5; -2; 1.5]\,; \\
&[-0.4; 0.5; 0.8; -2.3; 1; -2.3; -0.5; -1.5; -2; 1.5; -2; 1.5]\,; \\
&[0.5; -0.2; 1; 1.3; 2; 1.3; 1.8; 2; 0.4; 1.5; -2; 1.5]\,; \\
&[-0.5; 0.2; 2; -2.3; 1.3; -2.3; 0.6; -0.2; -0.4; 1.5; -2; 1.5]\,; \\
&[0.8; -0.5; -2; -1.3; 2; -1.3; -1.5; 0.5; -2.6; 1.5; -2; 1.5]\,; \\
&[-0.6; 0.9; -0.5; -2.3; -2; -2.3; -1.2; \\
&\quad -0.5; -2.3; 1.5; -2; 1.5]\,; \\
&[0.7; -0.7; -1.5; -3; 0.5; -1; -1.1; -1.8; -0.2; 1.5; -2; 1.5]\,.
\end{aligned}
\tag{90}
$$

Figures 1, 2, 3, 4, 5, and 6 depict the time responses of state variables of $u_1(t)$, $u_2(t)$, and $u_3(t)$ and $v_1(t)$, $v_2(t)$, and $v_3(t)$, of the system in Example 1, respectively.

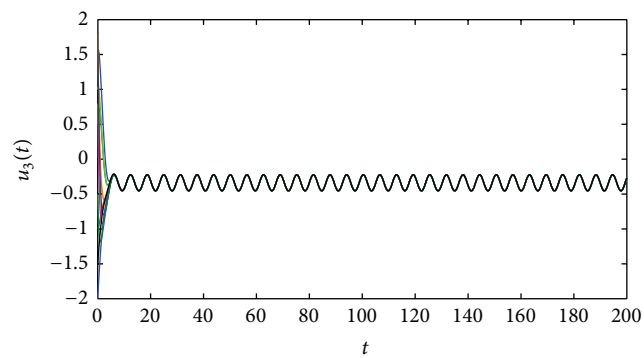


FIGURE 3: Transient response of state variables $u_3(t)$ of Example 1.

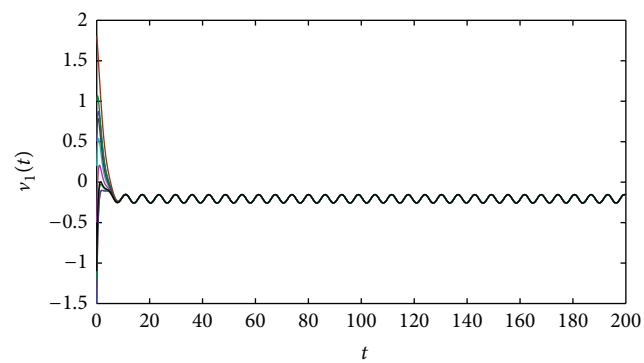


FIGURE 4: Transient response of state variables $v_1(t)$ of Example 1.

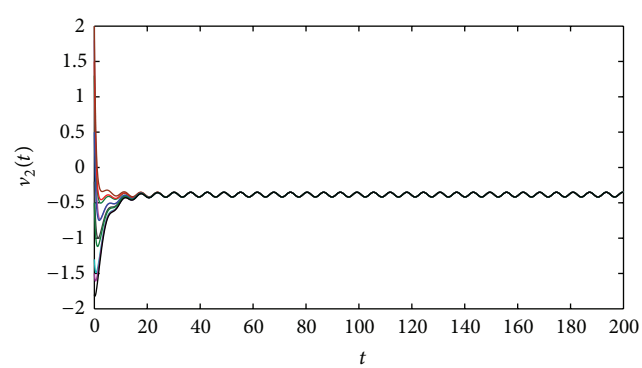


FIGURE 5: Transient response of state variables $v_2(t)$ of Example 1.

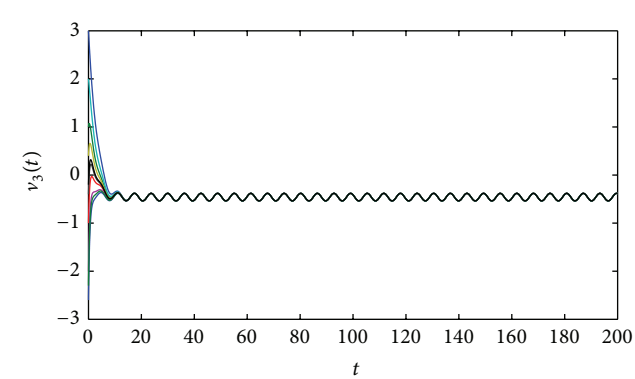


FIGURE 6: Transient response of state variables $v_3(t)$ of Example 1.

On the other hand, we have the following results by simple calculation:

$$\alpha_1 - K_1 = \frac{2}{5} > 0, \qquad \alpha_2 - K_2 = \frac{67}{80} > 0,$$

$$\alpha_3 - K_3 = \frac{14}{30} > 0, \qquad \beta_1 - K_1^* = \frac{81}{70} > 0,$$

$$\beta_2 - K_2^* = \frac{31}{24} > 0, \qquad \beta_3 - K_3^* = \frac{7}{5} > 0,$$

$$\alpha_1 - T_1 - 2 + A_1\left(\sum_{j=1}^{3}\left|c_{1j}\right|\overline{f}_j + \overline{I}_1\right) + \sum_{j=1}^{3}\overline{d}_j\frac{\left|h_{j1}\right|}{1-\overline{\sigma}_{j1}}k_1 = -\frac{218.5}{240} < 0,$$

$$\alpha_2 - T_2 - 2 + A_2\left(\sum_{j=1}^{3}\left|c_{2j}\right|\overline{f}_j + \overline{I}_2\right) + \sum_{j=1}^{3}\overline{d}_j\frac{\left|h_{j2}\right|}{1-\overline{\sigma}_{j2}}k_2 = -\frac{237.1}{768} < 0,$$

$$\alpha_3 - T_3 - 2 + A_3\left(\sum_{j=1}^{3}\left|c_{3j}\right|\overline{f}_j + \overline{I}_3\right) + \sum_{j=1}^{3}\overline{d}_j\frac{\left|h_{j3}\right|}{1-\overline{\sigma}_{j3}}k_3 = -\frac{7.19}{128} < 0,$$

$$2 - \alpha_1 - T_1 + \overline{a}_1\sum_{j=1}^{3}\left|c_{1j}\right|l_j + A_1\left(\sum_{j=1}^{3}\left|c_{1j}\right|\overline{f}_j + \overline{I}_1\right) = -\frac{25.1}{48} < 0,$$

$$2 - \alpha_2 - T_2 + \overline{a}_2\sum_{j=1}^{3}\left|c_{2j}\right|l_j + A_2\left(\sum_{j=1}^{3}\left|c_{2j}\right|\overline{f}_j + \overline{I}_2\right) = -\frac{23}{320} < 0,$$

$$2 - \alpha_3 - T_3 + \overline{a}_3\sum_{j=1}^{3}\left|c_{3j}\right|l_j + A_3\left(\sum_{j=1}^{3}\left|c_{3j}\right|\overline{f}_j + \overline{I}_3\right) = -\frac{1.85}{32} < 0,$$

$$\beta_1 - T_1^* - 2 + D_1\left(\sum_{i=1}^{3}\left|h_{1i}\right|\overline{g}_i + \overline{J}_1\right) + \sum_{i=1}^{3}\overline{a}_i\frac{\left|c_{i1}\right|}{1-\overline{\tau}_{i1}}l_1 = -\frac{2.3}{128} < 0,$$

$$\beta_2 - T_2^* - 2 + D_2\left(\sum_{i=1}^{3}\left|h_{2i}\right|\overline{g}_i + \overline{J}_2\right) + \sum_{i=1}^{3}\overline{a}_i\frac{\left|c_{i2}\right|}{1-\overline{\tau}_{i2}}l_2 = -\frac{85.1}{384} < 0,$$

$$\beta_3 - T_3^* - 2 + D_3\left(\sum_{i=1}^{3}\left|h_{3i}\right|\overline{g}_i + \overline{J}_3\right) + \sum_{i=1}^{3}\overline{a}_i\frac{\left|c_{i3}\right|}{1-\overline{\tau}_{i3}}l_3 = -\frac{33.5}{128} < 0,$$

$$2 - \beta_1 - T_1^* + \overline{d}_j\sum_{i=1}^{3}\left|h_{1i}\right|k_i + D_1\left(\sum_{i=1}^{3}\left|h_{ji}\right|\overline{g}_i + \overline{J}_1\right) = -\frac{87.9}{128} < 0,$$

$$2 - \beta_2 - T_2^* + \overline{d}_2\sum_{i=1}^{3}\left|h_{2i}\right|k_i + D_2\left(\sum_{i=1}^{3}\left|h_{2i}\right|\overline{g}_i + \overline{J}_2\right) = -\frac{83}{128} < 0,$$

$$2 - \beta_3 - T_3^* + \overline{d}_3\sum_{i=1}^{3}\left|h_{3i}\right|k_i + D_3\left(\sum_{i=1}^{3}\left|h_{3i}\right|\overline{g}_i + \overline{J}_3\right) = -\frac{41.55}{128} < 0. \tag{91}$$

Then, the conditions of Theorem 3 hold. From Theorem 3, system (86) has one $2\pi$-periodic solution, and all other solutions of system (86) exponentially converge to it as $t \to +\infty$.

Evidently, this consequence is coincident with the results of numerical simulation.

*Example 2.* For system (86), let $\alpha_1 = 2.12$, $\alpha_2 = 1.6$, $\alpha_3 = 1.7$, $\beta_1 = 2.2$, $\beta_2 = 5/3$, and $\beta_3 = 1.6$,; the other parameters are the same as those in Example 1.

Through numerical simulation, Figures 7, 8, 9, 10, 11, and 12 depict the time responses of state variables of $u_1(t)$, $u_2(t)$, and $u_3(t)$ and $v_1(t)$, $v_2(t)$, and $v_3(t)$, of the system in Example 2, respectively.

On the other hand, we have the following results by simple calculation:

$$\alpha_1 - 1 - T_1 = 0.1 > 0, \qquad \alpha_2 - 1 - T_2 = 0.1 > 0,$$

$$\alpha_3 - 1 - T_3 = 0.2 > 0, \qquad \beta_1 - 1 - T_1^* = 0.2 > 0,$$

$$\beta_2 - 1 - T_2^* = \frac{1}{3} > 0, \qquad \beta_3 - 1 - T_3^* = 0.1 > 0,$$

$$-T_1 + A_1\sum_{j=1}^{3}\overline{f}_j\left|c_{1j}\right| + A_1\overline{I}_1 + \overline{a}_1\sum_{j=1}^{3}l_j\left|c_{1j}\right| = -\frac{83}{96} < 0,$$

$$-T_2 + A_2\sum_{j=1}^{3}\overline{f}_j\left|c_{2j}\right| + A_2\overline{I}_2 + \overline{a}_2\sum_{j=1}^{3}l_j\left|c_{2j}\right| = -\frac{77}{256} < 0,$$

$$-T_3 + A_3\sum_{j=1}^{3}\overline{f}_j\left|c_{3j}\right| + A_3\overline{I}_3 + \overline{a}_3\sum_{j=1}^{3}l_j\left|c_{3j}\right| = -\frac{165}{640} < 0,$$

$$-T_1^* + D_1\sum_{i=1}^{3}\overline{g}_i\left|h_{1i}\right| + D_1\overline{J}_1 + \overline{d}_1\sum_{i=1}^{3}k_i\left|h_{1i}\right| = -\frac{115}{256} < 0,$$

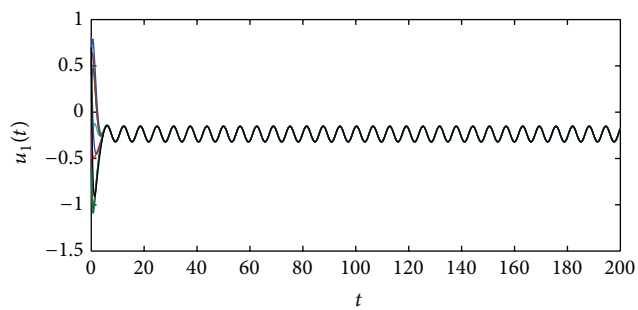


FIGURE 7: Transient response of state variables $u_1(t)$ of Example 2.

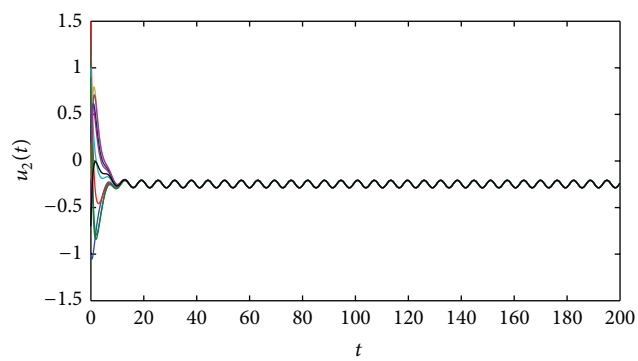


FIGURE 8: Transient response of state variables $u_2(t)$ of Example 2.

$$-T_2^* + D_2 \sum_{i=1}^{3} \overline{g}_i \left|h_{2i}\right| + D_2 \overline{J}_2 + \overline{d}_2 \sum_{i=1}^{3} k_i \left|h_{2i}\right| = -\frac{83}{384} < 0,$$
$$-T_3^* + D_3 \sum_{i=1}^{3} \overline{g}_i \left|h_{3i}\right| + D_3 \overline{J}_3 + \overline{d}_3 \sum_{i=1}^{n} k_i \left|h_{3i}\right| = -\frac{99}{256} < 0. \tag{92}$$

Then, the conditions of Theorem 4 hold. From Theorem 4, system (86) has one $2\pi$-periodic solution, and all other solutions of system (86) exponentially converge to it as $t \to +\infty$.

Evidently, this consequence is coincident with the results of numerical simulation.

*Remark 5.* Examples 1 and 2 showed that system (86) has one $2\pi$-periodic solution, which is exponentially stable. In Example 1, there is

$$\alpha_1 - 1 - T_1 = -0.3 < 0. \tag{93}$$

But this condition does not satisfy Theorem 4. While in Example 2, there is

$$2 - \beta_3 - T_3^* + \overline{d}_3 \sum_{i=1}^{3} \left|h_{3i}\right| k_i + D_3 \left( \sum_{i=1}^{3} \left|h_{3i}\right| \overline{g}_i + \overline{J}_3 \right) = \frac{35.25}{128} > 0. \tag{94}$$

This condition does not satisfy Theorem 3. It showed that Theorems 3 and 4 have different applications.

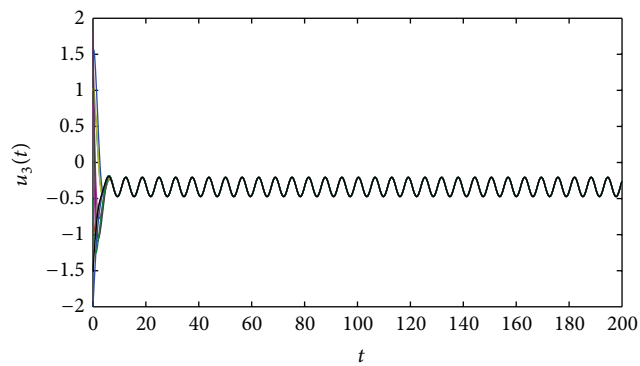


FIGURE 9: Transient response of state variables $u_3(t)$ of Example 2.

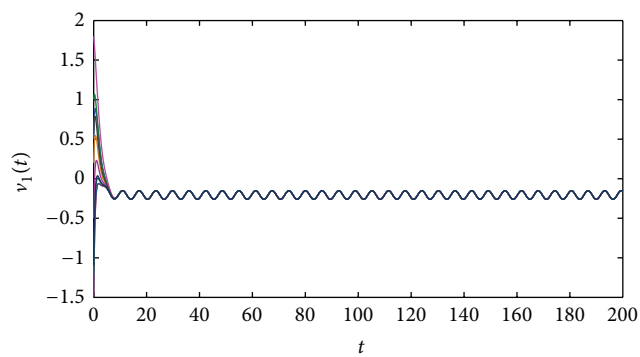


FIGURE 10: Transient response of state variables $v_1(t)$ of Example 2.

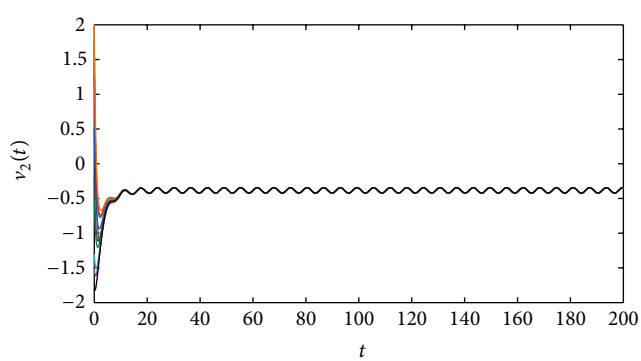


FIGURE 11: Transient response of state variables $v_2(t)$ of Example 2.

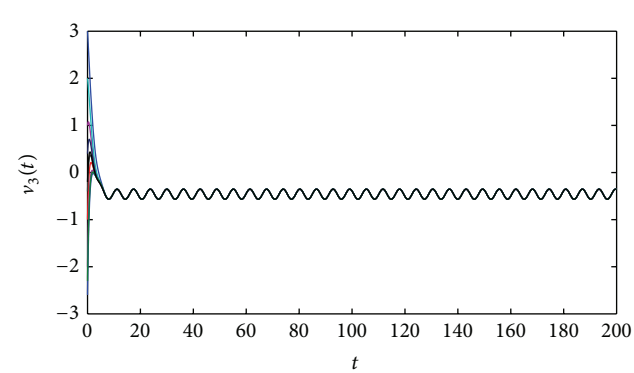


FIGURE 12: Transient response of state variables $v_3(t)$ of Example 2.

In fact, the parameter $\alpha_i$, $\beta_j$ in Theorem 3 must be satisfied

$$2 - T_i < \alpha_i < 2 + T_i, \qquad 2 - T_j^* < \beta_j < 2 + T_j^*. \tag{95}$$

For Theorem 4, it is only required to satisfy

$$\alpha_i > 1 + T_i, \qquad \beta_j > 1 + T_j^*. \tag{96}$$

Therefore, Theorems 3 and 4 can solve different problems.

## 5. Conclusion

In this paper, we give three theorems to ensure the existence and the exponential stability of the periodic solution for inertial Cohen-Grossberg-type BAM neural networks. Novel existence and stability conditions are stated with simple algebraic forms and their verification and applications are straightforward and convenient. Especially, we give different conditions in Theorems 3 and 4 to ensure the exponential stability of the periodic solution, which have different advantages in different problems and applications. Finally two examples illustrate the effectiveness in different conditions. The method used in this paper can be employed to study general neural network with time-varying delays.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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