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Research Article

Analytic and Approximate Solutions of the Space-Time Fractional Schrödinger Equations by Homotopy Perturbation Sumudu Transform Method

Sara H. M. Hamed,[1,2] Eltayeb A. Yousif,[2,3] and Arbab I. Arbab[4]

[1] *Department of Mathematics, Faculty of Mathematical Sciences and Statistics, Alneelain University, 11121 Khartoum, Sudan*
[2] *Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Khartoum, 11111 Khartoum, Sudan*
[3] *Department of Mathematics, Faculty of Science, Northern Border University, Arar 91431, Saudi Arabia*
[4] *Department of Physics, Faculty of Science, University of Khartoum, 11111 Khartoum, Sudan*

Correspondence should be addressed to Eltayeb A. Yousif; eltayebyousif@gmail.com

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A combination of homotopy perturbation method and Sumudu transform is applied to find exact and approximate solution of space and time fractional nonlinear Schrödinger equation. The fractional derivatives are described in the Caputo sense. The solutions are given in the form of convergent series with easily computable components. The results show that the method is effective and convenient for solving nonlinear differential equations of fractional order.

1. Introduction

Fractional calculus is a generalization of differentiations and integrations of integer order to arbitrary orders [1–7]. Fractional calculus has attracted much attention due to its appearance and numerous applications in science and engineering during the last decades. Many problems in physics, biology, and engineering are modulated in terms of fractional differential and integral equations such as acoustics, diffusion, signal processing, electrochemistry, systems identification, finance, fractional dynamics, nanotechnology, fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and maybe other physical phenomena [8–13]. Recently, there is a very comprehensive literature dealing with the problems of finding exact and approximate solutions of fractional differential equations. The solutions of fractional equations are investigated by many authors using powerful analytical methods. For example, the homotopy perturbation method [14–16], the Adomian decomposition method [17–19], the variational iteration method [20–22], the differential transform method [23–25], the fractional Riccati expansion method [26], the fractional subequation method [27, 28], the homotopy analysis method [29, 30], and the fractional complex transform [31]. Watugala [32] introduced Sumudu transform and used it in obtaining the solution of ordinary differential equations in control engineering problems. This method has been implemented by many authors in investigating various types of problems [33–40]. The homotopy perturbation Sumudu transform method (HPSTM) is a combination of the Sumudu transform method and homotopy perturbation method. It is applied to solve numerous linear and nonlinear partial differential equations [41–47].

Schrödinger equation is one of the most important models in mathematical physics; it arises in many physical systems with applications to numerous fields [48] such as nonlinear optics [49], dynamics of accelerators [50], mean-field theory of Bose-Einstein condensates [51, 52], and plasma physics

[53]. The fractional Schrödinger equation is a developing part of quantum physics which studies nonlocal quantum phenomena. Naber [54] studied time fractional Schrödinger equation in sense of Caputo derivative. Wang and Xu [55] generalized the linear Schrödinger equation to space-time fractional one and studied the problem by using integral transform technique. Jiang [56] obtained the time dependent solutions in terms of H-function to a linear space-time fractional Schrödinger equation containing a nonlocal term. Ford et al. [57] introduced a numerical method to solve a linear fractional Schrödinger equation in the case where the space has dimension two; they obtained the stability conditions for a finite difference scheme. In recent years, nonlinear fractional Schrödinger equation has attracted several researchers. The existence and uniqueness of the global solution to the periodic boundary value problem of fractional nonlinear Schrödinger equations are proved based on energy method [58] and Faedo-Galërkin method [59]. Analytical and numerical methods have been investigated for time fractional nonlinear Schrödinger equation [18, 60–62]. Very few theoretical and numerical analyses have been carried out for nonlinear Schrödinger equations with both space and time fractional derivatives. Herzallah and Gepreel [19] constructed an approximate solution for the cubic nonlinear fractional Schrödinger equation with time and space fractional derivatives using Adomian decomposition method. Hemida et al. [29] used a homotopy analysis method to construct approximate solutions for the space-time fractional nonlinear Schrödinger equation.

In this paper, we applied the homotopy perturbation Sumudu transform method (HPSTM) to obtain the analytical exact and approximate solutions for the fractional Schrödinger equation with space and time fractional derivatives of the form

$$iD_t^{\alpha}U(x,t) = c_1 D_x^{\beta}U(x,t) + V(x)U(x,t) + c_2 U(x,t)|U(x,t)|^2, \quad U(x,0) = f(x), \tag{1}$$

where α and β are parameters describing the order of the time and space fractional derivatives of $U(x,t)$, respectively, and they satisfy $0 < \alpha \le 1$, $1 < \beta \le 2$, $t > 0$, $V(x)$ is the trapping potential, c_1, c_2 are constants, and $f(x)$ is the initial condition [19]. The fractional derivatives are considered in the Caputo sense. In the case of $\alpha = 1$ and $\beta = 2$, (1) reduces to the classical Schrödinger equation. The solution of (1) is obtained for linear case when $c_2 = 0$ and nonlinear case when $c_2 \ne 0$. Moreover, this method is applied in approximating the solution of the problem in the case of nonzero trapping potential when $V(x) \ne 0$.

The rest of this work is organized as follows. In Section 2, we provide some preliminaries. Section 3 introduces the concept of homotopy perturbation method, while Section 4 gives the Sumudu transform. The homotopy perturbation Sumudu transform method (HPSTM) is analyzed in Section 5. Applications of numerical examples are provided in Section 6. The conclusions are given in Section 7.

2. Preliminaries

Definition 1. The Caputo fractional derivative of order $\alpha > 0$ of a function $f(x)$, $x > 0$, is defined by [4, 10]

$$D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, & n-1 < \alpha \le n \in \mathbb{N} \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \in \mathbb{N}, \end{cases} \tag{2}$$

where D_x^{α} is called the Caputo derivative operator.

Definition 2. The Mittag-Leffler function with two parameters is defined by [63, 64]

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \ge 0. \tag{3}$$

The Mittag-Leffler function $E(x, m, a)$ is expressed in terms of formula (3) as

$$E(x, m, a) = x^m E_{1,m+1}(ax). \tag{4}$$

Note 1. The following results are used further in this paper.

From Definition 2 and formula (4), these functions are directly obtained:

$$E(x, -m, \lambda) = x^{-m} E_{1,1-m}(\lambda x) = \lambda^m e^{\lambda x}, \quad m = 0, 1, 2, \ldots. \tag{5}$$

The special type of Mittag-Leffler function $E_{1/2,1}(a\sqrt{t})$ is given by [58]

$$E_{1/2,1}(a\sqrt{x}) = e^{a^2 t} erfc(-a\sqrt{x}), \tag{6}$$

where $erfc$ is a complementary error function.

The fractional derivatives of Mittag-Leffler function is given by [10, 64]

$$D_x^{\gamma}\left[x^{\beta-1} E_{\alpha,\beta}(ax^{\alpha})\right] = x^{\beta-\gamma-1} E_{\alpha,\beta-\gamma}(ax^{\alpha}), \quad \gamma \ge 0. \tag{7}$$

3. Homotopy Perturbation Method

To illustrate the concept of homotopy perturbation method, we consider the nonlinear differential equation,

$$L(U) + N(U) = f(r), \quad r \in \Omega, \tag{8}$$

with the boundary conditions,

$$B\left(U, \frac{\partial U}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{9}$$

where L is a linear operator, N is nonlinear operator, B is boundary operator, Γ is the boundary of the domain Ω, and $f(r)$ is a known analytic function.

He's homotopy perturbation technique [65–67] defines the homotopy $v(r,p):\Omega\times[0,1]\to R$ which satisfies

$$H(v,p)=(1-p)\left[L(v)-L(U_0)\right]+p\left[L(v)+N(v)-f(r)\right]=0, \tag{10}$$

or

$$H(v,p)=L(v)-L(U_0)+pL(U_0)+p\left[N(v)-f(r)\right]=0, \tag{11}$$

where $r\in\Omega$, $p\in[0,1]$ is an impending parameter, and U_0 is an initial approximation which satisfies the boundary condition. The basic assumption is that the solution of (10) and (11) can be expressed as power series in p as follows:

$$v=v_0+pv_1+p^2v_2+\cdots. \tag{12}$$

The approximate solution of (8) is given by

$$U=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots. \tag{13}$$

4. Sumudu Transform

Consider functions in the set A that are defined by

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,\text{ such }\left|f(t)\right|<Me^{|t|/\tau_j}\ \text{if } t\in(-1)^j\times[0,\infty)\right\}, \tag{14}$$

where M is a constant and must be finite and τ_1 and τ_2 need not simultaneously exist and each may be finite. The Sumudu transform is defined by [32]

$$G(u)=S(f(t))=\int_0^\infty f(ut)\,e^{-t}dt,\quad u\in(-\tau_1,\tau_2). \tag{15}$$

The Sumudu transform was shown to be the theoretical dual of the Laplace transform. For the details of the relationship between Sumudu and the Laplace transforms and the comparison between the two transformations, see [33–37].

Definition 3. The Sumudu transform of fractional order derivative is defined by [45, 46]

$$S\left[D_x^\alpha f(x)\right]=\frac{1}{u^\alpha}S\left[f(x)\right]-\sum_{k=0}^{n-1}\frac{1}{u^{\alpha-k}}\left[f^{(k)}(x)\right]_{x=0},\quad n-1<\alpha\le n,\quad n\in\mathbb{N}. \tag{16}$$

5. Homotopy Perturbation Sumudu Transform Method

In this section, we implement the homotopy perturbation Sumudu transform method to space-time cubic nonlinear fractional Schrödinger equation given by (1). By applying Sumudu transform on both sides of (1) with respect to t, we get the following:

$$\begin{aligned}
&S\left(iD_t^\alpha U(x,t)\right)\\
&\quad=S\Big(c_1D_x^\beta U(x,t)+V(x)U(x,t)+c_2U(x,t)\left|U(x,t)\right|^2\Big),\\
&\frac{i}{u^\alpha}S(U(x,t))-i\sum_{k=0}^{n-1}\frac{1}{u^{\alpha-k}}\frac{\partial^kU(x,0)}{\partial t^k}\\
&\quad=S\left(c_1D_x^{2\beta}U(x,t)+V(x)U(x,t)+c_2U(x,t)\left|U(x,t)\right|^2\right),\\
&S(U(x,t))\\
&\quad=\sum_{k=0}^{n-1}u^kf_k(x)-iu^\alpha\Big(S\Big(c_1D_x^\beta U(x,t)+V(x)U(x,t)\\
&\qquad+c_2U(x,t)\left|U(x,t)\right|^2\Big)\Big).
\end{aligned} \tag{17}$$

Taking the inverse Sumudu transform to (17), we have

$$\begin{aligned}
U(x,t)={}&\sum_{k=0}^{n-1}\frac{t^kf_k(x)}{\Gamma(k+1)}\\
&-iS^{-1}\Big(u^\alpha\Big(S\Big(c_1D_x^\beta U(x,t)+V(x)U(x,t)\\
&+c_2U(x,t)\left|U(x,t)\right|^2\Big)\Big)\Big).
\end{aligned} \tag{18}$$

Application of the homotopy perturbation method to (18) yields

$$\begin{aligned}
&(1-P)(U(x,t)-U(x,0))\\
&\quad+P\Bigg(U(x,t)-\sum_{k=0}^{n-1}\frac{t^kf_k(x)}{\Gamma(k+1)}\\
&\qquad+iS^{-1}\Big(u^\alpha\Big(S\Big(c_1D_x^\beta U(x,t)+V(x)U(x,t)\\
&\qquad+c_2U(x,t)\left|U(x,t)\right|^2\Big)\Big)\Big)\Bigg)=0,\\
&U(x,t)=U(x,0)+P\sum_{k=1}^{n-1}\frac{t^kf_k(x)}{\Gamma(k+1)}\\
&\qquad-iPS^{-1}\Big(u^\alpha\Big(S\Big(c_1D_x^\beta U(x,t)+V(x)U(x,t)\\
&\qquad+c_2U(x,t)\left|U(x,t)\right|^2\Big)\Big)\Big).
\end{aligned} \tag{19}$$

Let

$$U(x,t)=\sum_{n=0}^{\infty}P^nU_n(x,t),\quad U(x,0)=U_0(x,t). \tag{20}$$

Substituting (20) into (19), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} P^n U_n(x,t) \\
&\quad = U_0(x,t) + P\sum_{k=1}^{n-1}\frac{t^k f_k(x)}{\Gamma(k+1)} \\
&\qquad - iPS^{-1}\left(u^{\alpha}\left(S\left(c_1 D_x^{\beta}\sum_{n=0}^{\infty}P^n U_n(x,t)\right)\right.\right. \\
&\qquad\qquad + V(x) S\left(\sum_{n=0}^{\infty}P^n U_n(x,t)\right) \\
&\qquad\qquad \left.\left. + S\left(\sum_{n=0}^{\infty}P^n H_n(U(x,t))\right)\right)\right),
\end{aligned} \tag{21}$$

where

$$H_n(U_0, U_1, \ldots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i U_i\right)\right]_{p=0}, \quad n = 0, 1, 2, \ldots, \tag{22}$$

and then

$$\begin{aligned}
P^0 &: U_0(x,t) = f_0(x) = f(x), \\
P^1 &: U_1(x,t) = \sum_{k=1}^{n-1}\frac{t^k f_k(x)}{\Gamma(k+1)} \\
&\qquad - iS^{-1}\left(u^{\alpha}\left(S\left(c_1 D_x^{\beta} U_0(x,t)\right)\right.\right. \\
&\qquad\qquad + V(x) S(U_0(x,t)) \\
&\qquad\qquad \left.\left. + S(H_0(U(x,t)))\right)\right), \\
P^2 &: U_2(x,t) = -iS^{-1}\left(u^{\alpha}\left(S\left(c_1 D_x^{\beta} U_1(x,t)\right)\right.\right. \\
&\qquad\qquad + V(x) S(U_1(x,t)) \\
&\qquad\qquad \left.\left. + S(H_1(U(x,t)))\right)\right), \\
P^3 &: U_3(x,t) = -iS^{-1}\left(u^{\alpha}\left(S\left(c_1 D_x^{\beta} U_2(x,t)\right)\right.\right. \\
&\qquad\qquad + V(x) S(U_2(x,t)) \\
&\qquad\qquad \left.\left. + S(H_2(U(x,t)))\right)\right), \\
&\quad\vdots
\end{aligned} \tag{23}$$

The HPMST assumes a series solution to (1) in the form

$$U = U_0 + U_1 + U_2 + \cdots = \sum_{n=1}^{\infty} U_n. \tag{24}$$

This series solution generally converges rapidly [45–47].

6. Applications

Example 4. Consider the following time fraction nonlinear Schrödinger equation:

$$iD_t^{\alpha}U + U_{xx} + 2\left|U^2\right|U = 0, \tag{25}$$

subject to the initial condition

$$U(x,0) = e^{ix}. \tag{26}$$

Use (21)–(24) to get

$$\begin{aligned}
&U(x,t) \\
&= U(x,0) + iS^{-1}\left(u^{\alpha}S\left(\frac{\partial^2}{\partial x^2}(U_0(x,t)) + 2{U_0}^2\overline{U_0}\right)\right) \\
&\quad + iS^{-1}\left(u^{\alpha}S\left(\frac{\partial^2}{\partial x^2}(U_1(x,t)) + 2\left(2U_0U_1\overline{U_0} + {U_0}^2\overline{U_1}\right)\right)\right) \\
&\quad + iS^{-1}\left(u^{\alpha}S\left(\frac{\partial^2}{\partial x^2}(U_2(x,t))\right.\right. \\
&\qquad\qquad + 2\left(2U_0U_2\overline{U_0} + {U_1}^2\overline{U_0}\right. \\
&\qquad\qquad\quad \left.\left.\left. + 2U_0U_1\overline{U_1} + {U_0}^2\overline{U_2}\right)\right)\right) \\
&\quad + \cdots.
\end{aligned} \tag{27}$$

The terms of the series are calculated as

$$\begin{aligned}
U_0(x,t) &= e^{ix}. \\
U_1(x,t) &= iS^{-1}\left(u^{\alpha}S\left(\frac{\partial^2 U_0}{\partial x^2} + 2{U_0}^2\overline{U_0}\right)\right) \\
&= \frac{it^{\alpha}}{\Gamma(\alpha+1)}e^{ix}, \\
U_2(x,t) &= iS^{-1}\left(u^{\alpha}S\left(\frac{\partial^2 U_1}{\partial x^2} + 2\left(2U_0U_1\overline{U_0} + {U_0}^2\overline{U_1}\right)\right)\right) \\
&= iS^{-1}\left(u^{\alpha}S\left(-\frac{it^{\alpha}}{\Gamma(\alpha+1)}e^{ix}\right.\right. \\
&\qquad + 2\left(2e^{ix}\left(\frac{it^{\alpha}}{\Gamma(\alpha+1)}e^{ix}\right)e^{-ix}\right. \\
&\qquad\qquad \left.\left.\left. + e^{2ix}\left(-\frac{it^{\alpha}}{\Gamma(\alpha+1)}e^{-ix}\right)\right)\right)\right) \\
&= iS^{-1}\left(u^{\alpha}S\left(\frac{it^{\alpha}}{\Gamma(\alpha+1)}e^{ix}\right)\right) = -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{ix}. \\
&\quad\vdots
\end{aligned} \tag{28}$$

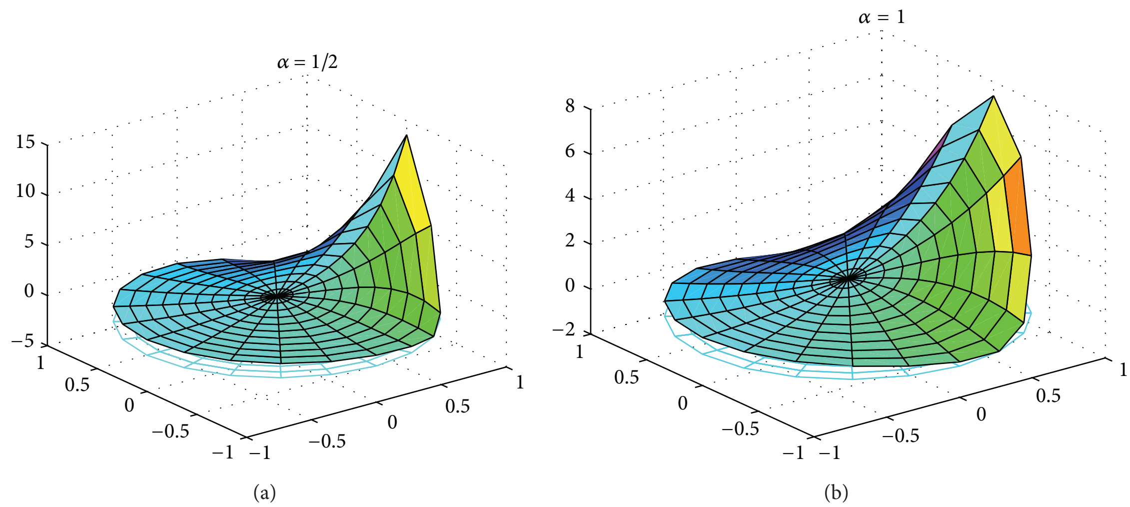


FIGURE 1: The surface plot of the solution $U(x,t)$ of Example 4: (a) $\alpha = 0.5$, (b) $\alpha = 1$.

Substituting $U_0(x,t), U_1(x,t), U_2(x,t), \ldots$ into (24) yields the solution

$$U(x,t) = e^{ix} + \frac{it^{\alpha}}{\Gamma(\alpha+1)}e^{ix} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}e^{ix} + \cdots = e^{ix}E_{\alpha,1}(it^{\alpha}). \tag{29}$$

Remark 5. If $\alpha \to 1/2$, then the solution is given by $U(x,t) = e^{ix}E_{1/2,1}(i\sqrt{t}) = e^{ix-t}erfc(-i\sqrt{t})$. The solution for the classical nonlinear Schrödinger equation is obtained as special case as $\alpha \to 1$, by $U(x,t) = e^{ix}E_{1,1}(it) = e^{i(x+t)}$.

Figure 1 shows the solution $U(x,t)$ of Example 4, Figure 1(a) the solution obtained for $\alpha = 0.5$, and Figure 1(b) that obtained for the value of $\alpha = 1$ (the solution of classical nonlinear Schrödinger equation).

Example 6. Consider the following linear space-time fractional Schrödinger equation:

$$iD_t^{\alpha}U(x,t) = cD_x^{\beta}U(x,t), \quad 0 < \alpha \le 1,\ 1 < \beta \le 2, \text{ and } c \text{ is a constsnt}, \tag{30}$$

subject to the initial condition

$$U(x,0) = e^{ix}. \tag{31}$$

By using (21) and (23), we have

$$i\sum_{n=0}^{\infty}P^nU_n(x,t) = iU_0(x,t) + cPS^{-1}\left(u^{\alpha}\left(S\left(\frac{\partial^{\beta}}{\partial x^{\beta}}\sum_{n=0}^{\infty}P^nU_n(x,t)\right)\right)\right), \tag{32}$$

and then

$$P^0 : U_0(x,t) = e^{ix},$$

$$\begin{aligned} P^1 : U_1(x,t) &= -icS^{-1}\left(u^{\alpha}S\left(\frac{\partial^{\beta}U_0}{\partial x^{\beta}}\right)\right) \\ &= -icS^{-1}\left(u^{\alpha}S\left(\frac{\partial^{\beta}}{\partial x^{\beta}}e^{ix}\right)\right) \\ &= \frac{ict^{\alpha}}{\Gamma(\alpha+1)}x^{2-\beta}E_{1,3-\beta}(ix) \\ &= \frac{ict^{\alpha}}{\Gamma(\alpha+1)}E(x,2-\beta,i), \end{aligned}$$

$$\begin{aligned} P^2 : U_2(x,t) &= -icS^{-1}\left(u^{\alpha}S\left(\frac{\partial^{\beta}U_1}{\partial x^{\beta}}\right)\right) \\ &= -icS^{-1}\left(u^{\alpha}S\left(\frac{ict^{\alpha}}{\Gamma(\alpha+1)}D_x^{\beta}(E(x,2-\beta,i))\right)\right) \\ &= \frac{c^2t^{2\alpha}}{\Gamma(2\alpha+1)}x^{2-2\beta}E_{1,3-2\beta}(ix) \\ &= \frac{c^2t^{2\alpha}}{\Gamma(2\alpha+1)}E(x,2-2\beta,i), \end{aligned}$$

$$\begin{aligned} P^3 : U_3(x,t) &= -icS^{-1}\left(u^{\alpha}S\left(\frac{\partial^{\beta}U_2}{\partial x^{\beta}}\right)\right) \\ &= -icS^{-1}\left(u^{\alpha}S\left(\frac{c^2t^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{\partial^{\beta}}{\partial x^{\beta}}E(x,2-2\beta,i)\right)\right)\right) \end{aligned}$$

$$= -\frac{ic^3 t^{3\alpha}}{\Gamma(3\alpha+1)} E(x, 2-3\beta, i).$$
$$\vdots \tag{33}$$

Substituting $U_0(x,t), U_1(x,t), U_2(x,t), \ldots$ into (24) gives the series solution as

$$\begin{aligned} U(x,t) &= e^{ix} + \frac{ict^{\alpha}}{\Gamma(\alpha+1)} E(x, 2-\beta, i) \\ &\quad + \frac{c^2 t^{2\alpha}}{\Gamma(2\alpha+1)} E(x, 2-2\beta, i) \\ &\quad - \frac{ic^3 t^{3\alpha}}{\Gamma(3\alpha+1)} E(x, 2-3\beta, i) + \cdots \\ &= e^{ix} + \frac{ict^{\alpha}}{\Gamma(\alpha+1)} E(x, 2-\beta, i) \\ &\quad - \frac{(ict^{\alpha})^2}{\Gamma(2\alpha+1)} E(x, 2-2\beta, i) \\ &\quad + \frac{(ict^{\alpha})^3}{\Gamma(3\alpha+1)} E(x, 2-3\beta, i) + \cdots \\ &= e^{ix} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(ict^{\alpha})^k}{\Gamma(k\alpha+1)} E(x, 2-\beta k, i). \end{aligned} \tag{34}$$

Remark 7. In (34), if we let $\alpha \to 1$, $\beta \to 2$, and $c = 1$, using the result given by formula (5), then we get the solution of the classical equation as

$$\begin{aligned} U_{\text{clas}}(x,t) &= e^{ix} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(it)^k}{\Gamma(k+1)} E(x, 2-2k, i) \\ &= e^{ix} + \sum_{k=1}^{\infty} \frac{(it)^k}{\Gamma(k+1)} e^{ix} = e^{i(x+t)}. \end{aligned} \tag{35}$$

Remark 8. Let $\widetilde{U}(x,t)$ be the solution when $\alpha \to 1/2$, $\beta \to 3/2$, and $c = 1$; then, from (34), we have

$$\begin{aligned} \widetilde{U}(x,t) &= e^{ix} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(i\sqrt{t})^k}{\Gamma((k/2)+1)} E\left(x, 2-\frac{3}{2}k, i\right) \\ &= \left[e^{ix} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{\Gamma(k+1)} E(x, 2-3k, i) \right] \\ &\quad + \left[\sum_{k=1}^{\infty} \frac{(i\sqrt{t})^k}{\Gamma(k+(1/2))} E\left(x, \frac{1-6k}{2}, i\right) \right] \\ &= \left[e^{ix} + \sum_{k=1}^{\infty} \left(\frac{(it)^k}{\Gamma(k+1)} \right) e^{ix} \right] \\ &\quad + \left[\sum_{k=1}^{\infty} \frac{(i\sqrt{t})^k}{\Gamma(k+(1/2))} E\left(x, \frac{1-6k}{2}, i\right) \right] \\ &= U_{\text{clas}}(x,t) + \left[\sum_{k=1}^{\infty} \frac{(i\sqrt{t})^k}{\Gamma(k+(1/2))} E\left(x, \frac{1-6k}{2}, i\right) \right]. \end{aligned} \tag{36}$$

It is noted that the function $\sum_{k=1}^{\infty}((i\sqrt{t})^k/\Gamma(k + (1/2)))E(x, ((1-6k)/2), i)$ represents the variation between the two solutions $U_{\text{clas}}(x,t)$ and $\widetilde{U}(x,t)$ given by (35) and (36), respectively.

Figure 2 compares the real part of the solution of Example 6. In Figure 2(a) we have the solution obtained for the value of $\alpha = 0.5$, $\beta = 1.5$ and in Figure 2(b) for the value of $\alpha = 1$, $\beta = 2$ (the solution of classical nonlinear Schrödinger equation). Figure 3 compares the imaginary part of the solution of Example 6. In Figure 3(a) we have the solution obtained for the value of $\alpha = 0.5$, $\beta = 1.5$ and in Figure 3(b) the solution obtained for the value of $\alpha = 1$, $\beta = 2$ (the solution of classical nonlinear Schrödinger equation).

Example 9. Consider the space-time fractional nonlinear Schrödinger equation:

$$iD_t^{\alpha}U + D_x^{\beta}U + 2\left|U^2\right|U = 0, \quad 0 < \alpha \le 1,\ 1 < \beta \le 2, \tag{37}$$

subject to the initial condition

$$U(x,0) = e^{ix}. \tag{38}$$

By applying (21)–(23), we have

$$\begin{aligned} & i\sum_{n=0}^{\infty} P^n U_n(x,t) \\ &= iU_0(x,t) - PS^{-1} \\ &\quad \times \left(u^{\alpha} \left(S\left(\frac{\partial^{\beta}}{\partial x^{\beta}} \sum_{n=0}^{\infty} P^n U_n(x,t) + 2\sum_{n=0}^{\infty} P^n H_n(U(x,t)) \right) \right) \right), \end{aligned}$$

$$P^0 : U_0(x,t) = e^{ix},$$

$$\begin{aligned} P^1 : U_1(x,t) &= iS^{-1}\left(u^{\alpha} S\left(\frac{\partial^{\beta} U_0}{\partial x^{\beta}} + 2U_0{}^2\overline{U_0} \right) \right) \\ &= iS^{-1}\left(u^{\alpha} S\left(\frac{\partial^{\beta} e^{ix}}{\partial x^{\beta}} + 2e^{2ix}e^{-ix} \right) \right) \\ &= \frac{-it^{\alpha}}{\Gamma(\alpha+1)} \left(E(x, 2-\beta, i) - 2e^{ix} \right), \end{aligned}$$

$$\begin{aligned} & P^2 : U_2(x,t) \\ &= iS^{-1}\left(u^{\alpha} S\left(\frac{\partial^{\beta} U_1}{\partial x^{\beta}} + 2\left(2U_0 U_1 \overline{U_0} + U_0{}^2\overline{U_1}\right) \right) \right) \end{aligned}$$

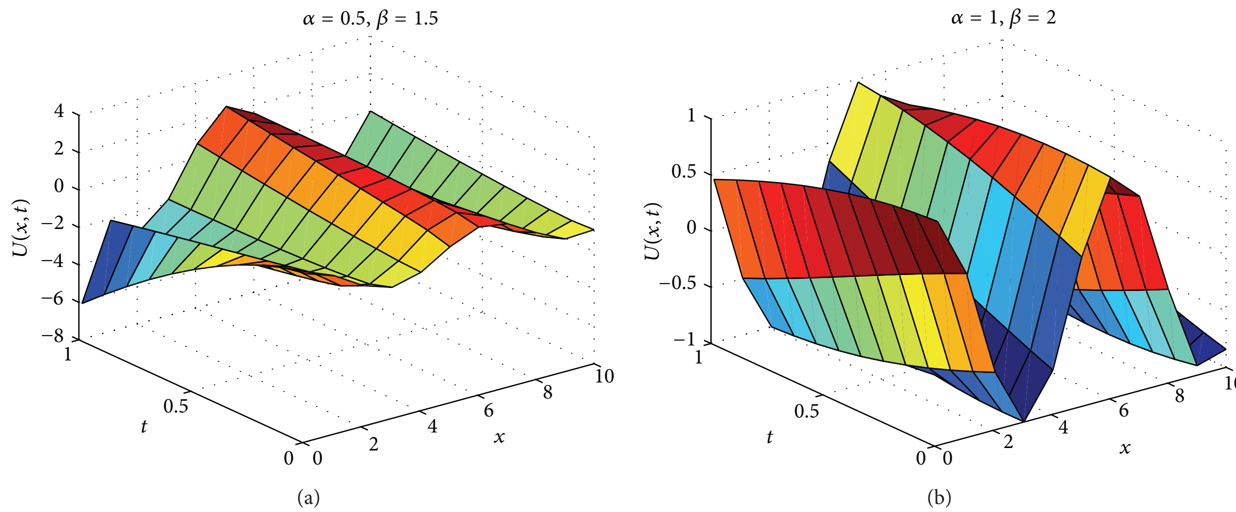


FIGURE 2: The surface plot of the real part of the solution $U(x,t)$ of Example 6: (a) $\alpha = 0.5$, $\beta = 1.5$, (b) $\alpha = 1$, $\beta = 2$.

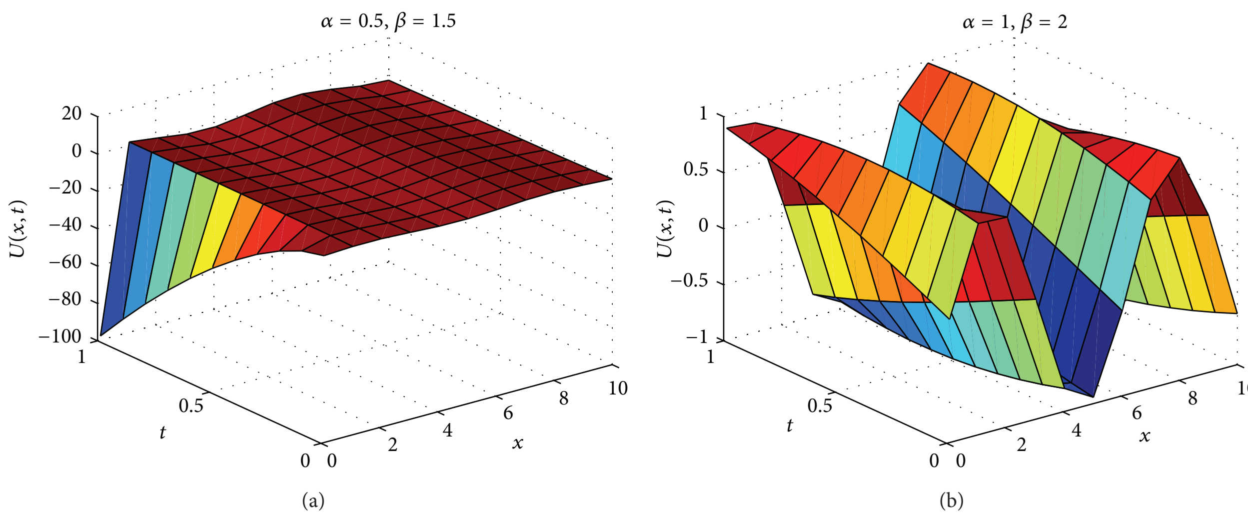


FIGURE 3: The surface plot of the imaginary part of the solution $U(x,t)$ of Example 6: (a) $\alpha = 0.5$, $\beta = 1.5$, (b) $\alpha = 1$, $\beta = 2$.

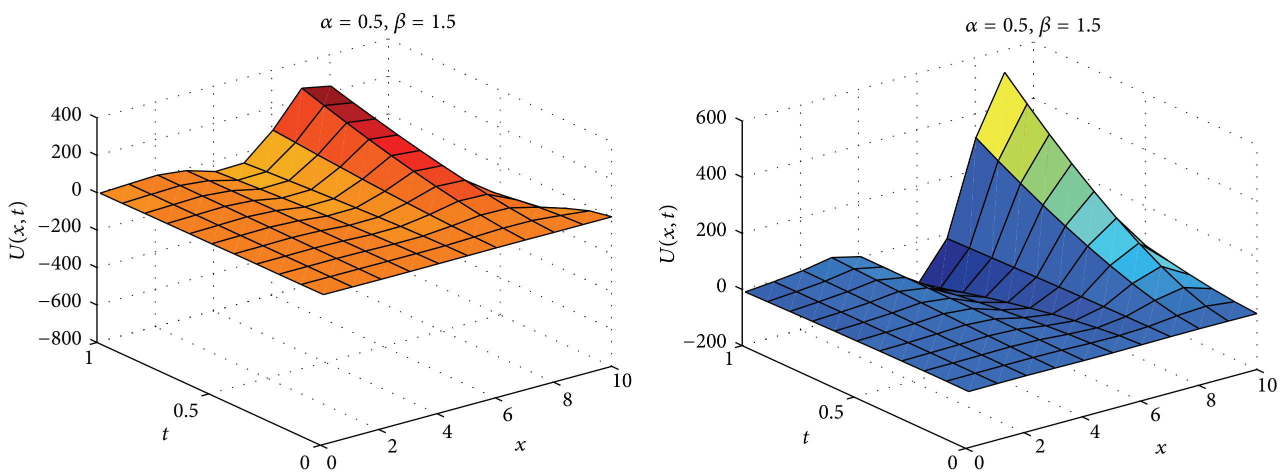
$$
\begin{aligned}
&= iS^{-1}\left(u^{\alpha}S\left(D_x^{\beta}\left(\frac{it^{\alpha}}{\Gamma(\alpha+1)}\left(-E(x,2-\beta,i)+2e^{ix}\right)\right)\right.\right.\\
&\qquad\left.\left. +2\left(2e^{ix}\left(\frac{it^{\alpha}}{\Gamma(\alpha+1)}\left(-E(x,2-\beta,i)+2e^{ix}\right)\right)\right.\right.\right.\\
&\qquad\left.\left.\left. \times\ e^{-ix}+e^{2ix}\overline{U_1}\right)\right)\right)\\
&= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(E(x,2-2\beta,i)+6E(x,2-\beta,i)\right.\\
&\qquad \left. -2e^{2ix}E(x,2-\beta,-i)-4e^{ix}\right).\\
&\qquad\vdots
\end{aligned}
\tag{39}
$$

Substituting $U_0(x,t), U_1(x,t), U_2(x,t), \ldots$ into (24), we obtain the following approximate solution to (37) and (38):

$$
\begin{aligned}
U_{\text{app}}(x,t) &= e^{ix} - \frac{it^{\alpha}}{\Gamma(\alpha+1)}\left(E(x,2-\beta,i)-2e^{ix}\right)\\
&\quad + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left(E(x,2-2\beta,i)+6E(x,2-\beta,i)\right.\\
&\qquad \left. -2e^{2ix}E(x,2-\beta,-i)-4e^{ix}\right)\\
&\quad + \cdots .
\end{aligned}
\tag{40}
$$

Remark 10. If $\alpha \to 1$ and $\beta \to 2$, then the solution given by (40) converged to $U_{\text{clas}}(x,t) = e^{i(x+t)}$, which is the solution of the classical equation.

Figure 4 shows the surface plot of the approximate solution of Example 9 for the values of $\alpha = 0.5$, $\beta = 1.5$. In



(a) (b)

FIGURE 4: The surface plot of the solution $U_{\text{app}}(x,t)$ of Example 9 for the values of $\alpha = 0.5$, $\beta = 1.5$. (a) The real part of the solution and (b) the imaginary parts of the solution.

Figure 4(a) the graph is given for the real part of the solution and in Figure 4(b) it is given for the imaginary part of the solution.

Table 1 shows the comparison of the absolute approximate solution of Example 9 between the homotopy analysis method [29], the homotopy perturbation Sumudu transform used in this paper, and the exact solution. It shows that an approximate solution obtained by HPSTM is in perfect agreement with the exact solution for $\alpha = 1$, $\beta = 2$; it is noted that the absolute error between HPSTM and the exact solution is sufficiently small compared with the homotopy analysis method. Furthermore, the approximate solutions for the values of $\alpha = 0.9$, $\beta = 1.8$ obtained by the HPSTM and homotopy analysis method are compared. Clearly, the results obtained by HPSTM are closer to the exact solution than the homotopy analysis method.

Example 11. Consider the space-time fractional nonlinear Schrödinger equation with nonzero trapping potential:

$$iD_t^{\alpha}U + \frac{1}{2}D_x^{\beta}U - U\cos^2 x - \left|U^2\right|U = 0, \qquad 0 < \alpha \le 1, \quad 1 < \beta \le 2, \tag{41}$$

subject to the initial condition

$$U(x,0) = \sin x. \tag{42}$$

By using (21)–(23), we get

$$\begin{aligned}
i\sum_{n=0}^{\infty} P^n U_n(x,t) &= iU_0(x,t) - PS^{-1} \\
&\quad \times \left(u^{\alpha}\left(S\left(\frac{1}{2}\frac{\partial^{\beta}}{\partial x^{\beta}}\sum_{n=0}^{\infty} P^n U_n(x,t)\right.\right.\right. \\
&\qquad - \cos^2 x \sum_{n=0}^{\infty} P^n U_n(x,t) \\
&\qquad \left.\left.\left. - \sum_{n=0}^{\infty} P^n H_n(U(x,t))\right)\right)\right),
\end{aligned}$$

$$\begin{aligned}
P^0 &: U_0(x,t) = \sin x, \\
P^1 &: U_1(x,t) \\
&= iS^{-1}\left(u^{\alpha}S\left(\frac{1}{2}\frac{\partial^{\beta}U_0}{\partial x^{\beta}} - U_0\cos^2 x - H_0(U(x,t))\right)\right) \\
&= iS^{-1}\left(u^{\alpha}S\left(\frac{1}{2}\frac{\partial^{\beta}}{\partial x^{\beta}}\sin x - \sin x\cos^2 x - \sin^3 x\right)\right) \\
&= \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad \times \left(\left(\frac{i}{2}\right)^2 \left(E(x, 2-\beta, i) - E(x, 2-\beta, -i)\right) - i\sin x\right), \\
P^2 &: U_2(x,t) \\
&= iS^{-1}\left(u^{\alpha}S\left(\frac{1}{2}\frac{\partial^{\beta}U_1}{\partial x^{\beta}} - U_1\cos^2 x - H_1(U(x,t))\right)\right),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^{\beta}U_1}{\partial x^{\beta}} - U_1\cos^2 x \\
&\quad = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad \times \left(\left(\left(\frac{i}{2}\right)^2 \left(E(x, 2-\beta, i) - E(x, 2-\beta, -i)\right) - i\sin x\right)\right.
\end{aligned}$$

Table 1: Compare the absolute solution between the homotopy analysis method, the homotopy perturbation Sumudu transform, and the exact solution of Example 9 for $0 < x < 1$, $0 < t < 1$.

$\alpha = 0.9,\ \beta = 1.8$		$\alpha = 1, \beta = 2$				
U_{HAM}	U_{HPSTM}	U_{HAM}	U_{HPSTM}	U_{clas}	$\text{abs}\lvert U_{\text{clas}} - U_{\text{HAM}}\rvert$	$\text{abs}\lvert U_{\text{clas}} - U_{\text{HPSTM}}\rvert$
1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000
1.0087	0.9682	1.0002	1.0000	1.0000	0.0002	0.0000
1.0159	0.9528	1.0032	1.0002	1.0000	0.0032	0.0002
1.0323	0.9539	1.0161	1.0010	1.0000	0.0161	0.0010
1.0718	0.9614	1.0500	1.0032	1.0000	0.0500	0.0032
1.1508	0.9658	1.1180	1.0078	1.0000	0.1180	0.0078
1.2840	0.9589	1.2322	1.0161	1.0000	0.2322	0.0161
1.4812	0.9338	1.4001	1.0296	1.0000	0.4001	0.0296
1.7462	0.8847	1.6243	1.0500	1.0000	0.6243	0.0500
2.0795	0.8074	1.9038	1.0789	1.0000	0.9038	0.0789
2.4798	0.6986	2.2361	1.1180	1.0000	1.2361	0.1180

$$-\left(\frac{i}{2}\right)^2 \left(E\left(x,2-\beta,i\right) - E\left(x,2-\beta,-i\right)\right) \times \sin^2 x + i\sin^3 x\Bigg), \tag{44}$$

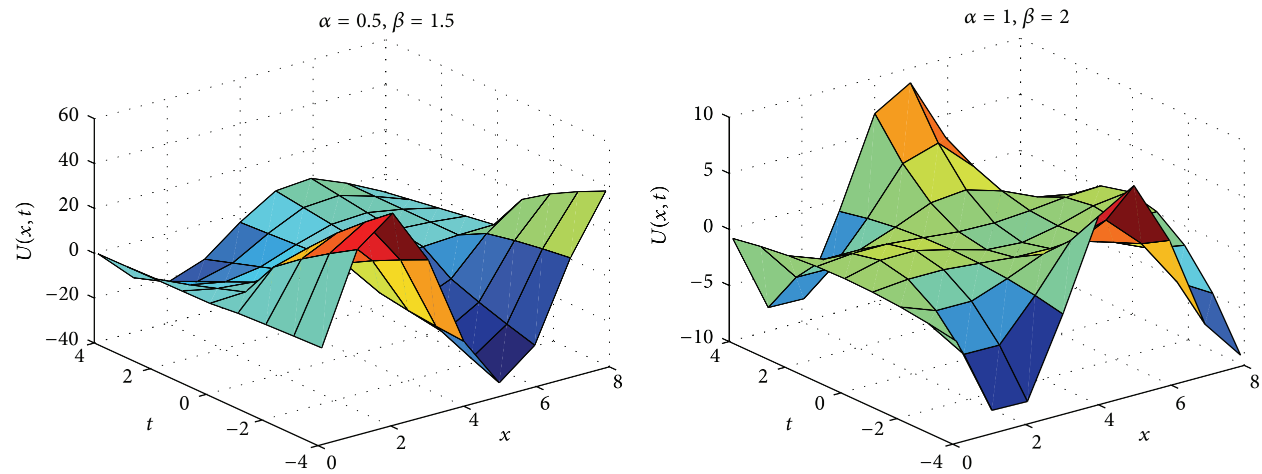
$$\begin{aligned}
H_1\left(U\left(x,t\right)\right) &= 2U_0U_1\overline{U_0} + {U_0}^2\overline{U_1} \\
&= 2\left(\sin x\left(\frac{t^\alpha}{\Gamma\left(\alpha+1\right)} \times\left(\left(\frac{i}{2}\right)^2\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right) - i\sin x\right)\right)\sin x\right) \\
&\quad - \sin^2 x\left(\frac{t^\alpha}{\Gamma\left(\alpha+1\right)} \times\left(\left(\frac{i}{2}\right)^2\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right) - i\sin x\right)\right) \\
&= \frac{t^\alpha}{\Gamma\left(\alpha+1\right)}\left(\left(\frac{i}{2}\right)^2\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right) \times \sin^2 x - i\sin^3 x\right).
\end{aligned} \tag{45}$$

Substituting (44) and (45) into (43), we obtain

$$\begin{aligned}
U_2\left(x,t\right) = \frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)} &\times\left(\left(\frac{i}{2}\right)^3\left(E\left(x,2-2\beta,i\right)-E\left(x,2-2\beta,-i\right)\right)\right. \\
&+\left(\left(\frac{i}{2}\right)\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right)\right) \\
&\left. - \sin x\right). \\
&\vdots
\end{aligned} \tag{46}$$

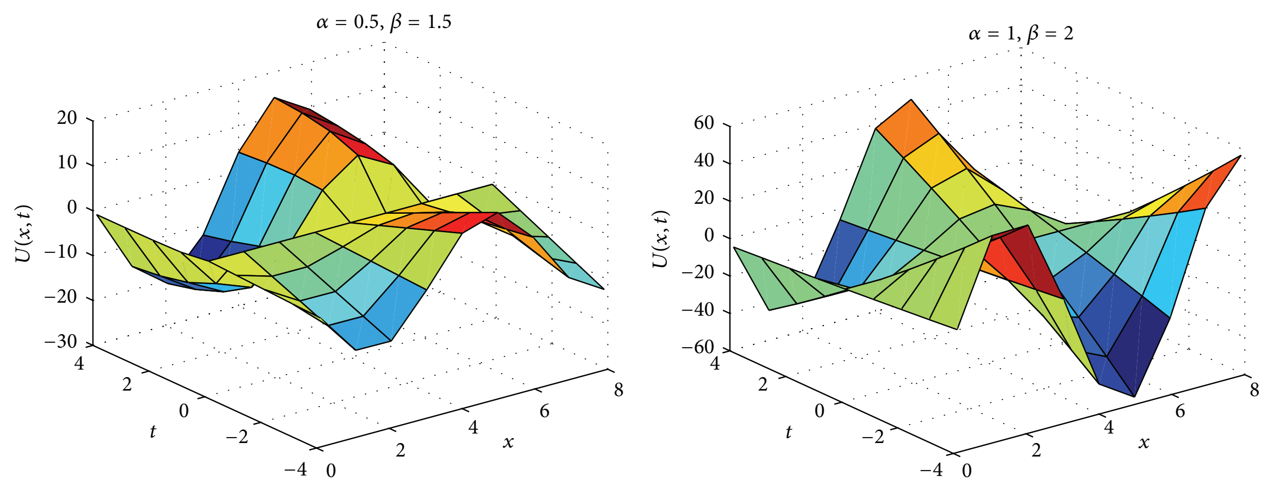
Substituting $U_0(x,t), U_1(x,t), U_2(x,t), \ldots$ into (24), we obtain the following approximate solution to (41) and (42):

$$\begin{aligned}
U_{\text{app}}\left(x,t\right) &= \sin x + \frac{t^\alpha}{\Gamma\left(\alpha+1\right)} \times\left(\left(\frac{i}{2}\right)^2\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right) - i\sin x\right) \\
&\quad + \frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)} \times\left(\left(\frac{i}{2}\right)^3\left(E\left(x,2-2\beta,i\right)-E\left(x,2-2\beta,-i\right)\right)\right. \\
&\qquad \left. +\left(\left(\frac{i}{2}\right)\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right)\right) - \sin x\right) \\
&\quad + \cdots \\
&= \sin x\left[1 + \frac{\left(-it^\alpha\right)}{\Gamma\left(\alpha+1\right)} + \frac{\left(-it^\alpha\right)^2}{\Gamma\left(2\alpha+1\right)} + \cdots\right] \\
&\quad + \left[\frac{t^\alpha}{\Gamma\left(\alpha+1\right)}\left(\left(\frac{i}{2}\right)^2\left(E\left(x,2-\beta,i\right)-E\left(x,2-\beta,-i\right)\right)\right)\right. \\
&\qquad + \frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}
\end{aligned}$$



(a) (b)

Figure 5: The surface plot of the real part of the approximate solution of Example 11 (a) for $\alpha = 0.5$, $\beta = 1.5$ and (b) for the exact solution $\alpha = 1$, $\beta = 2$.



(a) (b)

Figure 6: The surface plot of the imaginary part of the approximate solution of Example 11 (a) for $\alpha = 0.5$, $\beta = 1.5$ and (b) for the exact solution $\alpha = 1$, $\beta = 2$.

$$\times \left(\left(\frac{i}{2} \right)^3 \left(E\left(x, 2-2\beta, i\right) - E\left(x, 2-2\beta, -i\right) \right) \right. \\ \left. + \left(\left(\frac{i}{2} \right) \left(E\left(x, 2-\beta, i\right) - E\left(x, 2-\beta, -i\right) \right) \right) \right) + \cdots \Bigg]. \tag{47}$$

Figure 5 compares the surface plot of the real part of the approximate solution of Example 11. In Figure 5(a) we have the solution obtained for the values of $\alpha = 0.5$, $\beta = 1.5$ and in Figure 5(b) for the exact solution in case $\alpha = 1$, $\beta = 2$. Figure 6 compares the surface plot of the imaginary part of the approximate solution of Example 11. In Figure 6(a) we have the solution obtained for the values of $\alpha = 0.5$, $\beta = 1.5$ and in Figure 6(b) for the exact solution in case $\alpha = 1$, $\beta = 2$.

7. Conclusion and Discussion

Homotopy perturbation Sumudu transform method is applied successfully for finding exact solutions for linear space-time fractional Schrödinger equation and approximate solutions for nonlinear fractional Schrödinger equation with space and time fractional derivatives that are considered in the Caputo sense. We have demonstrated the efficiency of this method by four numerical expository examples for a variety of linear and nonlinear space-time fractional Schrödinger equations with zero and nonzero trapping potential. Example 4 is a time fractional nonlinear Schrödinger equation, in which we readily obtained the exact solution in a compact form. Furthermore, the solution is given for the values of $\alpha = 0.5$ and $\alpha = 1$, which are illustrated graphically in Figure 1. Example 6 is a linear space-time fractional Schrödinger equation with zero trapping potential, in which we obtained the solution in terms of Mittag-Leffler function; besides, the real and imaginary parts of the solution for the value of $\alpha = 0.5$, $\beta = 1.5$ are compared with classical solution graphically in Figures 2 and 3, respectively. In Example 9, the solution of a nonlinear space-time fractional Schrödinger equation with zero trapping potential is approximated and the comparison of the absolute approximate solutions between the homotopy analysis method [29], the homotopy perturbation Sumudu transform method used in this paper, and the exact solution is given in Table 1. It is shown that, for a sufficiently small number of components, the approximate solution given by homotopy perturbation Sumudu transform method becomes nearly more identical to the exact solution than the homotopy analysis method [29]. Example 11 calculated the approximate analytical solution of a nonlinear space-time fractional Schrödinger equation with nonzero trapping potential. To the best of our knowledge, the approximate solution for nonlinear space-time fractional Schrödinger equation with nonzero trapping potential has not been reported in the literature by using the homotopy perturbation method, the Adomian decomposition method, the variational iteration method, and the differential transform method. In conclusion, homotopy perturbation Sumudu transform method is reliable, effective, and easy to implement and produces accurate results. Thus, the method can be applied to solve other nonlinear fractional partial differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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