

## **Research Article**

# Integral $\varphi_0$ -Stability in terms of Two Measures for Impulsive Differential Equations with "Supremum"

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This paper establishes a criterion on integral  $\varphi_0$ -stability in terms of two measures for impulsive differential equations with "supremum" by using the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. Meantime, an example is given to illustrate our result.

#### 1. Introduction

In this paper, we discuss the integral  $\varphi_0$ -stability in terms of two measures for impulsive differential equations with "supremum":

$$x' = F\left(t, x(t), \sup_{s \in [t-r,t]} x(s)\right) \text{ for } t \ge 0, \ t \ne \tau_k,$$
  

$$x(\tau_k + 0) = I_k(x(\tau_k - 0)) \text{ for } k = 1, 2, \dots,$$
  

$$x(t) = \phi(t), \quad t \in [t_0 - r, t_0],$$
(1)

and its perturbed impulsive differential equations with "supremum"

$$x' = F\left(t, x(t), \sup_{s \in [t-r,t]} x(s)\right) + G\left(t, x(t), \sup_{s \in [t-r,t]} x(s)\right)$$
  
for  $t \ge 0$ ,  $t \ne \tau_k$ ,  
 $x(\tau_k + 0) = I_k(x(\tau_k - 0)) + J_k(x(\tau_k - 0))$   
for  $k = 1, 2, ...,$   
 $x(t) = \phi(t), \quad t \in [t_0 - r, t_0],$   
(2)

where  $x \in \mathbb{R}^n$ ,  $F, G : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ ,  $F(t, 0, 0) = G(t, 0, 0) \equiv 0$ ,  $I_k, J_k : \mathbb{R}^n \to \mathbb{R}^n$ ,  $I_k(0) = J_k(0) \equiv 0$ , k = 1, 2, ..., r > 0,  $t_0 \in \mathbb{R}^+$ , and  $\phi \in (PC[t_0 - r, t_0], \mathbb{R}^n)$ . Let  $\mathbb{R}^n$  be *n*-dimensional Euclidean space with norm ||x||,  $\mathbb{R}^+ = [0, \infty)$ , and  $\{\tau_k\}_1^\infty$  a sequence of fixed points in  $\mathbb{R}^+$  such that  $\tau_{k+1} > \tau_k$  and  $\lim_{k\to\infty} \tau_k = \infty$ . We denote by  $x(t; t_0, \phi)$  the solution of (1). In our further investigation we will assume that solution  $x(t; t_0, \phi)$  is defined on  $[t_0 - r, \infty)$  for any initial function  $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ .

The research on impulsive differential equations with "supremum" problem, Bainov et al. [1] justified the partial averaging for impulsive differential equations, He et al. [2] discussed the periodic boundary value problem for first order impulsive differential equations, Agarwal and Hristova [3] studied the strict stability in terms of two measures for impulsive differential equations, Stamova and Stamov [4] investigated the global stability of models based on impulsive differential equations, and Hristova [5, 6] obtained the  $\varphi_0$ -stability in terms of two measures for measures for impulsive differential equations.

In recent years, the integral stability theory has been rapid development (see [7–12]). For example, Soliman and Abdalla [10] introduced integral  $\varphi_0$ -stability of perturbed system of ordinary differential equations. Hristova [12] studied the integral stability in terms of two measures for impulsive differential equations with "supremum." However, the corresponding theory of impulsive differential equations with

"supremum" is still at an initial stage of its development, especially for integral  $\varphi_0$ -stability in terms of two measures. Motivated by the idea of [5, 6, 10, 12], in this work, by employing the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method, we extend the notions of  $\varphi_0$ -stability in terms of two measures to integral  $\varphi_0$ -stability in terms of two measures for impulsive differential equations with "supremum."

#### 2. Preliminaries

Denote by PC(X, Y)  $(X \in R, Y \in R^n)$  the set of all functions  $u : X \to Y$  which are piecewise continuous in X with points of discontinuity of the first kind at the points  $\tau_k \in X$  and which are continuous from the left at the points  $\tau_k \in X$ ,  $u(\tau_k) = u(\tau_k - 0)$ .

We denote by  $PC^{1}(X, Y)$  the set of all function  $u \in PC(X, Y)$  which are continuously differentiable for  $t \in X$ ,  $t \neq \tau_{k}$ .

Let  $x, y \in \mathbb{R}^n$ . Denote by  $(x \cdot y)$  the dot product of both vectors x and y.

Let  $\mathscr{K} \subset \mathbb{R}^n$  be a cone, and  $\mathscr{K}^* = \{ \varphi \in \mathbb{R}^n : (\varphi \cdot x) \ge 0 \text{ for any } x \in \mathscr{K} \}$  is adjoint cone.

We give the following notations for convenience:

$$K = \{a \in C(R^+, R^+):$$
  

$$a(s) \text{ is strictly increasing, } a(0) = 0\};$$
  

$$CK = \{b \in C[R^+ \times R^+, R^+]:$$

$$b(t, \cdot) \in K$$
 for any fixed  $t \in [0, \infty)$ ; (3)

$$\Gamma = \left\{ h \in C\left[ [-r, \infty) \times \mathbb{R}^n, \mathscr{H} \right]:$$
$$\inf_{x \in \mathbb{R}^n} h\left( t, x \right) = 0 \text{ for each } t \in [-r, \infty) \right\}.$$

Let  $h_0, h \in \Gamma$ ,  $\varphi_0 \in \mathscr{K}^*$ ,  $t \in \mathbb{R}^+$ , and  $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ . Define

$$H_0(t,\phi,\varphi_0) = \sup\left\{\left(\varphi_0 \cdot h_0(t+s,\phi(t+s))\right) : s \in [-r,0]\right\},$$
(4)

$$H(t,\phi,\varphi_0) = \sup\left\{\left(\varphi_0 \cdot h_0\left(t+s,\phi\left(t+s\right)\right)\right) : s \in [-r,0]\right\}.$$
(5)

Let  $\rho, t$  and T > 0 be constants,  $\varphi_0 \in \mathscr{K}^*, h \in \Gamma$ . Define sets:

$$S(h, \rho, \varphi_0) = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : (\varphi_0 \cdot h(t, x)) < \rho\};$$

$$S^c(h, \rho, \varphi_0) = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : (\varphi_0 \cdot h(t, x)) \ge \rho\};$$

$$\Omega(t, T, \rho) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n:$$

$$(\varphi_0 \cdot h(t, x)) < \rho \text{ for } s \in [t, t + T],$$

$$(\varphi_0 \cdot h(t, y)) < \rho \text{ for } s \in [t - r, t + T]\}.$$
(6)

In our further investigations we use the following comparison scalar impulsive ordinary differential equation:

$$u' = g_{1}(t, u), \quad t \neq \tau_{k},$$

$$u(\tau_{k} + 0) = \xi_{k}(u(\tau_{k})),$$

$$u(t_{0}) = u_{0},$$

$$k = 1, 2, ...,$$
(7)

the scalar impulsive ordinary differential equation:

$$w' = g_2(t, w), \quad t \neq \tau_k,$$
  

$$w(\tau_k + 0) = \eta_k(w(\tau_k)),$$
  

$$w(t_0) = w_0,$$
  

$$k = 1, 2, \dots,$$
  
(8)

and its perturbed scalar impulsive ordinary differential equation:

$$w' = g_{2}(t, w) + q(t), \quad t \neq \tau_{k},$$
  

$$w(\tau_{k} + 0) = \eta_{k}(w(\tau_{k})) + \gamma_{k}(w(\tau_{k})),$$
  

$$w(t_{0}) = w_{0},$$
  

$$k = 1, 2, ...,$$
(9)

where  $u, w \in R$ ,  $g_1(t, 0) = g_2(t, 0) \equiv 0$ ,  $\xi_k(0) = 0$ ,  $\eta_k(0) = 0$ , k = 1, 2, ...

Assume that solutions of the scalar impulsive equations (7), (8), and (9) exist on  $[t_0, \infty)$  for any initial values. Meanwhile, we give some definitions and lemmas. The details can be found in [5].

Definition 1 (see [5]). We say that function  $V(t, x) : [-r, \infty) \times R^n \to \mathcal{K}, V = (V_1, V_2, \dots, V_n)$ , belongs to the class  $\Lambda$  if

(A1) 
$$V(t, x) \in PC^{1}([-r, \infty) \times \mathbb{R}^{n}, \mathscr{K});$$

(A2) for each k = 1, 2, ... and  $x \in \mathbb{R}^n$  there exist the finite limits

$$V(\tau_{k}-0,x) = \lim_{t\uparrow\tau_{k}} V(t,x), \qquad V(\tau_{k}+0,x) = \lim_{t\downarrow\tau_{k}} V(t,x);$$
(10)

(A3) there exist constants  $M_i > 0$ , i = 1, 2, ..., n, such that  $|V_i(t, x) - V_i(t, y)| \le M_i ||x - y||$  for any  $t \in \mathbb{R}^+$ ,  $x, y \in \mathbb{R}^n$ .

Definition 2 (see [5]). Let  $\varphi_0 \in \mathcal{K}^*$ ,  $h \in \Gamma$  be given. The function  $V(t, x) \in \Lambda$  is said to be  $\varphi_0$ -strongly *h*-decrescent if there exist a constant  $\delta > 0$  and a function  $a \in K$  such that  $(t, x) \in [-r, \infty) \times \mathbb{R}^n : (\varphi_0 \cdot h(t, x)) < \delta$  implies that  $(\varphi_0 \cdot V(t, x)) \leq a(\varphi_0 \cdot h(t, x)).$ 

Let  $V(t, x) \in \Lambda$ ,  $t \in \Omega$ ,  $t \neq \tau_k$ ,  $x \in \mathbb{R}^n$ , and  $\phi \in PC([t - r, t], \mathbb{R}^n)$ . We define a derivative of the function V(t, x) along the trajectory of solution of (1) as follows:

$$D_{(1)}V(t,\phi(t)) = \lim_{\epsilon \to 0} \sup \frac{1}{\epsilon} \left\{ V\left(t + \epsilon, \phi(t) + \epsilon F\left(t, \phi(t), \sup_{s \in [-r,0]} \phi(t + s)\right)\right) - V(t, \phi(t)) \right\}.$$
(11)

Similarly we define a derivative of the function  $V(t, x) \in \Lambda$  along the trajectory of solution of the perturbed system (2) for  $t \in \Omega$ ,  $t \neq \tau_k$ ,  $x \in \mathbb{R}^n$ , and  $\phi \in PC([t-r,t],\mathbb{R}^n)$  as follows:

$$D_{(2)}V(t,\phi(t)) = \lim_{\epsilon \to 0} \sup \frac{1}{\epsilon} \left\{ V\left(t+\epsilon,\phi(t) + \epsilon\left(F\left(t,\phi(t),\sup_{s\in[-r,0]}\phi(t+s)\right)\right) + G\left(t,\phi(t),\sup_{s\in[-r,0]}\phi(t+s)\right)\right) \right\} - V(t,\phi(t)) \right\}.$$
(12)

Definition 3 (see [5]). Let  $\varphi_0 \in \mathcal{K}^*$ ,  $h, h_0 \in \Gamma$  be given. The function  $h_0$  is  $\varphi_0$ -uniformly finer than h if there exist a constant  $\delta > 0$  and a function  $a \in K$ , such that for any point  $(t, x) \in [0, \infty) \times \mathbb{R}^n : (\varphi_0 \cdot h_0(t, x)) < \delta$  the inequality  $(\varphi_0 \cdot h(t, x)) \leq a(\varphi_0 \cdot h_0(t, x))$  holds.

**Lemma 4** (see [5]). Let  $h, h_0 \in \Gamma, \varphi_0 \in \mathcal{K}^*$  be given, and  $h_0(t, x)$  is  $\varphi_0$ -uniformly finer than h(t, x) with a constant  $\delta$  and a function  $a \in K$ . Then for any  $t \in \mathbb{R}^+$  and  $\phi \in PC([t - r, t], \mathbb{R}^n)$  inequality  $H_0(t, \phi, \varphi_0) < \delta$  implies  $H(t, \phi, \varphi_0) \leq a(H_0(t, \phi, \varphi_0))$ , where functions H and  $H_0$  are defined by (4), (5).

In our further investigations we use the following comparison result.

**Lemma 5** (see [5]). Let the following conditions be fulfilled.

- (B1) The vector  $\varphi_0 \in \mathscr{K}^*$  and function  $V \in \Lambda$  are such that
  - (i) for any number  $t \ge 0$ :  $t \ne \tau_k$  and any function  $\psi \in PC([t-r,t], \mathbb{R}^n)$  such that  $(\varphi_0 \cdot V(t, \psi(t))) \ge$  $(\varphi_0 \cdot V(t+s, \psi(t+s)))$  for  $s \in [-r, 0)$  the inequality

$$\left(\varphi_{0} \cdot D_{(1)}V\left(t, \psi\left(t\right)\right)\right) \leq g_{1}\left(t, \left(\varphi_{0} \cdot V\left(t, \psi\left(t\right)\right)\right)\right)$$
(13)

holds, where 
$$g_1 \in PC(R^+ \times R^+, R^+)$$
.

- (ii)  $(\varphi_0 \cdot V(\tau_k + 0, I_k(x))) \leq \xi_k(\varphi_0 \cdot V(\tau_k, x)), k = 1, 2, \dots, x \in \mathbb{R}^n$ , and  $\tau_k \in [t_0, T]$ , where functions  $\xi_k \in K$ .
- (B2) Function  $x(t;t_0,\phi)$  is a solution of (1) that is defined for  $t \in [t_0 - r, T]$ , where  $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ .
- (B3) Function  $u^*(t) = u^*(t; t_0, u_0)$  is the maximal solution of (7) with initial condition  $u^*(t_0) = u_0$  that is defined for  $t \in [t_0, T]$ .

Then the inequality  $\sup_{s \in [-r,0]} (\varphi_0 \cdot V(t_0 + s, \phi(t_0 + s))) \le u_0$ implies the validity of the inequality  $(\varphi_0 \cdot V(t, x(t))) \le u^*(t)$ for  $t \in [t_0, T]$ .

Definition 6. Let  $h_0, h \in \Gamma$ . System of impulsive differential equations with "supremum" (1) is said to be

(S1)  $(H_0, h)$ -equi-integral  $\varphi_0$ -stable if for every  $\alpha \ge 0$  and for any  $t_0 \ge 0$  there exists a positive function  $\beta = \beta(t_0, \alpha) \in CK$  which is continuous in  $t_0$  for each  $\alpha$ and such that for maximal solution  $y^*(t; t_0, \phi)$  of the perturbed system of impulsive differential equations with "supremum" (2) the inequality

$$\left(\varphi_0 \cdot h\left(t, y^*\left(t; t_0, \phi\right)\right)\right) < \beta, \quad t \ge t_0 \tag{14}$$

holds, provided that

$$H_0\left(t_0,\phi,\varphi_0\right) \le \alpha,\tag{15}$$

and for every T > 0,

$$\int_{t_0}^{t_0+T} \sup_{(x,y)\in\Omega(t_0,T,\beta)} \left\| G\left(s,x,y^*\right) \right\| ds$$

$$+ \sum_{t_0 \leq \tau_k \leq t_0+Tx: h(\tau_k,x) < \beta} \left\| J_k\left(x\right) \right\| \leq \alpha,$$
(16)

where  $H_0(t_0, \phi, \varphi_0)$  is defined by (4) and  $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ ;

(S2)  $(H_0, h)$ -uniform-integrally  $\varphi_0$ -stable if (S1) is satisfied, where  $\delta$  is independent on  $t_0$ .

*Remark 7.* We note that in the case when  $h_0(t, x) \equiv ||x||$  and  $h(t, x) \equiv ||x||$  the  $(H_0, h)$ -equi-integral (uniform-integral)  $\varphi_0$ -stability reduces to equi-integral (uniform-integral)  $\varphi_0$ -stability.

#### 3. Main Result

**Theorem 8.** Let the following conditions be fulfilled.

- (H1) Functions  $h_0, h \in \Gamma$ ;  $h_0$  is  $\varphi_0$ -uniformly finer than h.
- (H2) There exists a function  $V_1 \in \Lambda$  that is  $\varphi_0$ -strongly  $h_0$ -decrescent and
  - (i) for any number  $t \ge 0$ ,  $t \ne \tau_k$ , and any function  $\psi \in PC([t-r,t], \mathbb{R}^n)$ , such that  $(\varphi_0 \cdot V_1(t, \psi(t))) >$   $(\varphi_0 \cdot V_1(t + s, \psi(t + s)))$  for  $s \in [-r, 0)$  and  $(t, \psi(t)) \in S(h, \rho, \varphi_0)$  the inequality

$$(\varphi_0 \cdot D_{(1)}V_1(t, \psi(t))) \le g_1(t, (\varphi_0 \cdot V_1(t, \psi(t))))$$
(17)

*holds, where* 
$$\rho > 0$$
 *is a constant.*

- (ii)  $(\varphi_0 \cdot V_1(\tau_k + 0, I_k(x))) \le \xi_k(\varphi_0 \cdot V_1(\tau_k, x)), \text{ for } (\tau_k, x) \in S(h, \rho, \varphi_0), k = 1, 2, \dots$
- (H3) For any number  $\mu > 0$  there exists a function  $V_2^{(\mu)} \in \Lambda$  such that
  - (iii)  $b(\varphi_0 \cdot h(t, x)) \leq (\varphi_0 \cdot V_2^{(\mu)}(t, x)) \leq a(\varphi_0 \cdot h_0(t, x))$ for  $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ , where  $a, b \in K$  and  $\lim_{u \to \infty} b(u) = \infty$ .
  - (iv) For any number  $t \ge 0$ ,  $t \ne \tau_k$ , and any function  $\psi \in PC([t - r, t], \mathbb{R}^n)$ , such that  $(t, \psi(t)) \in S(h, \rho, \varphi_0) \cap S^c(h_0, \mu, \varphi_0)$  and  $(\varphi_0 \cdot (V_1(t, \psi(t)) + V_2^{(\mu)}(t, \psi(t)))) > (\varphi_0 \cdot (V_1(t + s, \psi(t + s)) + V_2^{(\mu)}(t + s, \psi(t + s))))$  for  $s \in [-r, 0)$  the inequality

$$\left( \varphi_0 \cdot \left( D_{(1)} V_1(t, \psi(t)) + D_{(2)} V_2^{(\mu)}(t, \psi(t)) \right) \right)$$

$$\leq g_2 \left( t, \varphi_0 \cdot \left( V_1(t, \psi(t)) + V_2^{(\mu)}(t, \psi(t)) \right) \right)$$
(18)

holds.

- $\begin{array}{l} (\mathrm{v}) \ (\varphi_0 \cdot (V_1(\tau_k + 0, I_k(x)) + V_2^{(\mu)}(\tau_k + 0, I_k(x)))) \leq \\ \eta_k(\varphi_0 \cdot (V_1(\tau_k, x) + V_2^{(\mu)}(\tau_k, x))) \ for \ (\tau_k, x) \ \in \\ S(h, \rho, \varphi_0) \cap S^c(h_0, \mu, \varphi_0), \ k = 1, 2, \dots \end{array}$
- (H4) Zero solution of the scalar impulsive differential equation (7) is equi-stable.
- (H5) Zero solution of the scalar impulsive differential equation (8) is uniform-integrally stable.

Then system of impulsive differential equations with "supremum" (1) is  $(H_0, h)$ -uniform-integrally  $\varphi_0$ -stable.

*Proof.* Since function  $V_1(t, x)$  is  $\varphi_0$ -strongly  $h_0$ -decrescent, there exist a constant  $\rho_1 \in (0, \rho)$  and a function  $\psi_1 \in K$  such that  $(\varphi_0 \cdot h_0(t, x)) < \rho_1$  implies that

$$\left(\varphi_0 \cdot V_1\left(t, x\right)\right) \le \psi_1\left(\left(\varphi_0 \cdot h_0\left(t, x\right)\right)\right). \tag{19}$$

Since  $h_0(t, x)$  is  $\varphi_0$ -uniformly finer than h(t, x), there exist a constant  $\rho_0 \in (0, \rho_1)$  and a function  $\psi_2 \in K$  such that  $(\varphi_0 \cdot h_0(t, x)) < \rho_0$  implies that

$$\left(\varphi_{0}\cdot h\left(t,x\right)\right) \leq \psi_{2}\left(\varphi_{0}\cdot h_{0}\left(t,x\right)\right),\tag{20}$$

where  $\psi_2(\rho_0) < \rho_1$ .

According to Lemma 4, the inequality  $H_0(t, \phi, \varphi_0) < \rho_0$ implies

$$H(t,\phi,\varphi_0) \le \psi_2(H_0(t,\phi,\varphi_0)), \quad \phi \in PC([t-r,t], \mathbb{R}^n).$$
(21)

Let  $t_0 \ge 0$  be a fixed point. Choose a number  $\alpha > 0$  such that  $\alpha < \rho_0$ .

According to condition (*H*3) of Theorem 8, there exists a function  $V_2^{(\alpha)}(t, x)$  that is Lipshitz with a constant  $M_2$ . Let  $M_1$  be the Lipshitz constant of function V(t, x).

Denote  $(M_1 + M_2)\alpha = \alpha_1$ . Without loss of generality we assume  $\alpha_1 < b(\rho)$ .

Since the zero solution of the scalar impulsive differential equation (7) is equi-stable, there exists a function  $\delta_1 = \delta_1(t_0, \alpha_1) > 0$  such that the inequality  $|u_0| < \delta_1$  implies

$$|u(t;t_0,u_0)| < \frac{\alpha_1}{2}, \quad t \ge t_0,$$
 (22)

where  $u(t; t_0, u_0)$  is a solution of (7).

Since the function  $\psi_1 \in K$  there exists a  $\delta_2 = \delta_2(\delta_1) > 0$ ,  $\delta_2 < \rho_1$ , such that for  $|u| < \delta_2$  the inequality

$$\psi_1(u) < \delta_1 \tag{23}$$

holds.

Since the zero solution of the scalar impulsive differential equation (8) is uniform-integrally stable, there exists a function  $\beta_1 = \beta_1(\alpha_1) \in CK$ ,  $b(\rho) > \beta_1 \ge \alpha_1$ , such that for every solution of the perturbed impulsive equation (9) the inequality

$$|w(t;t_0,w_0)| < \beta_1, \quad t \ge t_0,$$
 (24)

holds, provided that

$$|w_0| < \alpha_1 \tag{25}$$

and for every T > 0,

$$\int_{t_{0}}^{t_{0}+T} |q(s)| \, ds + \sum_{t_{0} \le \tau_{k} \le t_{0}+T} |\gamma_{k}| \le \alpha_{1}.$$
(26)

Since the function  $b \in K$ ,  $\lim_{s \to \infty} b(s) = \infty$ , and  $\psi_2(\alpha) < \psi_2(\rho_0) < \rho_1 < \rho$ , we could choose a constant  $\beta = \beta(\beta_1) > 0$ ,  $\rho > \beta > \alpha$ ,  $\beta > \psi_2(\alpha)$ , such that

$$b\left(\beta\right) \ge \beta_1. \tag{27}$$

Since the function  $a, \psi_2 \in K$ , and  $\beta > \psi_2(\alpha)$ , we can find a  $\delta_3 = \delta_3(\alpha_1, \beta) > 0$ ,  $\alpha < \delta_3 < \min(\delta_2, \rho_0)$ , such that the inequalities

$$a(\delta_3) < \frac{\alpha_1}{2}, \qquad \psi_2(\delta_3) < \beta$$
 (28)

hold.

From (21) and (28) it follows that  $H_0(t_0, \phi, \varphi_0) < \alpha$  implies

$$H(t_{0},\phi,\varphi_{0}) \leq \psi_{2}(H_{0}(t_{0},\phi,\varphi_{0})) < \psi_{2}(\alpha) < \psi_{2}(\delta_{3}) < \beta;$$
(29)

that is,  $h(t, \phi, \varphi_0) < \beta$  for  $t \in [t_0 - r, t_0]$ .

Now let the initial functions  $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$  be such that

$$H_0\left(t_0,\phi,\varphi_0\right) < \alpha \tag{30}$$

and let the perturbed functions in impulsive equation with "supremum" (2) be such that

$$\int_{t_0}^{t_0+T} \sup_{x,y \in \Omega(t_0,T,\beta)} \|G(s,x,y^*)\| ds$$
  
+ 
$$\sum_{t_0 \le \tau_k \le t_0+T} \|J_k(x)\| \le \alpha$$
 (31)

for every T > 0.

Let  $y^*(t) = y^*(t; t_0, \phi)$  be a solution of (2), where the initial function and the perturbed functions satisfy (30) and (31); then

$$\left(\varphi_0 \cdot h\left(t, y^*\left(t; t_0, \phi\right)\right)\right) < \beta, \quad t \ge t_0. \tag{32}$$

Suppose it is not true. There exists a point  $t^* > t_0$  such that

$$(\varphi_{0} \cdot h(t^{*}, y^{*}(t^{*}; t_{0}, \phi))) = \beta,$$
  

$$(\varphi_{0} \cdot h(t, y^{*}(t; t_{0}, \phi))) < \beta,$$
  

$$t \in [t_{0}, t^{*}).$$
(33)

*Case 1.* Let  $t^* \neq \tau_k$ , k = 1, 2, ... Then from the continuity of the maximal solution  $y^*(t; t_0, \phi)$  at point  $t^*$  follows that  $(\varphi_0 \cdot h(t^*, y^*(t^*; t_0, \phi))) = \beta$ .

If we assume that  $(\varphi_0 \cdot h_0(t^*, y^*(t^*))) \leq \delta_3$  then from the choice of  $\delta_3$  and inequality (28) it follows  $(\varphi_0 \cdot h(t^*, y^*(t^*; t_0, \phi))) \leq (\varphi_0 \cdot h_0(t^*, y^*(t^*; t_0, \phi))) \leq \psi_2(\delta_3) < \beta$  that contradicts (33).

Therefore

$$\left(\varphi_{0} \cdot h_{0}\left(t^{*}, y^{*}\left(t^{*}\right)\right)\right) \leq \delta_{3}, \qquad H_{0}\left(t_{0}, \phi, \varphi_{0}\right) < \alpha < \delta_{1}.$$
(34)

*Case 1.1.* Let there exist a point  $t_0^* \in (t_0, t^*)$ ,  $t_0^* \neq \tau_k$ , k = 1, 2, ..., such that  $\delta_3 = (\varphi_0 \cdot h_0(t^*, y^*(t^*)))$  and  $(t, y^*(t)) \in S(h, \beta, \varphi_0) \cap S^c(h_0, \delta_3, \varphi_0)$ . Since  $\beta < \rho$  and  $\delta_3 > \alpha$  it follows that

$$(t, y^{*}(t)) \in S(h, \rho, \varphi_{0}) \cap S^{c}(h_{0}, \alpha, \varphi_{0}), \quad t \in [t_{0}^{*}, t^{*}).$$
  
(35)

Define a function  $\phi^*(t) = y^*(t)$  for  $t \in [t_0^* - r, t_0^*]$ and let  $r_1(t; t_0^*, u_0)$  be the maximal solution of impulsive scalar differential equation (7) where  $u_0 = \sup_{s \in [-r,0]} (\varphi_0 \cdot V_1(t_0^*, \phi^*(t_0^*)))$ . Let  $x^*(t) \equiv x^*(t; t_0^*, \phi^*)$  be the solution of the impulsive equations (1),  $t \in [t_0^* - r, t_0^*]$ . From conditions (i), (ii) of Theorem 8, according to Lemma 5, it follows that

$$(\varphi_0 \cdot V_1(t, x^*(t))) \le r_1(t; t_0^*, u_0), \quad t \in [t_0^*, t^*].$$
(36)

From the choice of the point  $t_0^*$  it follows that  $(\varphi_0 \cdot h_0(t_0^*, \phi^*(t_0^*))) = (\varphi_0 \cdot h_0(t_0^*, y^*(t_0^*))) = \delta_3 < \delta_2$ .

According to inequalities (19) and (23) we obtain

$$u_{0} = (\varphi_{0} \cdot V_{1}(t_{0}^{*}, \phi^{*}(t_{0}^{*})))$$
  

$$\leq \psi_{1}(\varphi_{0} \cdot h_{0}(t_{0}^{*}, \phi^{*}(t_{0}^{*}))) < \delta_{1}.$$
(37)

From inequalities (22) and (36) it follows that  $(\varphi_0 \cdot V_1(t, x^*(t))) \le r_1(t; t_0^*, u_0) < \alpha_1/2$  for  $t \in [t_0^*, t^*]$ , or

$$\left(\varphi_{0} \cdot V_{1}\left(t_{0}^{*}, y^{*}\left(t_{0}^{*}\right)\right)\right) < \left(\varphi_{0} \cdot V_{1}\left(t_{0}^{*}, x\left(t_{0}^{*}\right)\right)\right) < \frac{\alpha_{1}}{2}.$$
 (38)

From inequality (28) and condition (iii) of Theorem 8, it follows that

$$\left( \varphi_0 \cdot V_2^{(\alpha)} \left( t_0^*, y^* \left( t_0^* \right) \right) \right) < a \left( \varphi_0 \cdot h_0 \left( t_0^* + s, y^* \left( t_0^* + s \right) \right) \right)$$
  
=  $a \left( \delta_3 \right) < \frac{\alpha_1}{2}.$  (39)

Consider function  $V_2^{(\alpha)}(t, x)$  that is defined in condition (*H7*) of Theorem 8, and define the function

$$V(t, x) = V_1(t, x) + V_2^{(\alpha)}(t, x), \qquad (40)$$

the function V(t, x) satisfies the conditions of Lemma 5. Let point  $t \in [t_0^*, t^*]$ ,  $t \neq t_k$ , and function  $\psi \in PC([t - r, t], R^n)$  be such that  $(t, \psi(t)) \in S(h, \beta, \varphi_0) \cap S^c(h_0, \alpha, \varphi_0)$ ,  $(\psi(t), \sup_{s \in [-r,0]} \psi(t + s)) \in \Omega(t_0^*, T^*, \beta)$ , and  $V(t, \psi(t)) > V(t + s, \psi(t + s))$  for  $s \in [-r, 0)$ . Then using the Lipshitz conditions for functions  $V_1(t, x)$  and  $V_2^{(\alpha)}(t, x)$ , and condition (iv) of Theorem 8, we obtain

$$\begin{aligned} \left(\varphi_{0} \cdot D_{(2)}V\left(t,\psi\left(t\right)\right)\right) \\ &= \left(\varphi_{0} \cdot \left(D_{(2)}V_{1}\left(t,\psi\left(t\right)\right) + D_{(2)}V_{2}^{(\alpha)}\left(t,\psi\left(t\right)\right)\right)\right) \\ &\leq \left(\varphi_{0} \cdot D_{(1)}V_{1}\left(t,\psi\left(t\right)\right) + D_{(1)}V_{2}^{(\alpha)}\left(t,\psi\left(t\right)\right)\right) \\ &+ \left(M_{1} + M_{2}\right) \left\|G\left(t,\psi\left(t\right),\sup_{s\in[-r,0]}\psi\left(t+s\right)\right)\right\| \quad (41) \\ &\leq g_{2}\left(t,\left(\varphi_{0} \cdot V\left(t,\psi\left(t\right)\right)\right)\right) + \left(M_{1} + M_{2}\right) \\ &\times \sup_{(x,y)\in\Omega\left(t_{0}^{*},T^{*},\beta\right)}\left\|G\left(t,x,y^{*}\right)\right\|, \end{aligned}$$

where  $T^* = t^* - t_0^*$ .

Let  $\tau_k \in (t_0^*, t^*)$ ,  $x \in \mathbb{R}^n$  be such that  $(\tau_k, x) \in S(h, \beta, \varphi_0) \cap S^c(h_0, \alpha, \varphi_0)$ . According to condition (v) of Theorem 8, we have

$$\begin{aligned} \left(\varphi_{0} \cdot V\left(t_{k}+0, I_{k}\left(x\right)+J_{k}\left(x\right)\right)\right) \\ &= \left(\varphi_{0} \cdot V\left(t_{k}+0, I_{k}\left(x\right)\right)\right) \\ &+ \left(\varphi_{0} \cdot \left(V\left(t_{k}+0, I_{k}\left(x\right)+J_{k}\left(x\right)\right)-V\left(t_{k}+0, I_{k}\left(x\right)\right)\right)\right) \\ &\leq \eta_{k}\left(\varphi_{0} \cdot V\left(t_{k},x\right)\right) + \left(M_{1}+M_{2}\right) \left\|J_{k}\left(x\right)\right\| \\ &\leq \eta_{k}\left(\varphi_{0} \cdot V\left(t_{k},x\right)\right) + \left(M_{1}+M_{2}\right) \\ &\times \sup_{x:h(\tau_{k},x) < \beta} \left\|J_{k}\left(x\right)\right\|. \end{aligned}$$

$$(42)$$

According to inequalities (41), (42) and Lemma 5, the inequality

$$(\varphi_0 \cdot V(t, y^*(t))) \le r^*(t; t_0^*, w_0^*), \quad t \in [t_0^*, t^*]$$
(43)

holds.

Consider the scalar impulsive differential equation (9), where

$$q(t) = (M_{1} + M_{2}) \sup_{(x,y)\in\Omega(t_{0}^{*},T^{*},\beta)} \|G(t,x,y^{*})\|,$$

$$\gamma_{k} = (M_{1} + M_{2}) \sup_{x:h(\tau_{k},x)<\beta} \|J_{k}(x)\|.$$
(44)

According to above notations and inequality (31) for  $T^* = t^* - t^*_0$ , we obtain

$$\int_{t_0^*}^{t^*} q(s) \, ds + \sum_{\substack{t_0^* \le \tau_k \le t^*}} \gamma_k \le \left(M_1 + M_2\right) \alpha = \alpha_1. \tag{45}$$

Let  $r^*(t; t_0^*, w_0^*)$  be the maximal solution of (9) through the point  $(t_0^*, w_0^*)$ , where  $w_0^* = V(t_0^* + s, y^*(t_0^* + s))$ , and perturbations q(t) and  $\gamma_k$  are defined above and satisfy inequality (45).

Choose a point  $T^* > t^*$  such that

$$\int_{t_0^*}^{t^*} q(s) \, ds + \frac{1}{2} \left( T^* - t^* \right) q(t^*) < \alpha_1. \tag{46}$$

Now define the continuous function  $q^*(t) : [t_0^*, \infty) \rightarrow R$ :

$$q^{*}(t) = \begin{cases} q(t) & \text{for } t \in [t_{0}^{*}, t^{*}] \\ \frac{q(t^{*})}{t^{*} - T^{*}} (t - T^{*}) & \text{for } t \in [t^{*}, T^{*}] \\ 0 & \text{for } t \ge T^{*}, \end{cases}$$
(47)

and the sequence of numbers  $\{\gamma_k^*\}_1^\infty$ :

$$\gamma_k^* = \begin{cases} \gamma_k & \text{for } k : \tau_k \in (t_0^*, t^*] \\ 0 & \text{for } k : \tau_k > t^*. \end{cases}$$
(48)

From (45), it follows that for every T > 0

$$\int_{t_{0}^{*}}^{t_{0}^{*}+T} q^{*}(s) ds + \sum_{t_{0}^{*} \leq \tau_{k} \leq t_{0}^{*}+T} \gamma_{k}^{*}$$

$$\leq \int_{t_{0}^{*}}^{t_{0}^{*}+T} q(s) ds + \sum_{t_{0}^{*} \leq \tau_{k} \leq t_{0}^{*}+T} \gamma_{k} \leq \alpha_{1}.$$
(49)

Let  $R(t; t_0^*, w_0^*)$  be the maximal solution of the scalar impulsive differential equation (9) through the point  $(t_0^*, w_0^*)$ , where perturbations of the right parts are defined above function  $q^*(t)$  and numbers  $\gamma_k^*$ . We note that

$$R(t;t_0^*,w_0^*) = r^*(t;t_0^*,w_0^*), \quad t \in [t_0^*,t^*).$$
(50)

From inequalities (38) and (39), the definition of point  $w_0^*$ , and inequality (49) follows the validity of (24) for the solution  $R(t; t_0^*, w_0^*)$ ; that is,

$$R(t;t_0^*,w_0^*) < \beta_1, \quad t \ge t_0^*.$$
(51)

From inequalities (43) and (51), equality (50), the choice of point  $t^*$ , and condition (iii) of Theorem 8, we obtain

$$b(\beta) \ge \beta_{1} > R(t^{*}; t^{*}_{0}, w^{*}_{0})$$
  

$$\ge (\varphi_{0} \cdot V(t^{*}, y^{*}(t^{*}; t_{0}, \phi)))$$
  

$$\ge (\varphi_{0} \cdot V^{(\alpha)}_{2}(t^{*}, y^{*}(t^{*}; t_{0}, \phi)))$$
  

$$\ge b((\varphi_{0} \cdot h(t^{*}, y^{*}(t^{*}; t_{0}, \phi))))$$
  

$$= b(\beta).$$
(52)

The obtained contradiction proves the validity of the inequality (32) for  $t \ge t_0$ .

*Case 1.2.* Let there exist a point  $\tau_k \in (t_0, t^*)$  such that  $\delta_3 < (\varphi_0 \cdot h_0(\tau_k + 0, y^*(\tau_k + 0; t_0, x_0))), \delta_3 > (\varphi_0 \cdot h_0(\tau_k, y^*(\tau_k; t_0, x_0))),$ and (35) is true.

We choose a number  $\widetilde{\delta_3} : \delta_3 < \widetilde{\delta_3} < \beta$  such that  $\widetilde{\delta_3} = (\varphi_0 \cdot h_0(t_0^*, y^*(t_0^*; t_0, x_0)))$  and  $t_0^* \in (t_0, t^*), t_0^* \neq \tau_k, k = 1, 2, \dots$ We repeat the proof of Case 1.1, where instead of  $\delta_3$  we use  $\widetilde{\delta_3}$  and obtain a contradiction.

*Case 2.* Let there exist a natural number k such that  $(\varphi_0 \cdot h(t, y^*(t))) < \beta$  for  $t \in [t_0, \tau_k]$  and  $(\varphi_0 \cdot h(\tau_k, y^*(\tau_k + 0))) = (\varphi_0 \cdot h(\tau_k, I_k(y^*(\tau_k)) + J_k(y^*(\tau_k)))) > \beta$ .

We repeat the proof of Case 1 as in this case we choose the constant  $\beta = \beta(\beta_1) > 0$ , such that  $b(\beta) \ge \sup_k \{\eta_k(\beta_1)\}$ .

As in the proof of Case 1.1, we obtain the validity of inequalities (51) and (43). We apply conditions (iii) and (v) of Theorem 8 and obtain

$$b(\beta) \geq \eta_{k} \left(r^{*}(\tau_{k}; t_{0}^{*}, w_{0}^{*})\right)$$

$$\geq \eta_{k} \left(\varphi_{0} \cdot V(\tau_{k}, y^{*}(\tau_{k}))\right)$$

$$= \eta_{k} \left(\left(\varphi_{0} \cdot \left(V_{1}(\tau_{k}, y^{*}(\tau_{k})) + V_{2}^{(\alpha)}(\tau_{k}, y^{*}(\tau_{k}))\right)\right)\right)$$

$$\geq \left(\varphi_{0} \cdot V_{1}(\tau_{k}, I_{k}(y^{*}(\tau_{k})) + J_{k}(y^{*}(\tau_{k})))\right)$$

$$+ V_{2}^{(\alpha)}(\tau_{k}, I_{k}(y^{*}(\tau_{k})) + J_{k}(y^{*}(\tau_{k}))))$$

$$\geq \left(\varphi_{0} \cdot V_{2}^{(\alpha)}(\tau_{k}, I_{k}(y^{*}(\tau_{k})) + J_{k}(y^{*}(\tau_{k})))\right)$$

$$\geq b\left(\varphi_{0} \cdot h\left((\tau_{k}, I_{k}(y^{*}(\tau_{k})) + J_{k}(y^{*}(\tau_{k})))\right)\right)$$

$$\geq b\left(\beta\right),$$
(53)

and the obtained contradiction proves the validity of inequality (32) in this case. Inequality (32) proves  $(H_0, h)$ -uniformintegral  $\varphi_0$ -stabilities of the considered system of the impulsive differential equations with "supremum."

Next, we will provide an example which satisfies all the hypotheses of Theorem 8.

*Example 9.* Consider the system of impulsive differential equations with "supremum"

$$\begin{aligned} x' &= -e^{-t}x(t) + 2y(t) + e^{-t}\max_{s\in[t-r,t]}x(s), \quad t \neq k, \\ y' &= -x(t) - e^{-t}y(t) + \frac{1}{2}e^{-t}\max_{s\in[t-r,t]}y(s), \quad t \neq k, \\ x(k+0) &= \frac{1}{2^{k/2}}x(k), \\ y(k+0) &= \frac{1}{2^{k/2}}y(k), \\ k &= 1, 2, \dots, \\ x(t) &= \phi_1(t-t_0), \\ y(t) &= \phi_2(t-t_0) \\ t &\in [t_0 - r, t_0], \end{aligned}$$
(54)

and its perturbed impulsive differential equations with "supremum"

$$x' = -e^{-t}x(t) + 2y(t) + e^{-t}\max_{s \in [t-r,t]} x(s) + e^{-t}\max_{s \in [t-r,t]} x^{2}(s),$$
  
$$t \neq k,$$

$$y' = -x(t) - e^{-t}y(t) + \frac{1}{2}e^{-t}\max_{s\in[t-r,t]}y(s) + e^{-t}\max_{s\in[t-r,t]}y^{2}(s),$$
$$t \neq k,$$

$$x (k + 0) = \frac{1}{2^{k/2}} x (k),$$
  

$$y (k + 0) = \frac{1}{2^{k/2}} y (k),$$
  

$$k = 1, 2, \dots,$$
  

$$= \phi_{1} (t - t_{0}), \quad y (t) = \phi_{0} (t - t_{0}), \quad t \in [t_{0} - t_{0}],$$

$$x(t) = \phi_1(t - t_0), \quad y(t) = \phi_2(t - t_0) \quad t \in [t_0 - r, t_0],$$
(55)

where  $x, y \in R$ , r > 0 is enough small constant,  $t \ge t_0 \ge 0$ . Without loss of generality we will assume further that  $1 \ge t_0 \ge 0$ .

Let  $h_0(t, x, y) = (||x||, ||y||), h(t, x, y) = (x^2, y^2).$ 

Consider function  $V : \mathbb{R}^2 \to \mathcal{K}, V = (V_1, V_2), V_1(x, y) = (1/2)x^2, V_2(x, y) = (1/2)y^2$ , where  $\mathcal{K} = \{(x, y) : x \ge 0, y \ge 0\} \subset \mathbb{R}^2$  is a cone.

Now, let us consider the vector  $\varphi_0 = (1, 2)$ . It is easy to check that the function  $V_1(t, x, y) = V(x, y)$  is  $\varphi_0$ -strongly  $h_0$ -decrescent with a function  $\psi_2 = x \in K$  and the condition (iii) is satisfied for the function  $V_2^{(\mu)} = V(x, y)$ , where b(u) = (1/2)u and  $a(u) = u^2$ .

Let  $t \ge 0$ ,  $t \ne k$ ,  $k = 1, 2... \psi \in PC([t - r, t], R^2)$ ,  $\psi = (\psi_1, \psi_2)$  be such that the inequality

$$(\varphi_{0} \cdot V(\psi_{1}(t), \psi_{2}(t)))$$

$$\geq (\varphi_{0} \cdot V(\psi_{1}(t+s), \psi_{2}(t+s))), \quad s \in [-r, 0]$$
(56)

or

$$\frac{1}{2}\psi_{1}^{2}(t) + \psi_{2}^{2}(t) \ge \frac{1}{2}\psi_{1}^{2}(t+s) + \psi_{2}^{2}(t+s), \quad s \in [-r,0];$$
(57)

then

$$\psi_{1}(t) \max_{s \in [t-r,t]} \psi_{1}(s) \leq 2 \left( \varphi_{0} \cdot V \left( \psi_{1}(t), \psi_{2}(t) \right) \right),$$

$$\psi_{2}(t) \max_{s \in [t-r,t]} \psi_{2}(s) \leq \left( \varphi_{0} \cdot V \left( \psi_{1}(t), \psi_{2}(t) \right) \right).$$
(58)

Therefore if inequality (57) is satisfied then

$$\begin{aligned} \left(\varphi_{0} \cdot D_{(54)} V\left(\psi_{1}\left(t\right),\psi_{2}\left(t\right)\right)\right) \\ &= e^{-t} \left(-\left(\psi_{1}\left(t\right)\right)^{2} - 2\left(\psi_{2}\left(t\right)\right)^{2} \\ &+\psi_{1}\left(t\right) \max_{s \in [t-r,t]} \psi_{1}\left(s\right) + \psi_{2}\left(t\right) \max_{s \in [t-r,t]} \psi_{2}\left(s\right)\right) \\ &\leq e^{-t} \left(-\left(\psi_{1}\left(t\right)\right)^{2} - 2\left(\psi_{2}\left(t\right)\right)^{2} + 2\left(\varphi_{0} \cdot V\left(\psi_{1}\left(t\right),\psi_{2}\left(t\right)\right)\right) \\ &+ \left(\varphi_{0} \cdot V\left(\psi_{1}\left(t\right),\psi_{2}\left(t\right)\right)\right)\right) \\ &= e^{-t} \left(\varphi_{0} \cdot V\left(\psi_{1}\left(t\right),\psi_{2}\left(t\right)\right)\right) \end{aligned}$$
(59)

or

$$(\varphi_{0} \cdot D_{(54)} V (\psi_{1}(t), \psi_{2}(t)))$$

$$\leq e^{-t} (\varphi_{0} \cdot V (\psi_{1}(t), \psi_{2}(t))).$$
(60)

Inequality (60) proves the validity of condition (i) of Theorem 8 for the function  $V_1(t, x, y) = V(x, y)$ , where  $g_1(t, u) = ue^{-t}$ . Meanwhile, inequality (60) proves the validity of condition (iv) of Theorem 8 for the function  $V_2^{\mu}(t, x, y) = V(x, y)$ , where  $g_2(t, u) = 2ue^{-t}$ .

From jump conditions (54) and the choice of vector  $\varphi_0$ and function *V* we obtain the validity of conditions (ii) and (v) of Theorem 8 for the functions  $V_1(t, x, y) = V(x, y)$  and  $V_2^{\mu}(t, x, y) = V(x, y)$ , where  $\xi_k(u) = (1/2^k)u$  and  $\eta_k(u) = (1/2^k)u$ .

Consider following comparison scalar impulsive differential equation:

$$u' = ue^{-t}, \quad t \neq k, \quad u(k+0) = \frac{1}{2^k}u(k), \quad (61)$$

$$w' = 2we^{-t}, \quad t \neq k, \quad w(k+0) = \frac{1}{2^k}w(k).$$
 (62)

The solutions of the impulsive differential equation (61) and (62), correspondingly, are equi-stable and uniformintegrally stable. Thus, according to Theorem 8 the system of impulsive differential equations with "supremum" (54) is  $(H_0, h)$ -uniform-integrally  $\varphi_0$ -stable.

#### 4. Conclusion

This paper extends the notions of  $\varphi_0$ -stability in terms of two measures to integral  $\varphi_0$ -stability in terms of two measures for impulsive differential equations with "supremum" and establishes a criterion on integral  $\varphi_0$ -stability in terms of two measures for such system by using the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. Finally, an example is given to illustrate our result.

#### **Conflict of Interests**

The authors declare that they have no conflict of interests.

#### **Authors' Contribution**

All authors completed the paper together. All authors read and approved the final paper.

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