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Research Article

Solving Initial-Boundary Value Problems for Local Fractional Differential Equation by Local Fractional Fourier Series Method

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The initial-boundary value problems for the local fractional differential equation are investigated in this paper. The local fractional Fourier series solutions with the nondifferential terms are obtained. Two illustrative examples are given to show efficiency and accuracy of the presented method to process the local fractional differential equations.

1. Introduction

In various fields of physics, mathematics, and engineering, because of the different operators, there are classical differential equations [1], fractional differential equation [2–4], and local fractional differential equations [5, 6]. There are more techniques to achieve analytical approximations to the solutions to differential equations in mathematical physics, such as the decomposition method [7], the variational iteration method [8], the homotopy perturbation method [9], the heat-balance integral method [10], the Fourier transform [11], the Laplace transform [11], and the references therein.

Recently, a new Fourier series (local fractional Fourier series) via local fractional operator was proposed [6] and had various applications in the applied fields such as fractal wave problems in fractal string [12, 13] and the heat-conduction problems arising in fractal heat transfer [14, 15]. For a detailed description of the local fractional Fourier series method, we refer the readers to the recent works [14–16]. This is the main advantage of local fractional differential equations in comparison with classical integer-order and fractional-order models.

In the present paper we consider the local fractional differential equation:

$$\frac{\partial^{\alpha} u(x,y)}{\partial y^{\alpha}} - \frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} = 0, \tag{1}$$

subject to the initial-boundary value conditions:

$$\frac{\partial^{\alpha} u(0,t)}{\partial x^{\alpha}} = 0, \qquad \frac{\partial^{\alpha} u(L,t)}{\partial x^{\alpha}} = 0, \qquad u(x,0) = g(x), \tag{2}$$

where the operators are described by the local fractional differential operators [5, 6, 12–15]. The paper is organized as follows. In Section 2, the basic theory of the local fractional calculus and local fractional Fourier series is presented. In Section 3, we discuss the initial-boundary problems for local fractional differential equation. Finally, Section 4 is devoted to the conclusions.

2. Analysis of the Method

In this section, we present the basic theory of the local fractional calculus and analyze the local fractional Fourier series method.

Definition 1. Let F be a subset of the real line and be a fractal. The mass function $\gamma^{\alpha}[F,a,b]$ can be written as [5]

$$\gamma^{\alpha}[F,a,b] = \lim_{\max_{0<i<n-1}(x_{i+1}-x_i)\to 0} \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(1+\alpha)}. \tag{3}$$

The following properties are present as follows [5].

(i) If $F \cap (a,b) = \varnothing$, then $\gamma^{\alpha}[F,a,b] = 0$.

(ii) If $a < b < c$ and $\gamma^{\alpha}[F,a,b] < 0$, then $\gamma^{\alpha}[F,a,b] + \gamma^{\alpha}[F,b,c] = \gamma^{\alpha}[F,a,c]$.

If $f : (F,d) \rightarrow (\Omega', d')$ is a bi-Lipschitz mapping, then we have [5, 12]

$$\rho^{s} H^{s}(F) \leq H^{s}(f(F)) \leq \tau^{s} H^{s}(F) \tag{4}$$

such that

$$\rho^{\alpha}\left|x_1 - x_2\right|^{\alpha} \leq \left|f(x_1) - f(x_2)\right| \leq \tau^{\alpha}\left|x_1 - x_2\right|^{\alpha}. \tag{5}$$

In view of (5), we have

$$\left|f(x_1) - f(x_2)\right| \leq \tau^{\alpha}\left|x_1 - x_2\right|^{\alpha} \tag{6}$$

such that

$$\left|f(x_1) - f(x_2)\right| < \varepsilon^{\alpha}, \tag{7}$$

where α is the fractal dimension of F. This result is directly deduced from fractal geometry and relates to the fractal coarse-grained mass function $\gamma^{\alpha}[F,a,b]$, which reads [5, 13]

$$\gamma^{\alpha}[F,a,b] = \frac{H^{\alpha}(F \cap (a,b))}{\Gamma(1+\alpha)} \tag{8}$$

with

$$H^{\alpha}(F \cap (a,b)) = (b-a)^{\alpha}, \tag{9}$$

where H^{α} is α dimensional Hausdorff measure.

Definition 2. If there is [5, 6, 12–15]

$$\left|f(x) - f(x_0)\right| < \varepsilon^{\alpha} \tag{10}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then $f(x)$ is called local fractional continuous at $x = x_0$.

If $f(x)$ is local fractional continuous on the interval (a,b), then we can write it in the form [5, 6, 12]

$$f(x) \in C_{\alpha}(a,b). \tag{11}$$

Definition 3. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined as follows [5, 6, 12–15]:

$$f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \tag{12}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$.

From (12) we can rewrite the local fractional derivative as

$$f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{\gamma^{\alpha}[F, x_0, x]}, \tag{13}$$

where

$$\gamma^{\alpha}[F, x_0, x] = \frac{H^{\alpha}(F \cap (a,b))}{\Gamma(1+\alpha)}. \tag{14}$$

Definition 4. The partition of the interval $[a,b]$ is (t_j, t_{j+1}), $j = 0, \ldots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$. Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is given by [5, 6, 12–15]

$$\begin{aligned} {}_{a}I_{b}^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} f(t)(dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha}. \end{aligned} \tag{15}$$

Following (14), we have

$${}_{a}I_{b}^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)\gamma^{\alpha}[F, t_j, t_{j+1}], \tag{16}$$

where

$$\gamma^{\alpha}[F,a,b] = \lim_{\max_{0<i<n-1}(x_{i+1}-x_i) \to 0} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(1+\alpha)}. \tag{17}$$

If F are Cantor sets, we can get the derivative and integral on Cantor sets.

Some properties of local fractional integrals are listed as follows:

$$\begin{gathered} {}_{0}I_{x}^{(\alpha)} E_{\alpha}(x^{\alpha}) = E_{\alpha}(x^{\alpha}) - 1, \\ {}_{0}I_{x}^{(\alpha)} \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}, \\ {}_{0}I_{x}^{(\alpha)} \sin_{\alpha}(a^{\alpha}x^{\alpha}) = \frac{1}{a^{\alpha}}\left[\cos_{\alpha}(a^{\alpha}x^{\alpha}) - 1\right], \\ {}_{0}I_{x}^{(\alpha)} \cos_{\alpha}(a^{\alpha}x^{\alpha}) = \frac{1}{a^{\alpha}} \sin_{\alpha}(a^{\alpha}x^{\alpha}), \\ {}_{0}I_{x}^{(\alpha)} \frac{x^{\alpha}}{\Gamma(1+\alpha)} \sin_{\alpha}(a^{\alpha}x^{\alpha}) \\ = -\frac{1}{a^{\alpha}}\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)} \cos_{\alpha}(a^{\alpha}x^{\alpha}) - \frac{1}{a^{\alpha}} \sin_{\alpha}(a^{\alpha}x^{\alpha})\right], \\ {}_{0}I_{x}^{(\alpha)} \frac{x^{\alpha}}{\Gamma(1+\alpha)} \cos_{\alpha}(a^{\alpha}x^{\alpha}) \\ = \frac{1}{a^{\alpha}}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)} \sin_{\alpha}(a^{\alpha}x^{\alpha}) - \frac{1}{a^{\alpha}}\left[\cos_{\alpha}(a^{\alpha}x^{\alpha}) - 1\right]\right\}, \\ {}_{0}I_{x}^{(\alpha)} \left\{E_{\alpha}(x^{\alpha}) \sin_{\alpha}(a^{\alpha}x^{\alpha})\right\} \\ = \frac{E_{\alpha}(x^{\alpha})\left[\sin_{\alpha}(a^{\alpha}x^{\alpha}) - a^{\alpha}\cos_{\alpha}(a^{\alpha}x^{\alpha})\right] + a^{\alpha}}{1 + a^{2\alpha}}, \\ {}_{0}I_{x}^{(\alpha)} \left\{E_{\alpha}(x^{\alpha}) \cos_{\alpha}(a^{\alpha}x^{\alpha})\right\} \\ = \frac{E_{\alpha}(x^{\alpha})\left[\cos_{\alpha}(a^{\alpha}x^{\alpha}) + a^{\alpha}\sin(a^{\alpha}x^{\alpha})\right] - 1}{1 + a^{2\alpha}}. \end{gathered} \tag{18}$$

Definition 5. Local fractional trigonometric Fourier series of $f(t)$ is given by [6, 12–16]

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_k \sin_\alpha (k^\alpha {\omega_0}^\alpha t^\alpha) + \sum_{i=1}^{\infty} b_k \cos_\alpha (k^\alpha {\omega_0}^\alpha t^\alpha) \quad (19)$$

for $x \in R$ and $0 < \alpha \le 1$.

The local fractional Fourier coefficients of (19) can be computed by

$$\begin{aligned} a_0 &= \frac{1}{T^\alpha} \Gamma(1+\alpha)\ {}_0I_T f(t), \\ a_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0I_T \{f(t) \sin_\alpha (k^\alpha {\omega_0}^\alpha t^\alpha)\}, \\ b_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0I_T \{f(t) \cos_\alpha (k^\alpha {\omega_0}^\alpha t^\alpha)\}. \end{aligned} \quad (20)$$

If $\omega_0 = 1$, then we get

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_k \sin_\alpha (k^\alpha t^\alpha) + \sum_{i=1}^{\infty} b_k \cos_\alpha (k^\alpha t^\alpha), \quad (21)$$

where the local fractional Fourier coefficients can be computed by

$$\begin{aligned} a_0 &= \frac{1}{T^\alpha} \Gamma(1+\alpha)\ {}_0I_T f(t), \\ a_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0I_T \{f(t) \sin_\alpha (k^\alpha t^\alpha)\}, \\ b_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0I_T \{f(t) \cos_\alpha (k^\alpha t^\alpha)\}. \end{aligned} \quad (22)$$

3. The Initial-Boundary Problems for the Local Fractional Differential Equation

In this section, we consider (1) with the various initial-boundary conditions.

Example 6. The initial-boundary condition (2) becomes

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0, \qquad \frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0, \qquad u(x,0) = E_\alpha(x^\alpha). \quad (23)$$

Let $u = XY$ in (1). Separation of the variables yields

$$XY^{(\alpha)} = YX^{(2\alpha)}. \quad (24)$$

Setting

$$\frac{Y^{(\alpha)}}{Y} = \frac{X^{(2\alpha)}}{X} = -\lambda^{2\alpha}, \quad (25)$$

we obtain

$$\begin{aligned} X^{(2\alpha)} + \lambda^{2\alpha} X &= 0, \\ Y^{(\alpha)} + \lambda^{2\alpha} Y &= 0. \end{aligned} \quad (26)$$

Hence, we have their solutions, which read

$$\begin{aligned} X &= a \cos_\alpha (\lambda^\alpha x^\alpha) + b \sin_\alpha (\lambda^\alpha x^\alpha), \\ Y &= c E_\alpha \left(-\lambda^{2\alpha} y^\alpha\right). \end{aligned} \quad (27)$$

Therefore, a solution is written in the form

$$\begin{aligned} u(x,y) &= XY \\ &= E_\alpha \left(-\lambda^{2\alpha} y^\alpha\right) \left(A \cos_\alpha (\lambda^\alpha x^\alpha) + B \sin_\alpha (\lambda^\alpha x^\alpha)\right), \end{aligned} \quad (28)$$

where $A = ac$, $B = bc$.

For the given condition

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0, \quad (29)$$

there is $B = 0$, so that

$$u(x,y) = A E_\alpha \left(-\lambda^{2\alpha} y^\alpha\right) \cos_\alpha (\lambda^\alpha x^\alpha). \quad (30)$$

For the given condition

$$\frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0, \quad (31)$$

we obtain

$$\sin_\alpha (\lambda^\alpha x^\alpha) = 0, \quad (32)$$

$$\lambda^\alpha = \left(\frac{m\pi}{L}\right)^\alpha, \quad m \in Z^+ \cup 0. \quad (33)$$

Thus, from (33) we deduce that

$$u(x,y) = A E_\alpha \left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right) \cos_\alpha \left(\left(\frac{m\pi x}{L}\right)^\alpha\right), \quad m \in Z^+ \cup 0. \quad (34)$$

To satisfy the condition (23), (34) is written in the form

$$\begin{aligned} u(x,y) &= A_0 + \sum_{m=1}^{\infty} A_m E_\alpha \left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right) \cos_\alpha \left(\left(\frac{m\pi x}{L}\right)^\alpha\right). \end{aligned} \quad (35)$$

Then, we derive

$$A_0 = \frac{\Gamma(1+\alpha)}{L^\alpha}\left(E_\alpha\left(L^\alpha\right)-1\right),$$

$$A_m = \left(\frac{2}{L}\right)^\alpha \Gamma(1+\alpha) E_\alpha\left(x^\alpha\right) \times \frac{\left[\cos_\alpha\left((m\pi/L)^\alpha x^\alpha\right)+(m\pi/L)^\alpha \sin\left((m\pi/L)^\alpha x^\alpha\right)\right]-1}{1+(m\pi/L)^{2\alpha}},$$

$$\begin{aligned} u(x,y) &= \frac{\Gamma(1+\alpha)}{2L^\alpha}\left(E_\alpha\left(L^\alpha\right)-1\right) \\ &+ \sum_{m=1}^{\infty} \frac{\left[\cos_\alpha\left((m\pi/L)^\alpha x^\alpha\right)+(m\pi/L)^\alpha \sin\left((m\pi/L)^\alpha x^\alpha\right)\right]-1}{1+(m\pi/L)^{2\alpha}} \\ &\times E_\alpha\left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right)\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right) \\ &\times E_\alpha\left(x^\alpha\right)\left(\frac{2}{L}\right)^\alpha \Gamma(1+\alpha). \end{aligned} \tag{36}$$

Example 7. Let us consider (1) with the initial-boundary value condition, which becomes

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0, \qquad \frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0, \qquad u(x,0) = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{37}$$

Following (35), we have

$$u(x,y) = A_0 + \sum_{m=1}^{\infty} A_m E_\alpha\left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right)\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right), \tag{38}$$

where

$$\begin{aligned} A_0 &= \frac{1}{L^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha}, \\ A_m &= \frac{1}{(2m\pi)^\alpha}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)\right. \\ &\quad \left. -\left(\frac{L}{m\pi}\right)^\alpha\left[\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)-1\right]\right\}. \end{aligned} \tag{39}$$

Hence, we get

$$\begin{aligned} u(x,y) &= \frac{1}{L^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha} \\ &+ \sum_{m=1}^{\infty}\frac{1}{(2m\pi)^\alpha}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)\right. \\ &\quad \left. -\left(\frac{L}{m\pi}\right)^\alpha\left[\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)-1\right]\right\} \\ &\times E_\alpha\left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right)\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right). \end{aligned} \tag{40}$$

4. Conclusions

In this work, the initial-boundary value problems for the local fractional differential equation are discussed by using the local fractional Fourier series method. Analytical solutions for the local fractional differential equation with the nondifferentiable conditions are obtained.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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